Dual reflection monoids

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Symmetric inverse semigroups

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Symmetric inverse semigroups

groups

inverse semigroups
Theorem (Wagner-Preston)

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Coset monoids

groups

inverse semigroups
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Coset monoids

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- $C(G) = \{ Hg \mid H \leq G, \, g \in G \}$.
- $Hg \ast K \ell = \langle H \cup gKg^{-1} \rangle g \ell = (H \vee Kg^{-1})g \ell$.
- Proved by showing $\mathcal{I}_X$ embeds in $C(S_X \cup \{\infty\})$. 

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Dual symmetric inverse semigroups

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inverse semigroups
Theorem (FitzGerald-Leech)

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- $\mathcal{I}_X^* = \{ \text{bijections } A \rightarrow B \mid A, B \text{ quotients of } X \}$.
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Dual reflection monoids
Block bijections
Elements of $\mathcal{I}_X^*$ are called block bijections on $X$. 
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\[ \alpha = \in I_8^* \]
Elements of $\mathcal{I}_X^*$ are called \textit{block bijections} on $X$. For example

$$\alpha = \begin{tikzpicture}[scale=0.5]
\begin{scope}[rotate=45]
\node (1) at (0,0) {$1$}; \node (2) at (1,0) {$2$}; \node (3) at (2,0) {$3$}; \node (4) at (2.5,0.5) {$4$}; \node (5) at (3,0) {$5$}; \node (6) at (3.5,0.5) {$6$}; \node (7) at (4,0) {$7$}; \node (8) at (4.5,0.5) {$8$};
\end{scope}
\begin{scope}[rotate=-45]
\node (9) at (0,0) {$1$}; \node (10) at (1,0) {$2$}; \node (11) at (2,0) {$3$}; \node (12) at (2.5,0.5) {$4$}; \node (13) at (3,0) {$5$}; \node (14) at (3.5,0.5) {$6$}; \node (15) at (4,0) {$7$}; \node (16) at (4.5,0.5) {$8$};
\end{scope}
\node (a) at (0,1) {$\alpha = \in \mathcal{I}_8^*$};
dom(\alpha) = \{ \{1, 2\}, \{3\}, \{4, 6, 7\}, \{5, 8\} \}
Elements of $I^*_X$ are called **block bijections** on $X$. For example

$$
\alpha = \in I^*_8
$$

$$\text{dom}(\alpha) = \{ \{1, 2\}, \{3\}, \{4, 6, 7\}, \{5, 8\} \}$$

$$\text{im}(\alpha) = \{ \{1\}, \{2, 4\}, \{3\}, \{5, 6, 7, 8\} \}$$
Elements of $I^*_X$ are called block bijections on $X$. For example

$$\alpha = \in I^*_8$$

$$\text{dom}(\alpha) = \{ \{1, 2\} , \{3\} , \{4, 6, 7\} , \{5, 8\} \}$$

$$\text{im}(\alpha) = \{ \{1\} , \{2, 4\} , \{3\} , \{5, 6, 7, 8\} \}$$
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Elements of $\mathcal{I}_X^*$ are called block bijections on $X$. For example

\[
\alpha = \begin{array}{c}
\{1, 2\}, \{3\}, \{4, 6, 7\}, \{5, 8\}
\end{array}
\in \mathcal{I}_8^*
\]

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Elements of $\mathcal{I}_X^*$ are called block bijections on $X$. For example

$$\alpha = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \in \mathcal{I}_8^*
$$

$$\text{dom}(\alpha) = \{ \{1, 2\} , \{3\} , \{4, 6, 7\} , \{5, 8\} \}$$

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To multiply block bijections:
To multiply block bijections:

\[ \alpha = \]

\[ \beta = \]
To multiply block bijections:

\[ \alpha = \begin{array}{c}
\text{Diagram 1}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{Diagram 2}
\end{array} \]

\[ \beta = \begin{array}{c}
\text{Diagram 3}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{Diagram 4}
\end{array} \]

1. stack \( \alpha \) and \( \beta \),
To multiply block bijections:

1. stack $\alpha$ and $\beta$,
2. erase the middle row of dots,
To multiply block bijections:

1. stack $\alpha$ and $\beta$,
2. erase the middle row of dots,
3. calculate the connected components.

