3 Probability

3.1 Some definitions (P.30-31)

Consider the following experiments:

1. Toss a fair six-sided die once and observe the number that shows on the top.

2. Select a marble from a bag that contains 2 red, 1 black and 1 white balls and observe its colour.

Note: The word *fair* or *unbiased* means that all possible outcomes have the same chance to occur.

In an experiment of this type, we cannot state what a particular outcome will be at each throw. However, we can list all possible outcomes.

Examples

1. In 1, we observe: 1 or 2 or 3 or 4 or 5 or 6.

2. In 2, we observe: red or black or white.

Definition: An experiment of this type is called a *random experiment*, since we are not certain about its outcome(s).

On the contrary, a *deterministic* experiment yields the same outcome when repeated under the same conditions.

Definition: The set of all possible outcomes of a random experiment is called the *sample space* and is denoted S.

1. In 1, $S = \underline{\hspace{3cm}}$.
2. In 2, $S = \underline{\hspace{2.2cm}}$.

Definition: An *events* of a random experiment is a collection of outcomes with specified features.

Example: List the event A of observing a number less than 3 in experiment 1 above.

Ans: $A = \underline{\hspace{1.2cm}}$.

Example: A day is randomly selected from each of four consecutive weeks. List the events A of observing no weekday and B of observing at least 3 weekdays.

Ans: $A = \underline{\hspace{0.8cm}}$;

$B = \underline{\hspace{1.2cm}}$.

3.2 Probability for equal likely outcomes (P.32)

Definition: The *probability* of an event A is the relative frequency of its set of outcomes over an indefinitely large number of trials and is denoted $P(A)$.

Suppose we have a random experiment which has exactly c possible *equally likely* outcomes. We assign the probability to an event A by counting the number of simple outcomes that correspond to A. If the count is a then

$$P(A) = \frac{a}{c}.$$

Examples:

1. Throw a fair six-sided die. There are 6 equally likely possible outcomes. The sample space, S of this experiment is

$$S = \underline{\hspace{4cm}}.$$

If A denotes the event of observing an even number, ____________.

$$\text{Prob(an even number)} = P(A) = \frac{}{}.$$

2. Toss a fair coin 3 times. The sample space is

$$S = \underline{\hspace{12cm}}.$$

Let A be the event of observing exactly two heads in this experiment. Then $A =$ ____________________.

Therefore, the probability of observing exactly two heads is

$$P(A) = \frac{}{}.$$

Similarly, the probability of observing at least one head is

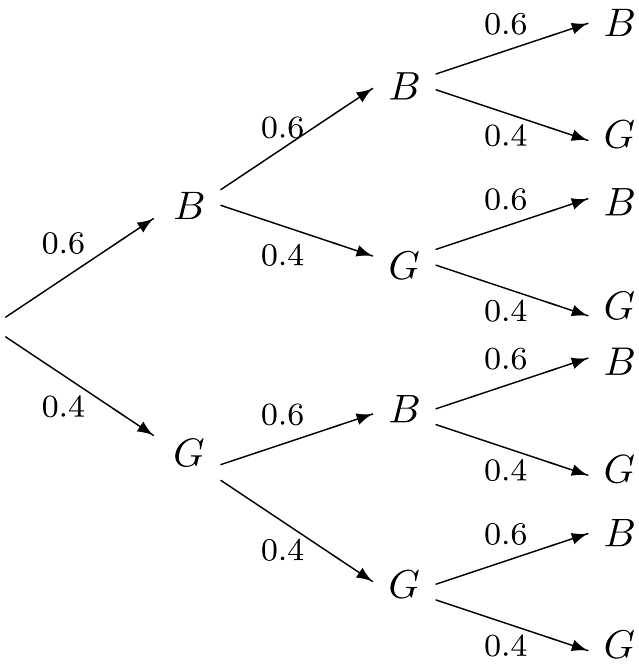
$$P(B) = \frac{}{}$$

since the event $B =$ ____.

