Algorhythms: Generating some Interesting Rhythms

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Rhythm

What IS Rhythm?

> **rhythm:** *a pattern of regular or irregular pulses caused in music or speech by the occurrence of strong and weak beats*[1]

For our purposes: a sequence of *onsets* (when a sound is played) and *rests*. Each possible onset or rest is called a *step*. Sequences will have a fixed length.

We can write rhythms like $(x - - - x - - - x - - - x - --)$. Each step lasts one *time unit*, the length of which will depend on the tempo.

[1] Macquarie Dictionary

Sequences and Modular Arithmetic

Each rhythm is an example of a "sequence" (note that in pitch terms, a sequence is something different...). As sequences repeat forever, the steps are an example of *modular arithmetic*.

If the length of our sequence is l, and

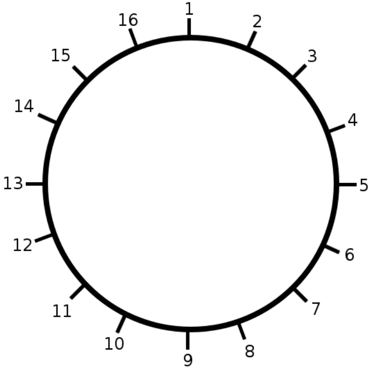
$$s \equiv t \pmod{l},$$

then

there is an onset at step s

$$\iff$$

there is an onset at step t.



Time Signatures

In our description of a sequence, a *time signature* essentially encodes the length of our sequence, or *bar*. Each step is a *semiquaver* or *sixteenth note* (even if it's not a sixteenth of the sequence!).

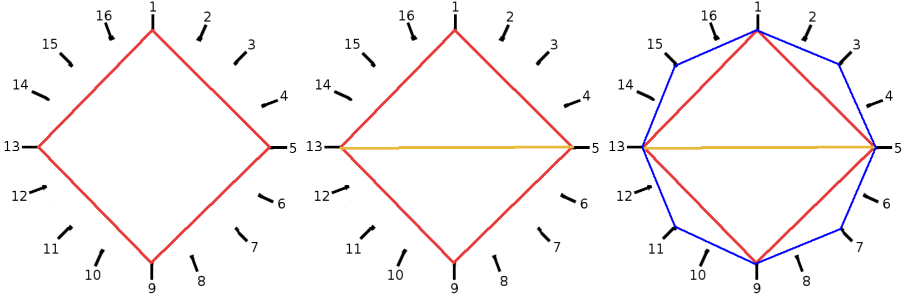
A traditional *time signature* is written as $\begin{matrix} n \\ k \end{matrix}$. This means a sequence/bar is n lots k^{th} steps/notes. For example, $\begin{matrix} 4 \\ 4 \end{matrix}$ means a sequence of 4 lots of quarter notes/steps. $\begin{matrix} 6 \\ 8 \end{matrix}$ means 6 lots of eighth notes.

A sequence in time signature $\begin{matrix} n \\ k \end{matrix}$ will have $\frac{16n}{k}$ steps.

Simple Rhythms

Say we have a sequence with l steps. The simplest rhythm we can create is to have an onset every k steps, where l is a factor of k.

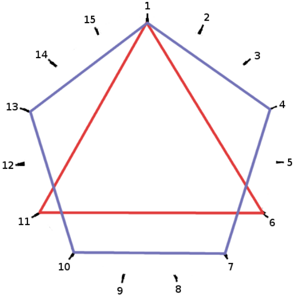
In our circular notation, sequences like this sketch out regular polygons:



Polyrhythms

Something interesting happens when we look at sequences of length l when l has a few different factors. For instance, in the case with 12 steps, we can view it as 3 sets of 4, or 4 sets of 3. If we play two rhythms like this over one another, this is called a *polyrhythm*, or *cross-rhythm*.

In the simplest case, a cross-rhythm can be constructed by making one sound play every α steps, and another sound play every β steps. This manifests as a polyrhythm when α and β are not multiples of each other.



Enumerating All Rhythms

We can write our sequences as *binary strings*, with an onset being 1 and a rest being 0. Then a rhythm $(x - -x - x-)$ would correspond to $1001010_2 = 74_{10}$.

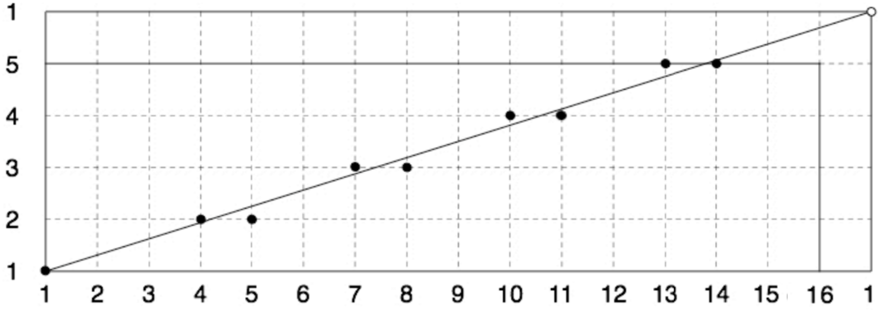
For a fixed sequence length l, we can list all the possible sequences of that length, of which there will be 2^l.