$$\alpha = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\end{array} = \alpha \beta$$
Block bijections
Units in $\mathcal{I}_X^*$ are permutations:
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$$\mathcal{G}(\mathcal{I}_X^*) = S_X.$$
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Idempotents in $\mathcal{I}_X^*$ are equivalences:
Units in $I^*_X$ are permutations:

$$\mathcal{G}(I^*_X) = S_X.$$  

Idempotents in $I^*_X$ are equivalences:

$$\mathcal{E}(I^*_X) \cong (\mathcal{EQ}_X, \vee).$$
Units in $\mathcal{I}_X^*$ are permutations:

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Idempotents in $\mathcal{I}_X^*$ are equivalences:

$$\mathbb{E}(\mathcal{I}_X^*) \cong (\mathcal{E}q_X, \vee).$$ 

$$\varepsilon = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array} \in \mathbb{E}(\mathcal{I}_8)^*$$
- **Units** in $\mathcal{I}_X^*$ are permutations:
  \[ G(\mathcal{I}_X^*) = S_X. \]

- **Idempotents** in $\mathcal{I}_X^*$ are equivalences:
  \[ E(\mathcal{I}_X^*) \cong (\mathcal{E}q_X, \lor). \]

Here $\varepsilon \lor \eta = \langle \varepsilon \cup \eta \rangle$ for $\varepsilon, \eta \in \mathcal{E}q_X$. 

\[ \varepsilon = \begin{array}{c}
\text{\includegraphics{diagram.png}}
\end{array} \in E(\mathcal{I}_8)^* \]
The factorizable part of $I_X^*$
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$$\mathcal{F}(\mathcal{I}_X^*) = \mathfrak{S}_X^* = \{ \text{uniform block bijections on } X \}.$$
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$$F(\mathcal{I}_X^*) = \mathfrak{S}_X^* = \{ \text{uniform block bijections on } X \}.$$ 

- Uniform $\equiv$ blocks in the domain are mapped to blocks of equal cardinality in the image.
The factorizable part of $\mathcal{I}_X^*$ is

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\mathcal{F}(\mathcal{I}_X^*) = \mathfrak{S}_X^* = \{ \text{uniform block bijections on } X \}.
$$

- Uniform $\equiv$ blocks in the domain are mapped to blocks of equal cardinality in the image
- $\equiv$ restriction of a permutation to an equivalence.
The factorizable part of $\mathcal{I}_X^*$ is

$$\mathbb{F}(\mathcal{I}_X^*) = \mathcal{S}_X^* = \{ \text{uniform block bijections on } X \}.$$  

- Uniform $\equiv$ blocks in the domain are mapped to blocks of equal cardinality in the image
- $\equiv$ restriction of a permutation to an equivalence.

- Write $\pi|_\varepsilon$ for the restriction of a permutation $\pi \in S_X$ to an equivalence $\varepsilon \in \mathcal{E}q_X$. 

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Uniform block bijections
Uniform block bijections

= 

Dual reflection monoids
Uniform block bijections

\[
\text{Diagram showing uniform block bijections.}
\]
Uniform block bijections
Uniform block bijections

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Uniform block bijections

Dual reflection monoids
Uniform block bijections

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\coordinate (A) at (0,0);
\coordinate (B) at (1,0);
\coordinate (C) at (2,0);
\coordinate (D) at (0,-1);
\coordinate (E) at (1,-1);
\coordinate (F) at (2,-1);
\coordinate (G) at (0,-2);
\coordinate (H) at (1,-2);
\coordinate (I) at (2,-2);
\draw (A) -- (B) -- (C);
\draw (D) -- (E) -- (F);
\draw (G) -- (H) -- (I);
\end{tikzpicture}
\end{array}
&= \\
\begin{array}{c}
\begin{tikzpicture}
\coordinate (A) at (0,0);
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\coordinate (I) at (2,-2);
\draw (A) -- (B) -- (C);
\draw (D) -- (E) -- (F);
\draw (G) -- (H) -- (I);
\end{tikzpicture}
\end{array}
\end{align*}
Uniform block bijections

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}
\]

\[ = \]

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array}
\]

\[ = \]

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array}
\]
Uniform block bijections

\[ \begin{array}{ccc}
\quad & = & \\
\quad & = & \\
\quad & \in S_6 & \\
\end{array} \]
Multiplication obeys the rule:

$$\pi|_\varepsilon \cdot \sigma|_\eta = (\pi \sigma)|_{\varepsilon \lor (\eta \cdot \pi^{-1})}.$$
Let $X$ be a set, and consider:
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A construction
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Let $X$ be a set, and consider:

