1. [25 marks] In this question, brief answers (a few lines at most) are required.

(i) Sketch the set \( S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 4\} \cup \{P\} \) where \( P \) is the point \((2, 0)\). Determine whether or not \( S \) is open in \( \mathbb{R}^2 \).

(ii) Let \( X = \{a, b\} \). List all the open sets in the discrete topology on \( X \).

(iii) Let \( S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \). Determine whether the set \( W = \{(x_1, x_2, x_3) \in S \mid x_3 = 0, x_1 \geq 0\} \) is

\( \text{(a)} \) open in \( S \)

\( \text{(b)} \) closed in \( S \).

(iv) Consider \( f : \mathbb{R} \to \mathbb{Z} \), where \( f(x) = \lfloor x \rfloor \), the greatest integer less than or equal to \( x \). Find \( f^{-1}[\{0\}] \) and use your answer to explain why \( f \) is not continuous on \( \mathbb{R} \).

(v) Find a homeomorphism from \([1, 3]\) to \([4, 20]\).

(vi) Let \( A \) and \( B \) be two subsets of a topological space. If \( A \subseteq B \), show that \( \overline{A} \subseteq \overline{B} \).

(vii) Determine whether or not the set

\[ S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \cup \{(1, 1)\} \]

is compact.

(viii) The sets \( A_k = (\frac{1}{k}, 1 - \frac{1}{k}) \), for all integers \( k \geq 3 \), are subsets of \( \mathbb{R} \). Find

\[ \bigcap_{k=3}^{\infty} A_k \quad \text{and} \quad \bigcup_{k=3}^{\infty} A_k. \]

(ix) Let \( A = (0, 2) \), a subset of \( \mathbb{R} \). Clearly, \( A \) is connected. Explain what’s wrong with the following reasoning.

Let \( A_1 = (0, 1) \) and \( A_2 = [1, 2) \). Then \( A = A_1 \cup A_2 \), \( A_1 \neq \emptyset \), \( A_2 \neq \emptyset \) and \( A_1 \cap A_2 = \emptyset \). Therefore \( A \) is disconnected.

(x) Which standard surface is obtained when two discs are pasted together onto a cylinder, one onto each rim?

(xi) Give a simple triangulation for a disc and find its Euler characteristic.

(xii) Let \( f : X \to Y \) be a function from set \( X \) to set \( Y \). Let \( A \) and \( B \) be subsets of \( Y \). Show that \( f^{-1}[B \setminus A] = f^{-1}[B] \setminus f^{-1}[A] \).
2. [9 marks]

Let \( d \) be defined as follows:

\[
d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|
\]

for all \((x_1, x_2)\) and \((y_1, y_2)\) in \(\mathbb{R}^2\).

(i) Show that \( d \) is a metric on \(\mathbb{R}^2\).

(ii) Find and sketch the 2-neighbourhood of the point \((1, 1)\) in the metric space \((\mathbb{R}^2, d)\).

(iii) Is the 2-neighbourhood that you found in the previous part open in \(\mathbb{R}^2\) with the usual Euclidean metric? Give a reason for your answer.

3. [11 marks]

(i) For each of the following sets \( X \subseteq \mathbb{R} \) and \( Y \subseteq \mathbb{R}^2 \), either explain why they are not homeomorphic or find a homeomorphism between them.

(a) \( X = [1, 4] \) and \( Y = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\} \)

(b) \( X = (1, 4] \) and \( Y = \{(x_1, x_2) \mid x_2 = 5x_1\} \)

(c) \( X = (-\infty, 0) \) and \( Y = \{(x_1, x_2) \mid x_1 > 0, x_2 = \ln x_1\} \)

(ii) Show that the set

\[
A = \{(t, e^t, t^2) \in \mathbb{R}^3 \mid t \in [0, 1]\}
\]

is compact and connected.

4. [10 marks] Let \( X = \{a, b, c, d\} \) and let

\[
\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}.
\]

(i) Explain why \( \mathcal{T} \) is not a topology on \( X \).

(ii) Find the smallest topology \( \mathcal{T}' \) which contains \( \mathcal{T} \) and draw the Hasse diagram for \( \mathcal{T}' \).

(iii) Is \((X, \mathcal{T}')\) connected or disconnected? Explain your answer.

(iv) Consider the subset \( A = \{b, d\} \) of \((X, \mathcal{T}')\). Find \( \text{Int}(A) \), \( \overline{A} \) and \( \text{Fr}(A) \).
5. [10 marks]

(i) A surface $M$ has edge equation $bacd^{-1}a^{-1}b^{-1}dc = 1$.

(a) Explain why the surface is rimless.

(b) Using the edge equation, draw a polygonal representation of $M$. Find the number of edges, the number of faces, the number of vertices and hence find its Euler characteristic.

(c) Determine the standard surface to which $M$ is homeomorphic.

(ii) Consider the surface represented by the following diagram:

Identify the surface by cutting along $w$ and pasting along $x$.

(iii) Find the Euler characteristic and an edge equation in canonical form for a torus with 3 crosscaps.

6. [10 marks]

(i) Prove that if $A$ and $B$ are connected sets such that $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

(ii) Consider the function $f : \mathbb{R} \to \mathbb{Z}$ defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Assuming that $\mathbb{Z}$ has the usual topology, find the inverse images of the open sets of $\mathbb{Z}$ and show that this collection of subsets of $\mathbb{R}$ gives a topology $T$ for $\mathbb{R}$. List all the open sets in $T$ and hence demonstrate that $(\mathbb{R}, T)$ is disconnected.

THIS IS THE END OF THE EXAMINATION PAPER