3.3 Probability using tree diagrams (P.33)

Draw a tree diagram for the gender of three children in a family.

Tree diagram for the gender of three children



	(a)	(b)
$P(BBB) = 0.6 \times 0.6 \times 0.6$		$\surd$
$P(BBG) = 0.6 \times 0.6 \times 0.4$		$\surd$
$P(BGB) = 0.6 \times 0.4 \times 0.6$		$\surd$
$P(BGG) = 0.6 \times 0.4 \times 0.4$	$\surd$	$\surd$
$P(GBB) = 0.4 \times 0.6 \times 0.6$		$\surd$
$P(GBG) = 0.4 \times 0.6 \times 0.4$	$\surd$	$\surd$
$P(GGB) = 0.4 \times 0.4 \times 0.6$	$\surd$	$\surd$
$P(GGG) = 0.4 \times 0.4 \times 0.4$	$\surd$	

Example: Given that the probability of a boy is 0.6. Find the probability that there are (a) at most one boy and (b) at least one boy in a family of three children.

Solution:

(a) $P(\text{at most 1 boy}) = \underline{\hspace{6cm}}$

$= \underline{\hspace{8cm}}$

(b) $P(\text{at least 1 boy}) = \underline{\hspace{4cm}}$

$= \underline{\hspace{5cm}}$

3.4 Important rules of probability (P.34)

3.4.1 Important definitions

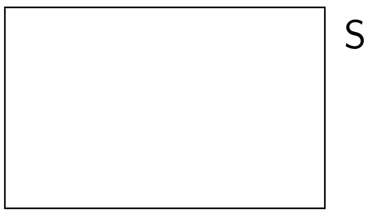
Definition: Two events A and B are said to be *mutually exclusive (m.e.)* if they both cannot happen at the same time.

Definition: Let $\bar{A}$ is the event that A does not occur. This is called the *complement* of A.

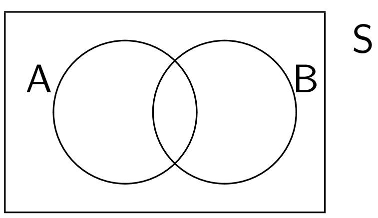
Definition: Two events A and B are said to be *independent* if A has no influence on B to occur and B has no influence on A to occur.

3.4.2 Diagrammatic representation of events (P.34)

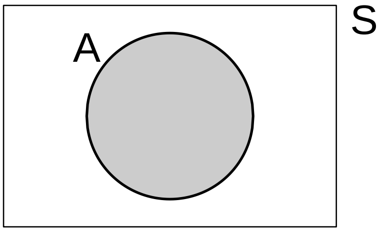
1. Rectangle is used to represent a sample space.



2. Circles are used to show the events.

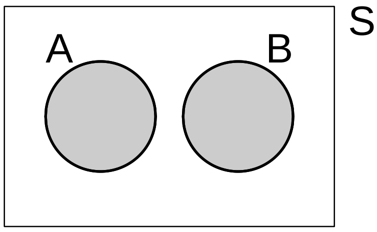


3. Shade a region as required.



4. **Mutually exclusive (m.e.) events**

If A and B have no common points, then they are m.e. events and $P(A \cap B) = 0$.



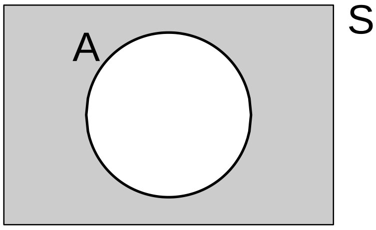
In this case, $P(A \text{ or } B \text{ occurs}) = P(A) + P(B)$.

Example: Let A be the set of even numbers less than 10 and B be the set of odd numbers less than 8. Show that they are m.e. events.

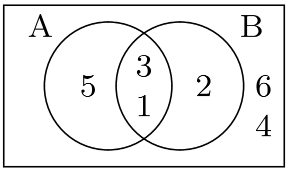
Solution: $A =$ __________, $B =$ __________.
Clearly, A and B have no common points. Hence, they are m.e. events.