The question is, can we list all *good* rhythms? Or at least describe a class of rhythms, most of which are pretty good?

Pentagons

Evenly spacing our onsets amongst the steps makes for a good rhythm. What if we can't? For example, what if we want to have five onsets in sixteen steps?

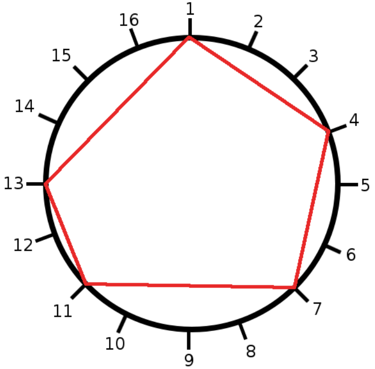
One possible way to approach this is to distribute the five onsets "as evenly as possible".



Nearly-Even Spacing

As there are 2 choices for each of the onsets except the first one, there are $2^4 = 16$ possibilities for an evenly-spaced set of onsets. Some of these choices generate very famous rhythms, including the Bo Diddley Beat/Clave Son :

$$(x - -x - -x - - - x - x - --)$$



The Euclidean Algorithm

Euclid's algorithm is a method for calculating the greatest common divisor (gcd) of two integers.

Given two integers a and b with $a > b$, find the quotient q and remainder r such that $a = qb + r$. Then $\gcd(a, b) = \gcd(b, r)$. We repeat this process for b and r, until we get a remainder of 0. Then the last non-zero remainder is the gcd of a and b.

Example: $\gcd(16, 6)$

$$16 = 2 \times 6 + 4. \quad r = \mathbf{4}.$$

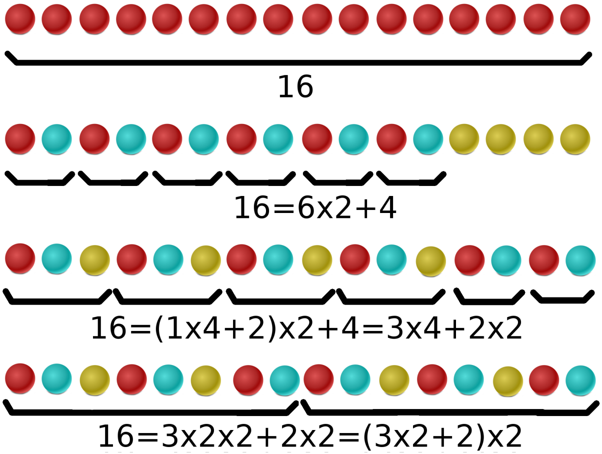
$$6 = 1 \times \mathbf{4} + 2. \quad r = \mathbf{2}.$$

$$4 = 2 \times \mathbf{2}. \quad r = 0.$$

So $\gcd(16, 6) = 2$.

The Euclidean Algorithm

As a picture, we can illustrate the Euclidean Algorithm like this:



Euclidean Rhythms

What if we do this with steps and onsets? Let's follow the Euclidean Algorithm, starting with 16 and 7

[1111111000000000]

[10][10][10][10][10][10][10][0][0]

[1010100][1010100][10]

[1010100101010010]

By following this procedure, we generate a string with k ones and $n - k$ zeros, so the k ones are spaced as evenly as possible. This was originally proposed by Bjorklund, who was looking at spallation neutron source accelerators.

Euclidean Rhythms

The rhythms we generate are called Euclidean Rhythms. Let's denote $E(k, n)$ as (a) Euclidean Rhythm with k onsets and n steps. For example, $E(4, 16)$, $E(2, 16)$, and $E(8, 16)$ are the kick, snare and high-hat patterns we saw earlier. $E(5, 8) = (x - xx - xx-)$ is a common pattern, called cinquillo in Cuba.

We should note that as they are cyclic, we can start these patterns from a different step to get new rhythms. We can also play the sequence backwards to get an equally evenly spaced rhythm. These *rotation* and *reflection* symmetries will be used later.

Euclidean Rhythms

Many common world music rhythms are examples of Euclidean Rhythms. For example:

- $E(2,5) = (x - x - -)$ is a rhythm used in Greek and African music, and used in Take Five and Mars.
- $E(5,6) = (xxxxx-)$ and $E(5,7) = (x - xx - xx)$ are popular Arabic rhythms.
- $E(5,12) = (x - -x - x - -x - x-)$ is a popular rhythm in parts of African, and starting on different steps generates many other common patterns.
- And many, many more (see references)

We can generate Euclidean Rhythms quickly using http://www.hisschemoller.com/2011/euclidean-rhythms-update/.

Classifying Rhythms

We've seen that we can classify rhythms by a binary string; a more compact way of writing them is by making a list of the time difference between consecutive onsets.