- $S_X$ — the symmetric group on $X$,
- $\mathcal{Eq}_X$ — the join semilattice of equivalence relations on $X$:
  \[
  \varepsilon \lor \eta = \langle \varepsilon \cup \eta \rangle = \varepsilon \cup (\varepsilon \circ \eta) \cup (\varepsilon \circ \eta \circ \varepsilon) \cup \cdots .
  \]
Let $X$ be a set, and consider:

- $S_X$ — the symmetric group on $X$,
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There is an action of $S_X$ on $\mathcal{Eq}_X$:
Let $X$ be a set, and consider:

- $S_X$ — the symmetric group on $X$,
- $\mathcal{Eq}_X$ — the join semilattice of equivalence relations on $X$:
  - $\epsilon \vee \eta = \langle \epsilon \cup \eta \rangle = \epsilon \cup (\epsilon \circ \eta) \cup (\epsilon \circ \eta \circ \epsilon) \cup \cdots$.
- There is an action of $S_X$ on $\mathcal{Eq}_X$:
  $$\epsilon \cdot \pi = \{ (x\pi, y\pi) \mid (x, y) \in \epsilon \}.$$
A construction

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Dual reflection monoids
Let $G \subseteq S_X$. 
A construction

- Let $G \subseteq S_X$.
- Let $E \subseteq \mathcal{Eq}_X$ be closed under the action of $G$. (Call such an $E$ a dual $G$-system.)
Let $G \subseteq S_X$.  

Let $E \subseteq \mathbb{E}q_X$ be closed under the action of $G$. (Call such an $E$ a dual $G$-system.)  

We define:

$$M^*(G, E) = \left\{ \pi|_\varepsilon \mid \pi \in G, \varepsilon \in E \right\}.$$
Let $G \subseteq S_X$.

Let $E \subseteq \mathcal{E}q_X$ be closed under the action of $G$. (Call such an $E$ a dual $G$-system.)

We define:

$$M^*(G, E) = \{ \pi|_\varepsilon \mid \pi \in G, \varepsilon \in E \}.$$ 

Note that $M^*(G, E) \leq \mathcal{S}^*_X$, since 

$$\pi|_\varepsilon \cdot \sigma|_\eta = (\pi \sigma)|_{\varepsilon \vee (\eta \cdot \pi^{-1})}.$$
A construction

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Dual reflection monoids
Theorem

The semigroup $M = M^*(G, E)$ is a factorizable inverse monoid.
Theorem

The semigroup \( M = M^*(G, E) \) is a factorizable inverse monoid. We have

\[
(\pi|_\varepsilon)^{-1} = \pi^{-1}|_{\varepsilon \cdot \pi},
\]
The semigroup $M = M^*(G, E)$ is a factorizable inverse monoid. We have

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The semigroup $M = M^*(G, E)$ is a factorizable inverse monoid. We have

- $(\pi|_\varepsilon)^{-1} = \pi^{-1}|_{\varepsilon \cdot \pi}$,
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- $E(M) = \{ 1|_\varepsilon \mid \varepsilon \in E \}$. 

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Dual reflection monoids
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- $\pi|_\varepsilon = 1|_\varepsilon \cdot \pi|_1$. 
Examples

Example

Extreme examples:
Example

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1. $M^*(G, \{1\}) \cong G$, 

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Examples

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1. $M^*(G, \{1\}) \cong G$,
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Examples

Example

Extreme examples:

1. $M^*(G, \{1\}) \cong G,$
2. $M^*(G, \{1, 0\}) \cong G^0,$
3. $M^*(\{1\}, E) \cong E,$
4. $M^*(S_X, \mathcal{E}q_X) = \mathcal{F}_X^*.$
Let $G$ be a group.
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The set $\text{Sub}(G)$ is a semilattice under $\lor$:

$$H \lor K = \langle H \cup K \rangle.$$
Coset monoids

- Let $G$ be a group.
- The set $\text{Sub}(G)$ is a semilattice under $\lor$:
  $$H \lor K = \langle H \cup K \rangle.$$ 
- Think of $G \leq S_G$ via the Cayley rep.
Let $G$ be a group.

The set $\text{Sub}(G)$ is a semilattice under $\vee$:

$$H \vee K = \langle H \cup K \rangle.$$ 

Think of $G \leq S_G$ via the Cayley rep.