5. **Non-mutually exclusive events**

(a) The event that A does not occur is denoted by $\bar{A}$. This is read as 'A does not occur" or 'the complement of A'.



Example: A fair die is thrown. Let A denote the event of observing an odd number and B the event of observing a number at most 3. Compute $P(A)$, $P(B)$ and $P(\overline{A})$.



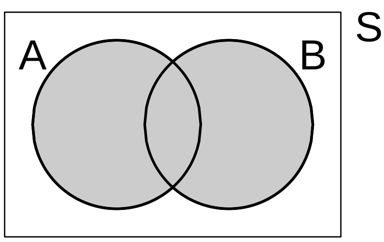
Solution: We have

$$\begin{aligned} A &= \underline{\qquad\qquad}, \; B = \underline{\qquad\qquad}, \; S = \underline{\qquad\qquad\qquad\qquad}, \\ \overline{A} &= \underline{\qquad\qquad}, \; \overline{B} = \underline{\qquad\qquad}, \end{aligned}$$

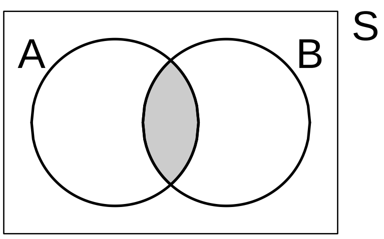
$$P(A) = P(B) = \underline{\qquad\qquad},$$

$$P(\overline{A}) = \underline{\qquad\qquad\qquad\qquad}.$$

(b) The event A or B or both occur is denoted by $A \cup B$. This is read as 'A or B or both occur' or 'at least one of A and B occur' or 'the union of A and B'.



(c) The event A and B both occur is denoted by $A \cap B$. This is read as 'A and B both occur' or 'the intersection of A and B'.

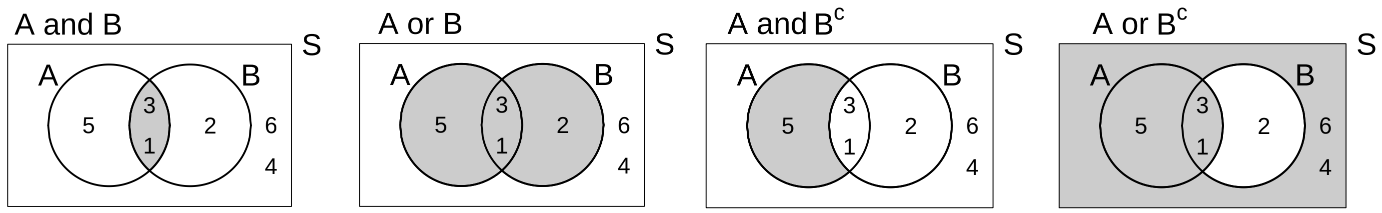


3.4.3 Important rules of probability (P.37)

1. For any event A, $0 \le P(A) \le 1$.
2. For two *mutually exclusive* events A and B, the probability that A or B occurs is $P(A) + P(B)$ and $P(A \cap B) = 0$.
3. The probability of the *complement* of an event A is
$$P(\bar{A}) = 1 - P(A).$$
4. *Addition Law of Probability:* In general, for any two events A and B
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
5. *Multiplication Law of Probability:* For two independent events A and B,
$$P(A \cap B) = P(A) \times P(B).$$

Example: Follow from the fair die example. Compute and interpret $P(A \cap B)$, $P(A \cup B)$ and $P(A \cap \overline{B})$. Verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Solution: Again, we have



$$
\begin{aligned}
A &= \underline{\qquad\qquad}, \; B = \underline{\qquad\qquad}, \; S = \underline{\qquad\qquad\qquad\qquad}, \\
A \cap B &= \underline{\qquad\quad}, \; A \cup B = \underline{\qquad\qquad\quad}, \\
A \cap \overline{B} &= \underline{\quad\;}, \quad A \cup \overline{B} = \underline{\qquad\qquad\qquad}. \\
P(A \cap B) &= \frac{}{\underline{\qquad\quad}}, \\
P(A \cup B) &= \frac{}{\underline{\qquad\quad}}, \\
&= P(A) + P(B) - P(A \cap B) = \underline{\qquad\qquad\qquad\qquad}. \\
P(A \cap \overline{B}) &= \frac{}{\underline{\quad\;}} \\
&= P(A) + P(\overline{B}) - P(A \cup \overline{B}) = \underline{\qquad\qquad\qquad\qquad}.
\end{aligned}
$$

