Moment Maps in Perturbed Semitoric Systems
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Summary
Semitoric integrable systems are a class of Hamiltonian systems with two degrees of freedom. Pelayo and Ngoc [1] studied (among other things) an example of a semitoric system, a coupled spin-oscillator. They calculated the polygonal symplectic invariants for such a system (shown in Figure 1).

D. and Pelayo[2] showed that a semitoric integrable Hamiltonian system with focus-focus singularities can be deformed by a Hamiltonian Hopf bifurcation to generate hyperbolic singularities.

A question raised by this work is how the invariants of the such a perturbed system behave. The polygonal invariants of the original spin-oscillator systems are replaced by a more complicated object. This poster presents a particular example of this based on a Jaynes-Cummings type system.

References

Background
An integrable Hamiltonian system with two degrees of freedom can be written \((M, \Omega, (J, E))\) where \(M\) is a symplectic 4-manifold with symplectic form \(\Omega\) and \(J\) and \(E\) are commuting integrals.

A semitoric integrable Hamiltonian system is an integrable system without hyperbolic singularities for which \(J\) is a Hamiltonian \(S^1\) action. Per D. and Pelayo [2].

Theorem (D. and Pelayo): A focus-focus singular point in a semitoric integrable system may be continuously deformed via a path of semitoric systems into a hyperbolic (elliptic-elliptic) singular point.

The resulting system is called a hyperbolic semitoric system, as it has hyperbolic singularities.

One can calculate the actions, and thus the moment map, of a non-hyperbolic semitoric system with a focus-focus singular point per [1].

For a hyperbolic semitoric system a new issue arises; in an area around the hyperbolic singularities, there are two tori in the preimage of the \((J,E)\) map, and thus two “possible actions.”

The question is how these new actions “fit together”, the relationship between the actions for the perturbed and unperturbed systems, and how to account for the monodromy. We approach this question by studying an example. The outer boundary of the polygonal invariant is the same for the deformed and undeformed system.

Example: Semitoric System
Consider the following system, defined on the phase space \(M = S^2 \times R^2\):
\[
E = \frac{1}{2}(ux + vy), \quad J = \frac{1}{2}(u^2 + v^2) + z.
\]
(1)

The system has the standard symplectic structure on \(S^2\) and \(R^2\). \(E\) and \(J\) commute. The system has a focus-focus point at \((1,0,\gamma)\) and no other singularities. \(J\) generates a \(S^1\) Hamiltonian action and so the system is semitoric.

This is an example of a spin-oscillator system, as studied in [1]. It is an example of Jaynes-Cummings type system. This model has been extensively studied in mathematics and physics communities. The preimage of \(E\) and \(J\), and the polygonal invariant for this system as calculated in [1] are shown in Figure 1.

Consider the perturbed system
\[
\tilde{E} = \frac{1}{2}(ux + vy + y^2), \quad \tilde{J} = \frac{1}{2}(u^2 + v^2) + z.
\]
(2)

\(E\) and \(J\) commute. For \(\gamma > 0.5\), this system has an elliptic-elliptic singularity at \((\tilde{J}, \tilde{E}) = (1, \gamma)\).

For this example, one can transform to canonical symplectic coordinates. Computing the actions in this setting is achieved via elliptic integrals, and the results are shown in Figures 6 and 8.

Figures
Figure 1: Left: the image of \((J, E)\) from Eq. 1 is the shaded area. There is a focus-focus point at \((1,0)\). The preimage of any point between these curves is a torus in \(M\). Right: the moment polytope generated by the image of the actions \((J, K)\).

Figure 2: The image of \((\tilde{J}, \tilde{E})\) for the perturbed Jaynes-Cummings system defined by Equation 2 is the shaded region. There is an elliptic-elliptic point at \((1, \gamma)\) indicated by a red dot. On the small blue triangle, there are families of isolated periodic orbits. Inside the blue triangle, there are two tori in the preimage (Figure 3).

Figure 3: Detail from Figure 2. There is an elliptic-elliptic fixed point marked EE at \((1, \gamma)\). For a fixed \(E\) and a small range of \(J\), there are two tori in the preimage. There are two tori in the preimage of points inside the blue “triangle”, and one torus in the preimage of points outside the triangle. There are two blue edges marked E corresponding to families of stable isolated periodic orbits, and a red edge labelled EE corresponding to a family of unstable isolated periodic orbits. At the lower edge, there is a reconnection bifurcation illustrated in Figure 5. At the intersections of these two edges, there are degenerate singularities.

Figure 4: The geometry described in Figure 3 shown in a reduced phase space. The blue surface is the submanifold created by fixing \(\tilde{J}\) and the orange surface is fixed \(\tilde{E}\). For the perturbed system, for the values of \((J, E)\) indicated in the interior of Figure 3 the preimage consists of two tori, shown in green here.

Figure 5: At the hyperbolic edge in Figure 3, there is a reconnection bifurcation, as illustrated in the bottom left of Figure 4. For such values of \(E\) and \(J\), the preimage is a “figure eight” of two pinched tori. This is illustrated above in green and yellow. This accounts for the two possible actions \(K_1, K_2\) associated with such values of \((J, E)\). In terms of the polygonal invariant, this creates a “hole” in the moment map that accounts for the symplectic area that now belongs to the \(K_2\) action. Compare this to Figure 6. The symplectic area of the section in yellow here corresponds to the height shown in yellow. The symplectic area of the section in green here corresponds to the height of the moment map of \(K_1\) in Figure 7. This is also the height of the “hole” in Figure 6.

Figure 6: The image of the actions \((\tilde{J}, \tilde{K}_i)\). The outer polytope is the same as the polygonal invariant for the unperturbed system, Figure 1. There is a hole in this figure where the preimage of the symplectic invariants is empty. This corresponds to the “jump” in action along the hyperbolic edge in Figure 3.

Figure 7: The second action \((\tilde{J}, \tilde{K}_2)\), calculated for the areas where the preimage of \((J, E)\) contains two tori shown in Figure 3.

Figure 8: To obtain a global definition of the action, one must make a “cut” in \((J, E)\) space. One such cut gives the actions shown in Figures 6,7. An alternative cut gives these actions. Note that for this cut, one forms a right angle along the lower edge of the polygon in the \((J, K_1)\) figure.