Let $\mathcal{N} \leq \text{Sub}(G)$ be closed under conjugation.
Let $G$ be a group.

The set $\text{Sub}(G)$ is a semilattice under $\lor$:

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Think of $G \leq S_G$ via the Cayley rep.

Let $\mathcal{N} \leq \text{Sub}(G)$ be closed under conjugation.

For $H \in \mathcal{N}$, define an equivalence

$$\varepsilon_H = \left\{ (x, y) \in G \times G \mid x^{-1}y \in H \right\}.$$
Let $G$ be a group.

The set $\text{Sub}(G)$ is a semilattice under $\lor$:

$$H \lor K = \langle H \cup K \rangle.$$ 

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For $H \in \mathcal{N}$, define an equivalence

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Put $E_{\mathcal{N}} = \{ \varepsilon_H \mid H \in \mathcal{N} \}$. 

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Dual reflection monoids
One shows:
Coset monoids

One shows:

- $\varepsilon\{1\} = 1$,
Coset monoids

One shows:

- $\varepsilon_{\{1\}} = 1,$

- $\varepsilon_H \lor \varepsilon_K = \varepsilon_{H \lor K}.$
Coset monoids

One shows:

- \( \varepsilon \{1\} = 1 \),

- \( \varepsilon_H \lor \varepsilon_K = \varepsilon_{H \lor K} \),

- \( \varepsilon_H \cdot g = \varepsilon_{Hg} \).
Coset monoids

One shows:

- $\varepsilon_{\{1\}} = 1$,
- $\varepsilon_H \lor \varepsilon_K = \varepsilon_{H \lor K}$,
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So $E_N$ is a dual $G$-system,
Coset monoids

One shows:

- $\varepsilon\{1\} = 1$,
- $\varepsilon_H \vee \varepsilon_K = \varepsilon_{H \vee K}$,
- $\varepsilon_H \cdot g = \varepsilon_{Hg}$.

So $E_N$ is a dual $G$-system, and we can form

$$M^*(G, E_N) = \{ g|_{\varepsilon_H} \mid g \in G, H \in \mathcal{N} \}.$$
Coset monoids

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One checks that the map $g|_{\varepsilon_H} \mapsto Hg$
Coset monoids

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So $E_N$ is a dual $G$-system, and we can form

$$M^*(G, E_N) = \{ g|_{\varepsilon_H} \mid g \in G, \ H \in \mathcal{N} \}.$$ 

One checks that the map $g|_{\varepsilon_H} \mapsto Hg$ determines an embedding

$$M^*(G, E_N) \rightarrow \mathcal{C}(G).$$
One shows:

- $\varepsilon\{1\} = 1$,
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So $E_N$ is a dual $G$-system, and we can form

$$M^*(G, E_N) = \left\{ g|_{\varepsilon_H} \mid g \in G, H \in N \right\}.$$ 

One checks that the map $g|_{\varepsilon_H} \mapsto Hg$ determines an embedding

$$M^*(G, E_N) \hookrightarrow \mathcal{C}(G).$$

This characterizes **cofull** submonoids of $\mathcal{C}(G)$!
Coxeter groups have presentations of the form:

\[ W = \langle S \mid (st)^{m_{st}} = 1 \ (\forall s, t \in S) \rangle, \]
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where \( m_{ss} = 1 \) and \( m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\} \) for all \( s \neq t \).
Coxeter groups have presentations of the form:

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They can be realised as reflection groups,
Coxeter groups have presentations of the form:

$$W = \langle S \mid (st)^{m_{st}} = 1 \quad (\forall s, t \in S) \rangle,$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ for all $s \neq t$.

They can be realised as reflection groups, i.e. groups of isometries of a vector space generated by reflections.
Coxeter groups
Example

\[ W = S_n = \left\langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1 \text{ for all } i \right. \]
\[ \left. (s_is_j)^2 = 1 \text{ if } |i - j| > 1 \right. \]
\[ (s_is_j)^3 = 1 \text{ if } |i - j| = 1 \right\rangle. \]
Example

\[ W = S_n = \left\langle s_1, \ldots, s_{n-1} \right| s_i^2 = 1 \quad \text{for all } i \\
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Here \( s_i \) corresponds to the simple transposition \((i, i + 1)\).
Example