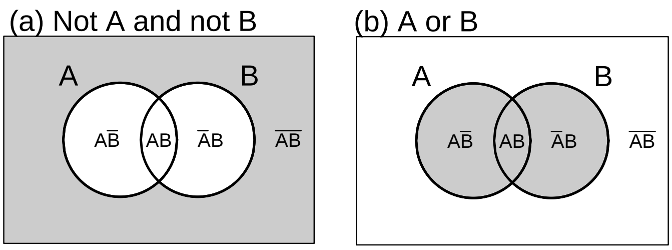
Example: Two students A and B are working at the same maths problem. The chances that A and B can work out it in half an hour are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. Find the probability that (a) both of them can't solve it and (b) at least one of them solve it in half an hour.

Solution: Let A and B denote the events of A and B solving the problem in half an hour respectively. A and B are independent events.
By tree diagram:

$$P(\text{both can't solve}) = P(\overline{AB}) = P(\overline{A})P(\overline{B}) = \underline{\hspace{3cm}}.$$

$$P(\text{at least one can solve}) = 1 - P(\text{both can't solve}) = \underline{\hspace{3cm}}.$$

By diagrammatic representation:



$$P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B}) = \underline{\hspace{5cm}}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \underline{\hspace{5cm}}$$

3.5 Conditional probability (P.38)

In real situation, the probability of the occurrence of certain event depends on situations. For example, the rate of influenza infection (A) depends on the age group and health condition (B) of a person.

Such probability is called the *conditional probability* and is denoted by $P(A|B)$, the probability of A given B (e.g. influenza infection given age group). This probability is calculated by

$$P(A|B) \quad = \quad \frac{P(A \cap B)}{P(B)}.$$

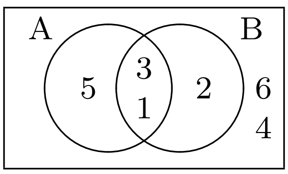
Equivalently, $P(A \cap B) \overset{\text{or}}{=} P(AB) = P(A|B) \times P(B)$.

If A and B are independent,

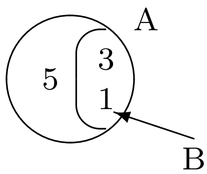
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

Example: Follow from the fair die example. Compute and interpret $P(B|A)$.

Solution: Again, we have



Restrict the sample space to A and look for the prob. of B



$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \underline{\quad\quad}.$$

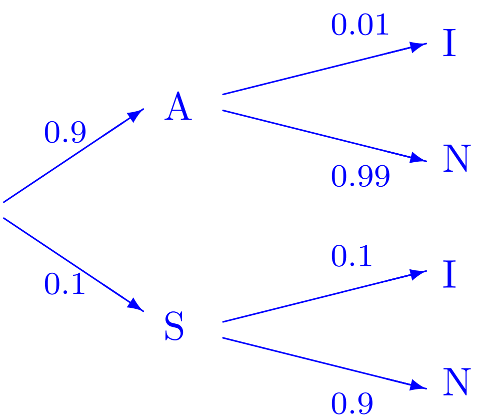
Example: In a certain district, the chance of getting influenza infection (I) is 0.01 among adults (A) and 0.1 among senior and children (S). It is known 90% of the population are adults. Find (a) the probability that a randomly selected person is infected and (b) given that a person is infected, the conditional probability that he/she is an adult.

