

# SOME QUESTIONS ON SUBGROUPS OF 3-DIMENSIONAL POINCARÉ DUALITY GROUPS

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ABSTRACT. Poincaré duality complexes model the homotopy types of closed manifolds. In the lowest dimensions the correspondence is precise: every connected  $PD_n$ -complex is homotopy equivalent to  $S^1$  or to a closed surface, when  $n = 1$  or  $2$ . Every  $PD_3$ -complex has an essentially unique factorization as a connected sum of indecomposables, and these are either aspherical or have virtually free fundamental group. There are many examples of the latter type which are not homotopy equivalent to 3-manifolds, but the possible groups are largely known. However the question of whether every aspherical  $PD_3$ -complex is homotopy equivalent to a 3-manifold remains open.

We shall outline the work which lead to this reduction to the aspherical case, mention briefly remaining problems in connection with indecomposable virtually free fundamental groups, and consider how we might show that  $PD_3$ -groups are 3-manifold groups. We then state a number of open questions on 3-dimensional Poincaré duality groups and their subgroups, motivated by considerations from 3-manifold topology.

The first half of this article corresponds to my talk at the Luminy conference *Structure of 3-manifold groups* (26 February – 2 March, 2018). When I was first contacted about the conference, it was suggested that I should give an expository talk on  $PD_3$ -groups and related open problems. Wall gave a comprehensive survey of *Poincaré duality in dimension 3* at the CassonFest in 2004, in which he considered the splitting of  $PD_3$ -complexes as connected sums of aspherical complexes and complexes with virtually free fundamental group, and the JSJ decomposition of orientable  $PD_3$ -groups along  $\mathbb{Z}^2$  subgroups. I have concentrated on the work done since then, mostly on  $PD_3$ -complexes with virtually free fundamental group, before considering possible approaches to showing that aspherical  $PD_3$ -complexes might all be homotopy equivalent to closed 3-manifolds.

The second half is an annotated list of questions about  $PD_3$ -groups and their subgroups, with relevant supporting evidence, mostly deriving from known results for 3-manifold groups. This began as an aide-memoire thirty years ago, and was put on the arXiv in 2016. The questions are straightforward, but have largely resisted answers, and suggest the limitations of our present understanding.

## 1. POINCARÉ DUALITY COMPLEXES

Poincaré duality complexes were introduced by Wall to model the homotopy types of closed manifolds [92].

Let  $X$  be a connected finite CW complex with fundamental group  $\pi$ , and let  $\Lambda = \mathbb{Z}[\pi]$  be the integral group ring, with its canonical involution determined by

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inversion:  $g \mapsto g^{-1}$ . Let  $C_*(X; \Lambda)$  be the cellular chain complex of the universal cover  $\tilde{X}$ , considered as a complex of finitely generated free left modules over  $\Lambda$ . Taking the  $\Lambda$ -linear dual modules and using the canonical involution of  $\Lambda$  to swap right and left  $\Lambda$ -module structures gives a cochain complex  $C^*(X; \Lambda)$  with

$$C^q(X; \Lambda) = \overline{\text{Hom}}_{\Lambda}(C_q(X; \Lambda), \Lambda).$$

Then  $X$  is an *orientable (finite)  $PD_n$ -complex* if there is a fundamental class  $[X] \in H_n(X; \mathbb{Z})$  such that slant product with a locally-finite 3-chain in  $C_3(\tilde{X}; \mathbb{Z})$  with image  $[X]$  induces a chain homotopy equivalence

$$- \cap [X] : C^{n-*}(X; \Lambda) \rightarrow C_*(X; \Lambda).$$

This definition can be elaborated in various ways, firstly to allow for non-orientable analogues, secondly to consider pairs (corresponding to manifolds with boundary), and thirdly to weaken the finiteness conditions. We shall focus on the orientable case, for simplicity, but  $PD_n$ -pairs arise naturally even when the primary interest is in the absolute case. (Examples of the third type may arise, for instance, as infinite cyclic covers of manifolds.)

Wall showed that every  $PD_n$ -complex is homotopy equivalent to an  $n$ -dimensional complex, and in all dimensions *except*  $n = 3$  we may assume that there is a single  $n$ -cell. Moreover, this top cell is essentially unique, and so there is a well-defined connected sum, for oriented  $PD_n$ -complexes. When  $n = 3$  we may write  $X$  as a union  $X = X' \cup e^3$ , where  $c.d.X' \leq 2$ . (Thus the exceptional case relates to the Eilenberg-Ganea Conjecture.)

