1. Let $X = (0, 1] \cup (2, 3)$ be a subset in $\mathbb{R}$ and $A = (0, 1]$.

Determine whether $A$ is

(a) open in $X$;  (b) closed in $X$;  (c) open in $\mathbb{R}$;  (d) closed in $\mathbb{R}$.

Solution.

(a) Since $A = (0, 2) \setminus X$ and $(0, 2)$ is open in $\mathbb{R}$, it follows that $A$ is open in $X$.

Or: For every point $x \in A$ (including $x = 1$), there is an open ball of $x$ such that the intersection of the open ball with $X$ is contained in $A$. (The open interval $(0, 2)$, for example, is an open ball of $1$, and its intersection with $X$ is contained in $A$.)

(b) We have $A = [0, 1] \cap X$ and $[0, 1]$ is closed in $\mathbb{R}$ so that $A$ is closed in $X$.

Or: $X \setminus A = (2, 3)$ is open in $X$, and so $A$ is closed in $X$.

(c) Since no open ball of $1$ in $\mathbb{R}$ can be totally contained in $A$, $A$ is not open in $\mathbb{R}$.

(d) The complement of $A$ in $\mathbb{R}$ is

$$\mathbb{R} \setminus A = (-\infty, 0] \cup (1, \infty)$$

which is not open in $\mathbb{R}$ and so $A$ is not closed in $\mathbb{R}$.

2. Determine whether the following subsets $A$ of the unit sphere

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

are (a) open in $S$; (b) open in $\mathbb{R}^3$; (c) closed in $S$.

(i) $A = \{(x_1, x_2, x_3) \in S \mid x_3 > 0\}$;

(ii) $A = \{(x_1, x_2, x_3) \in S \mid x_3 \geq 0\}$;

(iii) $A = \{(x_1, x_2, x_3) \in S \mid x_3 = 0\}$;

(iv) $A = \{(x_1, x_2, x_3) \in S \mid x_3 = 1\}$;

(v) $A = \{(x_1, x_2, x_3) \in S \mid x_3 = 0, x_1 > 0, x_2 > 0\}$.

Solution.

(i) (a) The set $A$ is the northern hemisphere of $S$, excluding the equator. If $a \in A$, then $B(a; \epsilon) \cap S$ is a portion of the sphere, bounded by a circle. For sufficiently small $\epsilon$, this portion of $S$ is entirely contained in $A$. Hence $A$ is open in $S$.

Or: $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\} \cap S$

and $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ is open in $\mathbb{R}^3$. 

(b) Clearly no open ball of $\mathbb{R}^3$ can be contained in $A$ (which is only a “surface”). Hence $A$ is not open in $\mathbb{R}^3$.

(c) The complement $S \setminus A$ is the southern hemisphere, including the equator. If $a$ is a point on the equator, then for all $\epsilon > 0$, $B(a; \epsilon) \cap S$ clearly contains points which are not in $S \setminus A$. Hence $S \setminus A$ is not open in $S$ so that $A$ is not closed in $S$.

(ii) The set $A$ is the northern hemisphere, including the equator.

(a) $A$ is not open in $S$. (See part (i) (c).)

(b) $A$ is not open in $\mathbb{R}^3$. (See part (i) (b).)

(c) $A$ is closed in $S$, since $S \setminus A$ is open in $S$. (See part (i) (a).)

(iii) The set $A$ is just the equator.

(a) Any open ball of a point on the equator clearly contains points in $S$ which are not in $A$, so $A$ is not open in $S$.

(b) Similarly, $A$ is not open in $\mathbb{R}^3$.

(c) The set $S \setminus A$ consists of the northern hemisphere without the equator and the southern hemisphere without the equator. As $S \setminus A$ is the union of two open sets, it follows that $S \setminus A$ is open in $S$. Hence $A$ is closed in $S$.

(iv) The set $A$ is just the north pole (a single point).

(a) $A$ is not open in $S$. (Same argument as in part (iii) (a).)

(b) $A$ is not open in $\mathbb{R}^3$.

(c) Since $A$ (a singleton set) is clearly closed in $\mathbb{R}^3$ and $A = A \cap S$, we see that $A$ is closed in $S$.

(v) The set $A$ is an open arc on $S$.

(a) $A$ is not open in $S$.

(b) $A$ is not open in $\mathbb{R}^3$.

(c) $S \setminus A$ is not open in $S$, since any open ball of the point $(1,0,0)$ will contain points in $A$. So $A$ is not closed in $S$.

3. Let $X = \{a, b, c, d, e\}$ and $\tau$ the topology on $X$ given by

\[ \tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,c,d\}, \{a,b,c,e\}, X\}. \]

Find the induced topology on each of the following subsets of $X$:

\[ \begin{align*}
(i) & \quad A = \{d, e\} & (ii) & \quad B = \{b, d, e\} & (iii) & \quad C = \{a, b, c\}
\end{align*} \]

Solution.

(i) Let $\tau_A$ be the induced topology on $A$. Then, for any $U \in \tau$, $U \cap A$ is in $\tau_A$. Therefore $\tau_A = \{\emptyset, \{d\}, \{e\}, A\}$.

(ii) $\tau_B = \{\emptyset, \{b\}, \{b, d\}, \{b, e\}, B\}$.

(iii) $\tau_C = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, C\}$. 

4. Let $Y$ be a topological space and let $H \subseteq X \subseteq Y$.
Prove that $H$ is closed in $X$ if and only if $H = C \cap X$, where $C$ is closed in $Y$.

**Solution.**
Suppose that $H$ is closed in $X$. So $X \setminus H$ is open in $X$, and hence there is an open set $U$ in $Y$ such that $X \setminus H = U \cap X$. Then $Y \setminus U$ is closed in $Y$, and
\[ H = X \setminus (X \setminus H) = X \setminus (U \cap X) = X \setminus U = (Y \setminus U) \cap X. \]

Now suppose that $H = C \cap X$, where $C$ is closed in $Y$.
Then $X \setminus H = X \setminus (C \cap X) = X \setminus C = (Y \setminus C) \cap X$, where $Y \setminus C$ is open in $Y$.
So $X \setminus H$ is open in $X$, and hence $H$ is closed in $X$.

5. (i) Let $X$ be an open subset of a topological space $Y$, and $G \subseteq X$ be open in $X$. Prove that $G$ is open in $Y$.
(ii) Let $X$ be a closed subset of a topological space $Y$, and $H \subseteq X$ be closed in $X$. Prove that $H$ is closed in $Y$.

**Solution.**
(i) Since $G$ is open in $X$, $G = G' \cap X$, where $G'$ is open in $Y$. Since $G'$ and $X$ are both open in $Y$, $G$ is open in $Y$.
(ii) Since $H$ is closed in $X$, $H = H' \cap X$, where $H'$ is closed in $Y$. Since $H'$ and $X$ are both closed in $Y$, $H$ is closed in $Y$. 