What is he universal enveloping algebra?

§1. Motivation. (Do not worms if M:s section makes no sense to you)

For a semi-simple simply connected be algebra of oner I we will Gushwet a unital association debra U(J). This algebra will be an IN-filtured algebra and ;

$$g - mod = \begin{cases} lowonorphisms \\ \beta:g \longrightarrow End(V) \\ of Lie zlgebuzs \\ for 2 C - vedev spzce V \end{cases}$$

 $\begin{cases} unital k-zlgebuz \\ lohonorphisms \\ \beta:U(g) \longrightarrow End(V) \\ for 3 C - vedev spzce V \end{cases}$
 $for 3 C - vedev spzce V \end{cases}$

Tune is a third equivalence by using the share of differential operators on the flag variety $\mathcal{FL}(\mathcal{I})$ of \mathcal{I} . let A be a C-absebra, $\operatorname{End}_{\mathcal{C}}(A)$ is a Lie absebra.

Ne Beilinson-Bernstein localisation Meaning ines the convenience

$$U(e_{j})_{2+} - mod = Diff (O_{JL(e_{j})} - mod Z_{+} = Am_{U(e_{j})}(c)$$

We can obtain irreducible J-mod by considering Verma modules Va por à weight à. <u>Philosophy</u>: Understand f.d. inducible J-mod by Using or-dimensional modules (he Va's) which are quotients of U(oJ). 52. Definition and first proporties of U(es).

let of be a Lie dyebra and \mathbb{C} . The tensor dyebra. T(g) is the predicted value $T(g) = \bigoplus_{i=0}^{\infty} g^{\otimes i}$ $\bigotimes_{i=0}^{\infty} (g^{\otimes i}) = \mathbb{C}$

i.e. elments sur V, O... OVn for some vie J 2nd some NEN ov in C, and muldiplication

$$(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n) \cdot (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_m) := \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n \otimes \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_m$$

It is a unital associative algebra area C.

The universal emeloping elseborn U(J) :s he prohent of vector spaces

$$T(\mathcal{A})/\langle \mathcal{A} \otimes \mathcal{B} - \mathcal{B} \otimes \mathcal{A} - [\mathcal{A}, \mathcal{B}] \notin \mathcal{P} \oplus (\mathcal{G} \otimes \mathcal{G}) | \mathcal{A}, \mathcal{B} \notin \mathcal{G} \rangle^{:T}$$

Where {X} nevers : lest encurated by X. W:th multiplication induced by the multiplication in T(S).

It is a unital associative algebra area C.

<u>Theorem</u> (Poincole' - Birkoff - Witt) let I be a Lie dyebra over \mathbb{C} with ordered bisss (e:) if I. An ordered tensor is a simple tensor of the form $e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_p}$ such that $e_{i_1} \leq e_{i_2} \leq \dots \leq e_{i_p}$. The allectron of all ordered tensors modulo I is a britis of U("S). <u>Proof</u>: Very hard! <u>Grollong</u>: let g be a Lie algebra. The natural map $i: g \longrightarrow U(G)$ $x \longmapsto x + I$ is injective Nearn: Neve is on equivalence of S- Ostegovics

Before proving this we need a lemma. <u>Remark</u>: A conjugation is notwally a Lie algebra by setting [x,y] := xy - yx.

A map ℓ between a hic algebra and a C-algebra is called a Lie algebra map if $\ell([x_i,y]) = \ell(x)\ell(y) - \ell(y)\ell(x)$.

lemma: The map $i: e_J \longrightarrow U(e_J)$ is a Lie algebra map and satisfies the following universal property:

> • For any unital associative algebra A and any Lie algebra map $\phi: g \longrightarrow A$ here is a map (*)

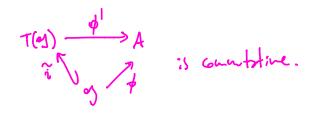
$$V(3) \xrightarrow{\phi} A$$

of associative p-algebras such that

 $\frac{p_{roop}}{p_{roop}}$: φ naturally extends to a morphism φ': T(-J) → A of k-dsebras by the whe

$$\phi^{\dagger}(a \otimes b \otimes \cdots \otimes c) = \phi^{\dagger}(a)\phi^{\dagger}(b) \cdots \phi^{\dagger}(c)$$

Ne map i: of \longrightarrow T(g), $x \mapsto x + I$ is injective and

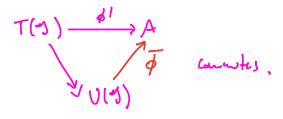


(et us show that ker $\phi' \supset I$, where I is the defining iteal of U(9), i.e., U(4) = T(4)/I. We have

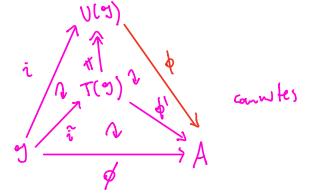
$$\phi'(ab - bba - [a, b]) = \phi(a)\phi(b) - \phi(b)\phi(a) - \phi([a, b])$$

= 0 by (*).

By the Andrewtil Theorem of homosophesens of zlyeboos here exists $\overline{\phi}: U(T) \longrightarrow A$ s.E.



Therefore

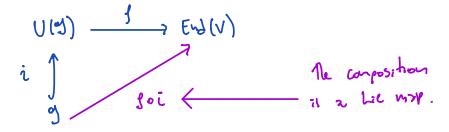


This lemma provides us an arrow

$$O_{\mathcal{L}}(\operatorname{Rep} \ \mathcal{I}) \longrightarrow O_{\mathcal{L}}(\operatorname{Rep} \ \mathcal{U}(\mathcal{I}))$$

$$\mathfrak{I}: \mathcal{I} \longrightarrow \operatorname{End}(\mathcal{V}) \longmapsto \overline{\mathfrak{I}}: \mathcal{U}(\mathcal{I}) \longrightarrow \operatorname{End}(\mathcal{V})$$

For an arrow to the left, note that : 1 g: U(2) -> End(V) :s a representation of U(2), we can compose



Neve are still missing some details I don't think they are particularly hard (please convect me if I am wrong):

• Showing the map between aways
Arr (Rep J)
$$\longrightarrow$$
 Arr (Rep U(J)).
• Showing the induced map Rep J \longrightarrow Rep U(D) is a O-Rindow,
which is an equivalence of categories.
Example: $sl_2(C) = \begin{cases} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a_ib_i c \in C \\ c & -a \end{bmatrix} : a_ib_i c \in C \end{cases}$
 $e = \begin{pmatrix} 0 & i \\ 0 & i \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ is a Uc algebra with
 $[h, e] = 2e$, $[h, f] = -2f$, $[e_i f] = h$.
Is a 3-dimensional Lie algebra.
 $U(sl_2(C)) = T(sl_2(C)) / (2e = hee - eeh_i, -2f = hef - feh_i, h = eef - fee)$
R: is an oo-dim algebra w/basis $\{e^{0i} \otimes f^{0i} \}_{iijk \in \mathbb{N}}$.

 \triangle wavning! $e \ge e = 0$, but $e \ge e + I \neq I$ motorix multiplication

Thanks for l:sten:ng l