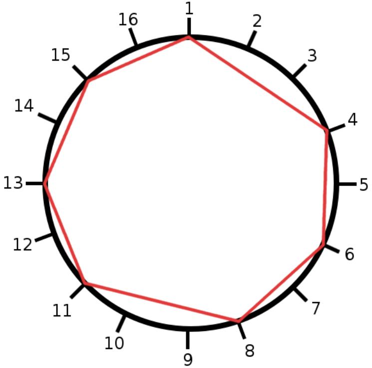
For example,

$E(7,16) = (x--x-x-x--x-x-x-)$

would be written as

$$[3223222]$$

(don't forget that we have to wrap around).



Euclidean Algorithm

Using this notation, we can write the Euclidean Algorithm quite nicely

Euclidean(k, n)

1. **If** k divides n evenly, **return** $\underbrace{\left(\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}\right)}_{k \text{ times}}$
2. Otherwise, let $a = n \bmod k$
3. Let $(x_1, x_2, \ldots, x_a) = \textbf{Euclidean}(a, k)$
4. **return**

$$\left(\underbrace{\left\lfloor\frac{n}{k}\right\rfloor, \ldots, \left\lfloor\frac{n}{k}\right\rfloor}_{x_1 - 1}, \left\lceil\frac{n}{k}\right\rceil, \underbrace{\left\lfloor\frac{n}{k}\right\rfloor, \ldots, \left\lfloor\frac{n}{k}\right\rfloor}_{x_2 - 1}, \left\lceil\frac{n}{k}\right\rceil, \ldots, \underbrace{\left\lfloor\frac{n}{k}\right\rfloor, \ldots, \left\lfloor\frac{n}{k}\right\rfloor}_{x_a - 1}, \left\lceil\frac{n}{k}\right\rceil\right)$$

Maximum Evenness

Definition (Chordal Distance)

Given two points x and y on a circle of a given circumference, the *chordal distance* $\bar{d}(x, y)$ is the Euclidean (straight-line) distance from x to y.

Definition (Evenness)

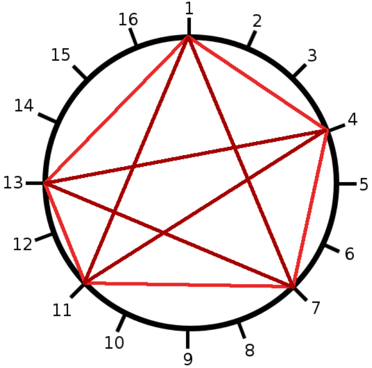
Given a rhythm with onsets at r_1, r_2,...,r_k, the *evenness* of R is sum of **all** the chordal distances between pairs of onsets

$$\sum_{1 \leq i < j \leq k} \bar{d}(r_i, r_j).$$

Maximum Evenness

So the evenness is the sum of all the **straight lines** between onsets. At right are all these chordal distances for the clave son.

Evenness is a "nice" property for rhythm to have. Which rhythms maximise the evenness, given a fixed length and number of onsets?



Theorem (Maximal Evenness)

Given $n \geq k \geq 2$, $k, n \in \mathbb{N}$, a rhythm with n steps and k onsets has maximal evenness **if and only if** *it is a Euclidean rhythm (up to rotation and reflection).*

Deep Rhythms

Definition (Geodesic Distance)

Given two points x and y on a circle of a given circumference, the *geodesic distance* $\tilde{d}(x, y)$ is the **shortest arc length** from x to y.

The *geodesic distance multiset* of a sequence with onsets at r_1, r_2,...,r_k is the set of all geodesic distances between any two numbers in R

$$\{\tilde{d}(r_i, r_j) | i \neq j\}.$$

Definition (Erdős Deep)

A rhythm with onsets at r_1, r_2,...,r_k is said to be *Erdős Deep* if its geodesic distance multiset has precisely one distance of multiplicity i for all $i = 1, 2, ..., k$.

Deep Rhythm Example

We saw that $E(7, 16)$ has onsets at 1, 4, 6, 8, 11, 13, and 15. Then the geodesic distance multiset is

$$\{3, 5, 7, 6, 4, 2, 2, 4, 7, 7, 5, 2, 5, 7, 7, 3, 5, 7, 2, 4, 2\}$$

(where all addition and subtraction is modulo 16).

Then

- 6 has multiplicity 1
- 3 has multiplicity 2
- 4 has multiplicity 3
- 5 has multiplicity 4
- 2 has multiplicity 5
- 7 has multiplicity 6

So this sequence is Erdős deep!

When are Euclidean Rhythms Erdős Deep?

Theorem

Given n, fix $k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Then the rhythm $E(k, n)$ is Erdős deep **if and only if** *n and k are coprime.*

So, if you want a rhythm that's even enough to not be jarring, but interesting enough to not be boring, pick n and k coprime and generate $E(k, n)$!

References

- *The Distance Geometry of Music*, Demaine et al
- *The Rhythm that Conquered the World*, Toussaint
- *The Euclidean Algorithm Generates Traditional Musical Rhythms*, Toussaint
- http://www.hisschemoller.com/2011/euclidean-rhythms-update