Closed PL  $n$ -manifolds are finite  $PD_n$ -complexes, but there are simply connected  $PD_n$ -complexes which are not homotopy equivalent to manifolds, for all  $n \geq 4$ .

## 2. POINCARÉ DUALITY GROUPS

The notion of Poincaré duality group of dimension  $n$  (or  $PD_n$ -group, for short) is an algebraic analogue of the notion of aspherical  $n$ -manifold.

A finitely presentable group  $G$  is a  *$PD_n$ -group* in the sense of Johnson and Wall if  $K(G, 1)$  is homotopy equivalent to a  $PD_n$ -complex [54]. Bieri and Eckmann gave an alternative purely algebraic formulation: a group  $G$  is a  $PD_n$ -group if the augmentation  $\mathbb{Z}[G]$ -module  $\mathbb{Z}$  has a finite projective resolution,  $H^i(G; \mathbb{Z}[G]) = 0$  for  $i < n$  and  $H^n(G; \mathbb{Z}[G])$  is infinite cyclic as an abelian group [6, 7]. The right action of  $G$  on this group determines the orientation character  $w_1(G) : G \rightarrow \mathbb{Z}^\times$ . The group  $G$  is *orientable* if  $H^n(G; \mathbb{Z}[G])$  is the augmentation module (i.e., if  $w_1(G)$  is the trivial homomorphism).

$PD_n$ -groups are *FP*, and thus finitely generated, but there are examples which are not finitely presentable, for all  $n \geq 4$  [22]. Indeed, there are uncountably many, by Theorem 18.1 of [71]. Whether there are  $PD_3$ -groups which are not finitely presentable remains unknown. (The case  $n = 3$  is critical; there are  $PD_n$ -groups with all sorts of bad behaviour when  $n > 3$ . See [22] and the references there.)

It is still an open question whether every finitely presentable  $PD_n$ -group is the fundamental group of a closed  $n$ -manifold. (This is one aspect of the circle of ideas around the Novikov Conjecture.)

[If we define a  *$PD_n$ -space* to be a space homotopy equivalent to a *CW-complex*  $X$  and such that  $C_*(X; \Lambda)$  is chain homotopy equivalent to a finite complex of finitely generated *projective* left  $\mathbb{Z}[\pi_1(X)]$ -modules and with a class  $[X]$  defining

a duality chain homotopy equivalence as for  $PD_n$ -complexes, then  $G$  is a  $PD_n$ -group in the sense of Bieri and Eckmann if and only if  $K(G, 1)$  is an (aspherical)  $PD_n$ -space. A  $PD_n$ -space  $X$  is finitely dominated if and only if  $\pi_1(X)$  is finitely presentable [15].

There is also a relative notion, of  $PD_n$ -pair of groups, which corresponds to an aspherical compact manifold with incompressible, aspherical boundary components. This notion arises naturally in connection with JSJ-decompositions of  $PD_3$ -groups.

Bieri and Eckmann defined a  $PD_n$ -pair of groups  $(G, \mathcal{S})$  as a group  $G$  of finite cohomological dimension with a finite family  $\mathcal{S}$  of monomorphisms  $j_S : S \rightarrow G$ , satisfying a modified form of Poincaré-Lefschetz duality. The “boundary” subgroups  $j_S(S)$  are  $PD_{n-1}$ -groups, and  $c.d.G = n - 1$  if  $\mathcal{S}$  is nonempty. (There is another formulation, in terms of a group  $G$  and a  $G$ -set  $W$ . In this formulation the boundary subgroups arise as stabilizers of points of  $W$ . See [23, page 138].)

### 3. LOW DIMENSIONS

When  $n = 1$  or  $2$  the modelling of  $n$ -manifolds by  $PD_n$ -complexes is precise: the only such complexes are homotopy equivalent to the circle or to a closed surface, and two such manifolds are homeomorphic if and only if their groups are isomorphic.

