Newton’s law of cooling

The temperature of a hot object decreases at a rate proportional to the temperature difference between the object and its surroundings.

- Let \( H = H(t) \) be the temperature of the object at time \( t \).
- Let \( R \) be the (constant) room temperature.

We obtain the DE

\[
- \frac{dH}{dt} = k(H - R), \quad \text{or}
\]

\[
\frac{dH}{dt} = -k(H - R).
\]
Newton’s law of cooling

The solution to the above DE is

\[ H = R + Ae^{-kt}, \quad \text{where } A \text{ is some constant.} \]

Here is the slope diagram (for \( k = R = 2 \)):

Curves: \( A = 2, \ A = 4, \ A = -2, \ A = 0. \)
When $A = 0$, we obtain the constant solution

$$y = R.$$ 

This is the equilibrium solution to the DE.

- It corresponds to the object being at the same temperature as the surrounding air (and never changing).
- If the initial temperature is greater than $R$, then the function $H = R + Ae^{-kt}$ decreases towards $R$ as $t$ increases.
- If the initial temperature is less than $R$, then $H$ increases towards $R$ as $t$ increases.

For this reason, $H = R$ is called a stable equilibrium solution.
Equilibrium solutions

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\[
\frac{dy}{dx} = F(x, y)
\]

is a constant function \( y = C \) which satisfies the DE.

To find the equilibrium solutions, sub \( y = C \) into the DE, and solve the resulting equation:

\[
0 = F(x, C).
\]
Equilibrium solutions

Example

Find all equilibrium solutions to the following DEs:

1. \[ \frac{dy}{dt} = 2y, \]

2. \[ \frac{dy}{dt} = 2y(20 - y), \]

3. \[ \frac{dy}{dt} = 2y(t - y). \]
Here is the slope diagram for the logistic equation $\frac{dy}{dt} = 2y(2 - y)$:

The equilibrium solutions are $y = 0$ and $y = 2$. The solution curves tend towards $y = 2$. This is a stable equilibrium.
The equilibrium solutions are $y = 0$ and $y = 2$. The lower solution curves tend away from $y = 0$. This is an \textit{unstable} equilibrium.
Stability of equilibrium solutions

**Definition**

Let \( y = C \) be an equilibrium solution to the DE

\[
\frac{dy}{dx} = F(x, y).
\]

We say that \( y = C \) is:

- **stable** if, when we shift the value of \( y \) slightly away from \( C \), \( y \) tends towards \( C \) as \( x \) increases, or
- **unstable** if, when we shift the value of \( y \) slightly away from \( C \), \( y \) tends away from \( C \) as \( x \) increases.
Stability of equilibrium solutions

Slope field for $\frac{dy}{dt} = 2y(2 - y)$:

- $y = 2$ — stable.
- $y = 0$ — unstable.
Stability of equilibrium solutions

Slope field for $\frac{dy}{dt} = 1.5(y - 2)^2(y - 4)$:

- $y = 4$ — stable.
- $y = 2$ — neither stable nor unstable.
Stability of equilibrium solutions

Slope field for \( \frac{dy}{dt} = 2y(t - y) \):

- \( y = 0 \) — unstable! As \( t \) increases, \( y \) will eventually move away from 0.
Stability of equilibrium solutions

To determine the stability of an equilibrium solution \( y = C \), we check the sign of the derivative near \( y = C \):

\[
\begin{array}{c|ccc}
    y & C^- & C & C^+ \\
    \frac{dy}{dt} & + & 0 & - \\
\end{array}
\]  \implies \text{stable}

\[
\begin{array}{c|ccc}
    y & C^- & C & C^+ \\
    \frac{dy}{dt} & - & 0 & + \\
\end{array}
\]  \implies \text{unstable}

\[
\begin{array}{c|ccc}
    y & C^- & C & C^+ \\
    \frac{dy}{dt} & \text{anything else} \\
\end{array}
\]  \implies \text{neither}
Example

Determine the stability of the equilibrium solutions to the following DEs:

1. \[
\frac{dy}{dt} = 2y,
\]

2. \[
\frac{dy}{dt} = 2y(20 - y),
\]

3. \[
\frac{dy}{dt} = 2y(t - y).
\]