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- Here \( s_i \) corresponds to the simple transposition \((i, i + 1)\).
- \( S_n \) acts on an \( n \)-dimensional vector space \( V \) by permuting the elements of a basis \( \{x_1, \ldots, x_n\} \).
Coxeter groups

Example

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- Here \( s_i \) corresponds to the simple transposition \((i, i + 1)\).
- \( S_n \) acts on an \( n \)-dimensional vector space \( V \) by permuting the elements of a basis \( \{x_1, \ldots, x_n\} \).
- \( s_i \) acts via reflection in the hyperplane \( x_i - x_{i+1} = 0 \).
Let $W$ be a Coxeter group acting on a vector space $V$. 
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It turns out that

$$T = \{ w^{-1}sw \mid s \in S, \ w \in W \}$$

is the set of all reflections in $W$. 
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is the set of all reflections in $W$.

We have the alternate presentation

$$W = \left\langle T \left| \begin{array}{l} t^2 = 1 \text{ for all } t \\ ts = st^s \text{ for all } s, t \end{array} \right. \right\rangle.$$
Dual reflection monoids

For \( t \in T \), define an equivalence \( \varepsilon_t \in \mathcal{Eq}_V \) by

\[
\varepsilon_t = \{ (x, y) \in V \times V \mid x = y \text{ or } y = xt \}.
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More generally, if $U \subseteq T$, define:

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Put $E_W = \{ \varepsilon_U \mid U \subseteq T \}$. We have:

- $1 = \varepsilon_\emptyset$,
- $\varepsilon_{U_1} \lor \varepsilon_{U_2} = \varepsilon_{U_1 \cup U_2}$,
- $\varepsilon_U \cdot W = \varepsilon_{w^{-1}Uw}$.
So $E_W$ is a dual $W$-system, and we may form

$$M_W = M^*(W, E_W) = \{ w|_{\varepsilon U} \mid w \in W, \ U \subseteq T \} \leq F^*_N,$$
So $E_W$ is a dual $W$-system, and we may form

$$M_W = M^*(W, E_W) = \{ w \mid_{\varepsilon U} \mid w \in W, \ U \subseteq T \} \leq S^*_V,$$

the dual reflection monoid of type $W$. 
So $E_W$ is a dual $W$-system, and we may form

$$\mathcal{M}_W = M^*(W, E_W) = \{ w_{|\varepsilon_U} \mid w \in W, \ U \subseteq T \} \leq \mathcal{F}_V^*,$$

the dual reflection monoid of type $W$.

**Theorem**

*Results include:*

- an embedding $\mathcal{M}_W \rightarrow C(W) : w_{|\varepsilon_U} \mapsto W_U w,$
So $E_W$ is a dual $W$-system, and we may form

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**Theorem**

Results include:

- an embedding $\mathcal{M}_W \to C(W) : w|_{ε_U} \mapsto W_U w$,
- presentations, and cardinalities, for $E_W$ and $\mathcal{M}_W$. 
So $E_W$ is a dual $W$-system, and we may form

$$M_W = M^*(W, E_W) = \{ w|_{\varepsilon_U} \mid w \in W, \ U \subseteq T \} \leq \mathfrak{F}_V^*,$$

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**Theorem**

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- an embedding $M_W \rightarrow C(W) : w|_{\varepsilon_U} \mapsto W_U w$,
- presentations, and cardinalities, for $E_W$ and $M_W$,
- realisations of $M_W$ as submonoids of $\mathfrak{F}_X^*$ for “nicer” sets $X$: 
So $E_W$ is a dual $W$-system, and we may form

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  - $X = C$, the Coxeter complex of $W$, 

So $E_W$ is a dual $W$-system, and we may form

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  - $X = \{1, \ldots, n\}$ for type $A$,
So $E_W$ is a dual $W$-system, and we may form

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**Theorem**

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- an embedding $M_W \rightarrow C(W) : w|_{\varepsilon_U} \mapsto W_U w$,
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- realisations of $M_W$ as submonoids of $\mathfrak{S}_X^*$ for “nicer” sets $X$:
  - $X = C$, the Coxeter complex of $W$,
  - $X = \{1, \ldots, n\}$ for type A,
  - $X = \{\pm 1, \ldots, \pm n\}$ for types B, C, D — “signed block bijections”, etc.
Still to do:
Dual reflection monoids

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- simplify presentations — investigate finite presentability?
- investigate representation theory, cellularity, etc.,
- find connections to other monoids — Renner monoids etc?