PHAR2813 - Mathematics for Pharmacy

Lecture 6
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Simultaneous DE’s

We have seen that the growth of a population is typically governed by a differential equation.

Examples.

- Exponential growth:
  \[ \frac{dy}{dt} = ky. \]

- Logistic growth:
  \[ \frac{dy}{dt} = ky(M - y). \]

It is very rare for a species to exist in isolation, however. In more realistic models, there will be another species (or several others) fighting for survival in the same environment. These groups might be working together, or might be in conflict.
Imagine two populations living together in the same environment. Suppose their growth functions are given by

\[ x = x(t) \quad \text{and} \quad y = y(t), \]

where \( t \) represents time. The presence of each species could have an impact on the other (positive or negative, or maybe none at all). We can often describe this situation with a pair of simultaneous DE’s (or linked DE’s):

\[
\frac{dx}{dt} = Ax + By \\
\frac{dy}{dt} = Cx + Dy.
\]
Simultaneous DE’s

Consider the pair of simultaneous DE’s:

\[
\frac{dx}{dt} = Ax + By
\]
\[
\frac{dy}{dt} = C x + Dy.
\]

Here \(A, B, C, D\) are constants:

- \(A\) represents the rate at which Species 1 reproduces,
- \(D\) represents the rate at which Species 2 reproduces,
- \(B\) represents the effect Species 2 has on the growth rate of Species 1, and
- \(C\) represents the effect Species 1 has on the growth rate of Species 2.
Consider the pair of simultaneous DE’s:

\[
\frac{dx}{dt} = 2x + y \\
\frac{dy}{dt} = -x + 2y.
\]

Here:

- Species 2 has an \textbf{excitatory effect} on Species 1,
- Species 1 has an \textbf{inhibitory effect} on Species 2.

Possible explanation:

- Species 1 eats Species 2!
Examples

Consider the pair of simultaneous DE’s:

\[
\frac{dx}{dt} = 2x - y
\]
\[
\frac{dy}{dt} = x + 2y.
\]

Here:

- Species 2 has an inhibitory effect on Species 1,
- Species 1 has an excitatory effect on Species 2.

Possible explanation:

- Species 2 eats Species 1! Reverse of previous case.
Consider the pair of simultaneous DE's:

\[
\frac{dx}{dt} = 2x - y
\]
\[
\frac{dy}{dt} = -x + 2y.
\]

Here:

- each species has an inhibitory effect on the other.

Possible explanation:

- they both compete for the same food source.
Consider the pair of simultaneous DE’s:

\[
\frac{dx}{dt} = 2x + y
\]

\[
\frac{dy}{dt} = x + 2y.
\]

Here:

- each species has an excitatory effect on the other.

Possible explanation:

- the two species work together for survival.

How do we solve such systems of DE’s?
Lions and cheetahs occupy the same region, competing for the same food source. Their populations are represented by the functions $x = x(t)$ and $y = y(t)$ respectively, and these functions satisfy the simultaneous DE’s:

\[ x' = 3x - 2y \]
\[ y' = -2x + 3y. \]

If there are initially 1,000 lions, and 2,000 cheetahs, what will happen in the long run?
General strategy of solution

We begin with the simultaneous DE’s:

\[ x' = Ax + By \]  \hspace{1cm} (1) \\
\[ y' = Cx + Dy. \] \hspace{1cm} (2)

**Step 1.** We rearrange Eq. 1 to make \( y \) the subject:

\[ y = \frac{1}{B} x' - \frac{A}{B} x. \]  \hspace{1cm} (3)

**Step 2.** We sub this into Eq. 2:

\[ y' = Cx + D \left( \frac{1}{B} x' - \frac{A}{B} x \right), \]

i.e.

\[ y' = \left( C - \frac{AD}{B} \right) x + \frac{D}{B} x'. \]  \hspace{1cm} (4)
General strategy of solution

**Step 3.** We differentiate Eq. 3:

\[ y' = \frac{1}{B}x'' - \frac{A}{B}x'. \]  

(5)

**Step 4.** We equate Eq. 4 and Eq. 5:

\[ \left( C - \frac{AD}{B} \right)x + \frac{D}{B}x' = \frac{1}{B}x'' - \frac{A}{B}x'. \]  

(6)

**Step 5.** We rearrange Eq. 6:

\[ \frac{1}{B}x'' - \left( \frac{A}{B} + \frac{D}{B} \right)x' + \left( \frac{AD}{B} - C \right)x = 0, \]

i.e.

\[ x'' - (A + D)x' + (AD - BC)x = 0. \]  

(7)
Step 6. Eq. 7 is a homogeneous second order linear DE with constant coefficients. We solve this for $x = x(t)$ — see Lecture 4.

Step 7. When we know $x$, we differentiate to obtain $x'$. We then sub $x$ and $x'$ in Eq. 3 to obtain $y$.

Step 8. Finally, we apply the initial conditions (if any) to determine the value of any constants.
Examples

Example

Solve the following systems of simultaneous DE’s:

1. \[ x' = 5x + y \]
   \[ y' = x + 5y, \quad \text{subject to } x(0) = 1,000 \text{ and } y(0) = 3,000. \]

2. \[ x' = x + y \]
   \[ y' = -x + 3y, \quad \text{subject to } x(0) = 1,000 \text{ and } y(0) = 2,000. \]

3. \[ x' = x + y \]
   \[ y' = -x + 3y, \quad \text{subject to } x(0) = y(0) = 1,000. \]