Math 3001 Topology

Solutions to 2002 Exam

1. (i) (a) A is open in \((Y, d)\) if there is, for each \(a \in A\), an open ball \(B(a; \varepsilon)\) contained in A.

(b) \((0, 2]\) is not open in \(\mathbb{R}\), as any ball \(B(2; \varepsilon)\) must contain numbers \(> 2\), and hence is not contained in \((0, 2]\).
\(\mathbb{R} \setminus (0, 2] = (-\infty, 0] \cup (2, \infty)\), and this is not open as any open ball \(B(0; \varepsilon)\) will contain numbers in \((0, 2]\). Hence \((0, 2]\) is not closed in \(\mathbb{R}\).

\((1, 2) \cup \{3\}\) is not open in \(\mathbb{R}\) as any open ball \(B(3; \varepsilon)\) will not be contained in the set.
\(\mathbb{R} \setminus ((1, 2) \cup \{3\}) = (-\infty, 1] \cup [2, 3) \cup (3, \infty)\), which is not open as it does not contain any \(B(1; \varepsilon)\). Hence \((1, 2) \cup \{3\}\) is not closed in \(\mathbb{R}\).

\([4, \infty)\) is not open as \(B(4; \varepsilon)\) is not contained in \([4, \infty)\) for any \(\varepsilon\).
\(\mathbb{R} \setminus [4, \infty) = (-\infty, 4)\) is open in \(\mathbb{R}\) as for any \(a \in (-\infty, 4)\), there is an open ball \(B(a; \varepsilon) \subseteq (-\infty, 4)\).
(For example, take \(\varepsilon = 4 - a\).)

\([4, \infty)\) is closed in \(\mathbb{R}\).
(ii) (a) A is open in X if for all $a \in A$, there exists an open ball $B(a; \varepsilon)$ such that $B(a; \varepsilon)_0 \subseteq A$.

(b) (1) \[ \text{A is not open in } X \]
\[ \text{A is closed in } X \]

(2) \[ \text{B is open in } X \]
\[ \text{B is not closed in } X \]

(3) \[ \text{C is not open in } X \]
\[ \text{C is not closed in } X \]

2. (i) $f$ is continuous if for every open set $W$ in $Y$, the inverse image $f^{-1}[W]$ is open in $X$.

(ii) First, $\text{Int} A \subseteq A$, and so $f^{-1}[\text{Int} A] \subseteq f^{-1}[A]$.

Now $f$ is continuous and $\text{Int} A$ is open in $X$, hence $f^{-1}[\text{Int} A]$ is open in $X$. Now $\text{Int}(f^{-1}[A])$ is the largest open set contained in $f^{-1}[A]$. Therefore $f^{-1}[\text{Int} A] \subseteq \text{Int}(f^{-1}[A])$. 

(iii) (a) If \( f(x) = f(y) \) then \((\cos x, \sin x, x) = (\cos y, \sin y, y)\), which implies that \( x = y \).
Hence \( f \) is injective.

(b) We have \( g: S \rightarrow \mathbb{R}^3 \), \( g(x) = (\cos x, \sin x, x) \)
for all \( x \in S \). As \( g \) is injective,
\( g^{-1}: W \rightarrow S \) is the function
\[ g^{-1}(\cos x, \sin x, x) = x. \]
\( g^{-1} \) is continuous (since each component function is continuous), so is it, restricting \( g^{-1} \).
Finally, since \( g^{-1} \) is a projection, it is also continuous. That is,
\( g: S \rightarrow W \) is a homeomorphism.

(c) The set \( \{ (\cos x, \sin x, x) \mid 0 \leq x \leq \pi/2 \} \) is
in fact equal to \( W \).
As \( W = g[S] \), \( W \) is the image
of the compact set \( [0, \pi/2] \) under
\( g \), hence \( W \) is compact.

3. (i) In \( S \), \( \{ a, b \} \cap \{ b, c \} = \{ b \} \) is
not an element of \( S \), \( e \in S \) is
not closed under finite intersection
and is therefore not a topology for \( X \).
However, $T$ is a topology for $X$ as $\emptyset, X$ are in $T$ and the union and intersection of any number of elements of $T$ are again in $T$.

(iii) $T_Y = \{\emptyset, \{b\}, \{c\}, \{b,c\}, Y\}$

(iv) $Y$, $\{c, e\}$, $\{b, e\}$, $\{e\}$, $\emptyset$ are closed.

(v) No proper subset of $Y$ is both open and closed, so $Y$ is connected.

(vi) Checking the inverse images of the open sets in $(Y, T_Y)$, we see that $f^{-1}[\{c\}] = \{c\}$, which is not open in $(X, T)$. Hence $f$ is not continuous.

4. (i) (a) A topological space is compact if every open covering has a finite subcover.

(b) The set $\bigcup_i A_i$, where

$$A_i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < i^2\}$$

for $i = 1, 2, \ldots$
is an open covering for \( \mathbb{R}^2 \), but no finite collection of \( A_i \) covers \( \mathbb{R}^2 \).

(c) For example, \( \{0\} \neq [0,1] \), although both are compact.