Solution: Let N denote a person not infected by influenza. We have

$$P(A) = \underline{\hspace{1cm}}, \quad P(S) = \underline{\hspace{1cm}}, \quad P(I|A) = \underline{\hspace{1.2cm}}, \quad P(I|S) = \underline{\hspace{1cm}}$$

where $P(I|A)$ refers to the probability of infected given that the person is an adult.

Population   Influenza   Event



$$P(AI) = P(A) \times P(I|A) = .9 \times .01 = .009$$

$$P(AN) = P(A) \times P(N|A) = .9 \times .99 = .891$$

$$P(SI) = P(S) \times P(I|S) = .1 \times .10 = .01$$

$$P(SN) = P(S) \times P(N|S) = .1 \times .90 = \underline{.09}$$

Sum to 1

Probability Tree

$$\begin{aligned} P(I) &= P(AI) + P(SI) = \underline{\hspace{5cm}}, \\ P(A|I) &= \frac{P(AI)}{P(I)} = \frac{P(AI)}{P(AI) + P(SI)} = \underline{\hspace{5cm}} \\ &= \underline{\hspace{1cm}}. \end{aligned}$$

Example: A bag contains 3 blue and 2 red balls. Two balls are drawn at random from the bag. Find the probability that (a) a blue and a red ball are drawn and (b) the second ball is not red.

Solution:

$P(\text{a red and blue balls}) = P(RB) + P(BR) =$ ____________________ .

$P(\text{the 2nd ball is not red}) = P(RB) + P(BB) =$ ____________________ .

Example: A bag contains 2 red balls, 1 black ball and 1 white ball. Andrew selects one ball from the bag and keeps it hidden. He then selects a second ball, also keeping it hidden.

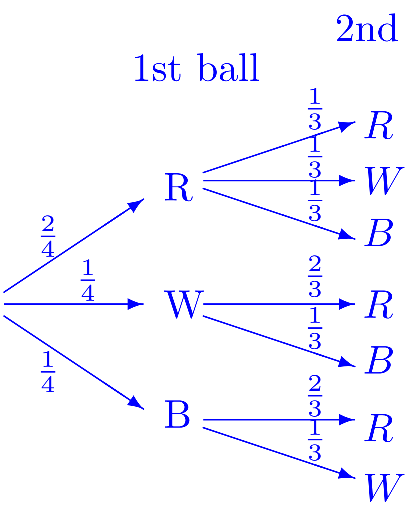
(i) Draw a tree diagram to show all possible outcomes.

(ii) Find the probability that both the selected balls are red.

(iii) Find the probability that at least one of the selected balls is red.

Solution:

(i) Tree diagram:



$$
\begin{array}{llllll}
 & & & & \text{(ii)} & \text{(iii)} \\
P(RR) & = P(R) \times P(R|R) & = \frac{2}{4} \times \frac{1}{3} & = \frac{1}{6} & \surd & \surd \\
P(RW) & = P(R) \times P(W|R) & = \frac{2}{4} \times \frac{1}{3} & = \frac{1}{6} & & \surd \\
P(RB) & = P(R) \times P(B|R) & = \frac{2}{4} \times \frac{1}{3} & = \frac{1}{6} & & \surd \\
P(WR) & = P(W) \times P(R|W) & = \frac{1}{4} \times \frac{2}{3} & = \frac{1}{6} & & \surd \\
P(WB) & = P(W) \times P(B|W) & = \frac{1}{4} \times \frac{1}{3} & = \frac{1}{12} & & \\
P(BR) & = P(B) \times P(R|B) & = \frac{1}{4} \times \frac{2}{3} & = \frac{1}{6} & & \surd \\
P(BW) & = P(B) \times P(W|B) & = \frac{1}{4} \times \frac{1}{3} & = \frac{1}{12} & &
\end{array}
$$

(ii) P(both red) $= P(RR) =$ __________ .

(iii) P(at least one red) $= P(R, \text{any}) + P(WR) + P(BR)$

$=$ ________________________ .

Exercise: Book Chapter 2, P.39, problems 1-4.