THE UNIVERSITY OF SYDNEY

Faculties of Arts, Economics, Education, Engineering & Science

MATH 3001 TOPOLOGY

June 2003

Time allowed: Two hours

Lecturer: Jenny Henderson

There are 4 pages and 6 questions in this examination.
All 6 questions may be attempted.
The exam is worth 75 marks.
No notes, calculators or books are allowed in the examination.
All working must be shown unless otherwise indicated.
1. [13 marks]

(i) Give the definition of an open set $U$ in a metric space $(X, d)$.

(ii) Classify the following sets as open or not open in $\mathbb{R}$, and closed or not closed in $\mathbb{R}$. No reasons need be given.

(a) $(-1, 1]$
(b) $(-1, 1] \cup [0, 2)$
(c) $(-1, 1] \cap [0, 2)$

(iii) $A$ is defined to be the set

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}.$$ Sketch each of the following subsets of $A$ and classify them as open or not open in $A$, and closed or not closed in $A$. No reasons need be given.

(a) $\{(x_1, x_2) \in A \mid x_1 x_2 = 0\}$
(b) $\{(x_1, x_2) \in A \mid 0 < x_1 \leq 1\}$

(iv) Sketch each of the following subsets of $\mathbb{R}^2$. Classify each one as compact or not compact, and connected or disconnected, giving brief reasons for your answers.

(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \geq 1\}$
(b) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < 1 \text{ and } |x_2| < 1\} \cup \{(1, 1)\}$
(c) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \cup \{(1, 1)\}$

2. [13 marks]

(i) Let $x$ and $y$ be any two points in $\mathbb{R}$. Define $d_1(x, y) = |x| + |y|$ if $x \neq y$ and $d_1(x, y) = 0$ if $x = y$.

(a) Show that $d_1$ is a metric on $\mathbb{R}$.
(b) Find the 5-neighbourhood of the point 2 in $(\mathbb{R}, d_1)$.
(c) Find the 1-neighbourhood of the point 2 in $(\mathbb{R}, d_1)$.
(d) Are the two neighbourhoods you found open sets in $\mathbb{R}$ with the usual metric?

(ii) For each of the parts below, either give a brief explanation of why $X$ is not homeomorphic to $Y$ or find a homeomorphism $f : X \to Y$.

(a) $X = [0, 1] \subseteq \mathbb{R}, \quad Y = \mathbb{R}$.
(b) $X = (0, 3] \subseteq \mathbb{R}, \quad Y = [1, 7) \subseteq \mathbb{R}$.
(c) $X = \mathbb{R}^2, \quad Y = \mathbb{R}$.
(d) $X = [0, 1] \subseteq \mathbb{R}, \quad Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. 

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3. [13 marks]

(i) Let \( f : \mathbb{R} \to \mathbb{R}^3 \) be the function \( f(t) = (t, t^2, t^4) \).

(a) Show that \( f \) is injective.

(b) Let \( B = [1, 2] \subseteq \mathbb{R} \), let \( g \) be the restriction of \( f \) to \( B \) and let \( W = g[B] \). Find a formula for \( g^{-1} : W \to B \).

(c) Prove that \( g \) is a homeomorphism.

(d) Explain why the set

\[ \{(t, t^2, t^4) \in \mathbb{R}^3 \mid 1 \leq t \leq 2\} \]

is compact and connected.

(ii) (a) Give the definition of a disconnected set \( A \) in a topological space \( X \).

(b) Suppose that \( X \) is a topological space, \( A \subseteq X \), and \( S \) is the set \( \{0, 1\} \) with the discrete topology. Prove that if

\[ f : A \to S \]

is a continuous surjection then \( A \) is disconnected.

4. [12 marks] Let \( X = \{a, b, c, d, e\} \) and let \( T \) be a topology on \( X \), where

\[ T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\} \]

(i) Draw the Hasse diagram for \( T \).

(ii) List the closed subsets of \((X, T)\) and explain why \((X, T)\) is connected.

(iii) If \( Y = \{a, b, e\} \), find \( \text{Int} \ Y \) and \( \overline{Y} \) in \((X, T)\).

(iv) Write down the induced topology \( T_Y \) on the set \( Y \).

(v) Consider \( f : (Y, T_Y) \to (Y, T_Y) \) given by

\[ f(a) = b, \quad f(b) = e, \quad f(e) = e \]

Is \( f \) continuous or not continuous? Give reasons.
5. [12 marks]

(i) Draw polygonal representations of and give edge equations for
(1) a Mobius strip   (2) a Klein bottle

(ii) Which surface is represented by the following triangulation?

\begin{verbatim}
124 145 125 235 345 234
\end{verbatim}

(iii) A surface has edge equation

\[ aba^{-1}cd^{-1}b^{-1}d^{-1}e = 1. \]

(a) Does the surface have a rim or is it rimless? Is it orientable or non-orientable? Give your reasoning.
(b) Draw a polygonal representation of the surface.
(c) Find the Euler characteristic of the surface.
(d) Determine the standard surface which is homeomorphic to this surface.

(iv) For each surface \( M \) given in (1) and (2) below,

(a) state which standard surface is homeomorphic to \( M \),
(b) write down an edge equation in canonical form for \( M \).

(1) \( M \) is a torus with 1 handle.
(2) \( M \) is a sphere with 2 crosscaps and 1 handle.

6. [12 marks]

(i) Show that in any topological space, every set containing a finite number of elements is compact.

(ii) Show that in a topological space with the discrete topology, every set containing infinitely many elements is not compact.

(iii) Find a homeomorphism from the set

\[ X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + 4x_2^2 = 4, -2 \leq x_1 < 2\} \]

to the set

\[ Y = (-\pi/2, \pi/2) \subseteq \mathbb{R}, \]

justifying your answer.

THIS IS THE END OF THE EXAMINATION PAPER