(d) (1) Not compact as not closed.
\[ \mathbb{R} \setminus A = (-\infty,0) \cup (1,\infty) \cup X, \]
where \( X \) is the set of all irrational numbers between 0 and 1. But any open ball centred on such an irrational number \( x \) will not be contained in \( \mathbb{R} \setminus A \), as it will contain rational points arbitrarily close to \( x \). Thus \( \mathbb{R} \setminus A \) is not open and hence \( A \) is not compact.

(2) \( B \) is not bounded and hence \( B \) is not compact.

(ii) (a) \( X \neq Y \), as \( X \) is compact and \( Y \) is not compact (as it's not closed).

(b) \( X \cong Y \), as the function \( f: X \to Y \) given by \( f(x) = 6-x \), \( \forall x \in X \), is a continuous bijection with continuous inverse \( f^{-1}: Y \to X \), \( f^{-1}(y) = 6 - y \).

(c) \( X \neq Y \), as if we had a homeomorphism \( f: X \to Y \), then removing points 2 and 4 from \( X \) will leave a connected set \( (2,4) \), whose image under a continuous
function must be connected. But removing f(2) and f(a) from Y would leave a disconnected set (2 points removed from a circle). Hence no such homeomorphism exists.

5 (i) (a) \[aba^{-1}c=1\]

(b) \[aba^{-1}b^{-1}=1\]

(c) \[aba^{-1}b^{-1}=1\]

(ii) Surface has a rim (edge d)
Surface is non-orientable, as c is paired with c.

(iv) The surface is homeomorphic to a sphere with one cross cap, C1, that is, the real projective plane, whose edge eqn is \[aa=1\].

(iii) \[26,61,12\] free edges
\[53,34,45\] free edges
Surface is a cylinder.
(v) \( bcd b^{-1} a^{-1} cad = 1 \)

Surface is non orientable, timless.

\[ e = 4 \]
\[ f = 1 \]
\[ v = 1 \]

Euler char. = \( 1 - 4 + 1 = -2 \)

\[ = 2 - n, \]

So \( n = 4 \)

The surface is \( \cong C_4 \), sphere with 4 cross caps.

(vi) Torus with one cross cap is homeomorphic to sphere with handle and one cross cap. But if a surface has a cross cap, then each handle can be converted to 2 cross caps. So we have a sphere with 3 cross caps, \( C_3 \).

The edge equ. in canonical form is \( a a b b c c = 1 \)

and the Euler characteristic is \( 2 - n = 2 - 3 = -1 \).

6. (i) (a) Since \( FrA = \overline{A} \cap X \setminus A \) we must prove that \( A \setminus FrA = \overline{A} \cap X \setminus A \).

Let \( x \in A \setminus FrA \). Then \( x \in \overline{A} \), so must show that \( x \in X \setminus A \). Now \( x \notin FrA \), and so for every open neighborhood \( U_x \) of \( x \), \( U_x \cap A \) \( \neq \) \( U_x \cap (X \setminus A) \) is not empty, so \( x \in X \setminus A \). Hence \( x \in FrA \).
Now let \( x \in \text{Fr } A \). Then \( x \in \overline{A} \) and \( x \in \overline{X \setminus A} \). We must show that \( x \in \text{Int } A \).

Now \( x \in \overline{X \setminus A} \), so every open neighborhood \( U_x \) of \( x \) has a non-empty intersection with \( X \setminus A \). Thus for each \( U_x \), \( U_x \cap A \neq \emptyset \). Therefore, \( x \in \text{Int } A \).

(b) Suppose \( \text{Fr } A = \emptyset \). We'll show that this implies that \( X \) is disconnected.

Now if \( \text{Fr } A = \emptyset \) then \( \overline{A} \subseteq \text{Int } A \) by (a).

But \( \text{Int } A \subseteq A \subseteq \overline{A} \), and so \( \overline{A} = \text{Int } A = A \).

Thus \( A \) is both open and closed, and as \( X = A \cup \overline{X \setminus A} \), we see that \( X \) is disconnected. This contradiction shows that \( \text{Fr } A \neq \emptyset \).

(ii) (a) Let \( W \) be any open set in \((Z, S)\). Then \( g^{-1}[W] \) is a subset of \( Y \), and since \( T \) is the discrete topology, \( g^{-1}[W] \) is open in \( Y \). Hence \( g \) is continuous.

(b) Let \( h : Z \to Y \) be a continuous function, and suppose that \( a \in Y \) is in \( h[Z] \), \( \exists a = h(x) \) for some \( x \in Z \). We'll show that \( h(x) = a \) for every \( x \in Z \), i.e., we'll show that \( h \) is a constant fn. First, \( [a] \) is open in \( Y \) as \( T \) is the discrete topology.
Now \( h^{-1} \{ \{a\} \} \) is also open in \( Z \) as \( h \) is continuous. But \( Z \) has topology \( S = \{ \emptyset, Z \} \). If \( h^{-1} \{ \{a\} \} = \emptyset \) we have a contradiction, as \( h(x) = a \) so \( x \in h^{-1} \{ \{a\} \} \). Therefore \( h^{-1} \{ \{a\} \} = Z \), so everything in \( Z \) maps to \( a \); hence \( h \) is constant on \( Z \).