The University of Sydney
School of Mathematics and Statistics

Honours essay suggestions

PM4 Honours: Essay topics 2014

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I would be happy to supervise fourth year students on various aspects of group theory, probability theory, and harmonic analysis on graphs and groups. Specific ideas include:

(a) *Groups with BN-pairs and buildings.* The axiomatic setup of a BN-pair in a group G allows for a uniform study of groups of 'Lie type', the simplest example being the group $GL_n(\mathbb{F})$ of $n \times n$ invertible matrices with entries in a field $\mathbb{F}$. This project could begin with a study of $GL_n(\mathbb{F})$, and then move on to the general theory of groups with BN-pairs and associated geometric structures, including an investigation of the important class of *Chevalley groups*. There are beautiful criteria for simplicity of groups with BN-pairs, thus facilitating a neat way to exhibit many of the infinite families of finite simple groups.

(b) *Harmonic analysis on graphs and groups.* Harmonic analysis is the abstract study of Fourier analysis. The classical setting is Fourier analysis on $\mathbb{R}^d$ or $\mathbb{Z}^d$, but there are far reaching generalisations to other groups and graphs. Examples where a rather explicit theory exists include lattices, trees, free groups, Coxeter groups, real Lie groups, and p-adic Lie groups. A detailed study of harmonic analysis on homogeneous trees (or semi-homogenous trees) would be a good introduction to this theory. Other options include studying harmonic analysis on the hyperbolic disc and the modular group $PSL_n(\mathbb{Z})$.

(c) *The groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$.* The group $SL_2(\mathbb{R})$ of 2×2 matrices with real entries and with determinant 1, and the discrete subgroup $SL_2(\mathbb{Z})$, arise in many areas including number theory and hyperbolic geometry. There are many issues to pursue here, including the representation theory of $SL_2(\mathbb{R})$ (see the 400 page book of Serge Lang) and the connections to hyperbolic geometry and Escher's famous artworks.

(d) *Automata for groups.* An automata for a group is a combinatorial object (essentially a directed graph with special properties) which reads the language of reduced expressions of elements in the group. The existence of such a structure has powerful implications, for example to the solution of the "word problem" for the group. One way to learn about these objects would be to study the automata for "Coxeter groups", where a rich theory is available.

(e) *Generalised Polygons.* Generalised polygons are beautiful graphs that play a central role in finite geometry. There are famous open problems concerning their classification, with landmark results including the Feit-Higman Theorem (proved using representation theory) and the the Bruck-Reyser-Chowla theorem (proved using a combination of number theory and representation theory). Proving these theorems, and investigating their consequences, would be an excellent introduction to the area.

(f) *Graph theory.* There are many accessible (although difficult) results in this theory. One option would be to investigate planar graphs (graphs that can be drawn in the plane without edge crossings). There is a famous theorem of Kuratowski, stating that a graph is planar if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Other questions include the asymptotic enumeration of planar graphs and colouring problems (the four and five colour theorems).

(g) *Representation theory and random walks.* Fourier analysis on groups can be used to study random walks on groups (and associated graphs). For example, a celebrated theorem of Dave Bayer and Persi Diaconis utilises representation theory of the symmetric group to prove that 7 shuffles of a deck of cards is required to sufficiently randomise the deck. This is an example of a "cutt-off" phenomena, and investigating this for other groups would be a great project.