It is easy to see that a  $PD_1$ -complex  $X$  must be aspherical, and  $\pi = \pi_1(X)$  has two ends and  $c.d.\pi = 1$ . Since  $\pi$  is free of finite rank  $r > 0$  and  $H_1(\pi; \mathbb{Z})$  is cyclic (or since  $\pi$  is torsion-free and has two ends),  $\pi \cong \mathbb{Z}$  and  $X \simeq S^1$ . (There are elementary arguments do not require cohomological characterizations of free groups or of the number of ends.)

Every  $PD_2$ -complex with finite fundamental group is homotopy equivalent to one of  $S^2$  or the real projective plane  $\mathbb{R}P^2$ . All others are aspherical. Eckmann and Müller showed that every  $PD_2$ -complex with  $\chi(X) \leq 0$  is homotopy equivalent to a closed surface, by first proving the corresponding result for  $PD_2$ -pairs with nonempty boundary and then showing that every  $PD_2$ -group splits over a copy of  $\mathbb{Z}$  [28]. Shortly afterwards, Eckmann and Linnell showed that there is no aspherical  $PD_2$ -complex  $X$  with  $\chi(X) > 0$  [27]. Much later Bowditch used ideas from geometric group theory to prove the stronger result that if  $G$  is a finitely generated group such that  $H^2(G; \mathbb{F}[G])$  has an  $\mathbb{F}[G]$ -submodule of finite dimension over  $\mathbb{F}$ , for some field  $\mathbb{F}$ , then  $G$  is commensurable with a surface group (i.e., the fundamental group of an aspherical closed surface) [14]. One might hope for a topological argument, based on improving a degree-1 map  $f : M \rightarrow X$  with domain a closed surface.

The first non-manifold example occurs in dimension  $n = 3$ . Swan showed that every finite group of cohomological period 4 acts freely on a finite-dimensional cell complex homotopy equivalent to  $S^3$  [85]. The quotient complexes are  $PD_3$ -complexes. (Swan’s result predates the notion of  $PD$ -complex!) However, if the group has non-central elements of order 2, it cannot act freely on  $S^3$  [75], and so is not a 3-manifold group. In particular, the symmetric group  $S_3$  has cohomological period 4, but is not a 3-manifold group. (By the much later work of Perelman, the finite 3-manifold groups are the fixed-point free finite subgroups of  $SO(4)$ .)

In these low dimensions  $n \leq 3$  it suffices to show that there is some chain homotopy equivalence  $C^{n-*}(X; \Lambda) \simeq C_*(X; \Lambda)$ ; that it is given by cap product with a fundamental class follows fairly easily [50].

The *fundamental triple* of a  $PD_3$ -complex  $X$  is  $(\pi, w, c_{X*}[X])$ , where  $\pi = \pi_1(X)$ ,  $w = w_1(X)$  is the orientation character,  $c_X : X \rightarrow K(\pi, 1)$  is the classifying map

and  $[X]$  is the fundamental class in  $H_3(X; \mathbb{Z}^w)$ . There is an obvious notion of isomorphism for such triples. (Note, however, that in the non-orientable case it is only meaningful to specify the sign of  $[X]$  if we work with pointed spaces.) Hendriks showed that this is a complete homotopy invariant for such complexes. (This was first used by Swarup, for orientable 3-manifolds, in [86].)

**Theorem.** [40] *Two  $PD_3$ -complexes are (orientably) homotopy equivalent if and only if their fundamental triples are isomorphic.*  $\square$

Turaev has characterized the possible triples corresponding to a given finitely presentable group and orientation character (the “Realization Theorem”). In particular, he gave the following criterion.

**Theorem.** [91] *A finitely presentable group  $\pi$  is the fundamental group of an orientable finite  $PD_3$ -complex if and only if  $I_\pi \oplus \Lambda^r \cong J_\pi \oplus \Lambda^s$  for some  $r, s \geq 0$ , where  $I_\pi$  is the augmentation ideal of  $\pi$ , with finite rectangular presentation matrix  $M$ , and  $J_\pi = \text{Coker}(\overline{M})$ .*  $\square$

C.B.Thomas gave an alternative set of invariants, for orientable 3-manifolds, based on the Postnikov approach [88]. This applies equally well for  $PD_3$ -complexes. When  $\pi$  is finite,  $X$  is orientable and  $\tilde{X} \simeq S^3$ , and  $X$  is determined by  $\pi$  and the first nontrivial  $k$ -invariant  $\kappa_2(X) \in H^4(\pi; \mathbb{Z})$ . Let  $\beta : H^3(\pi; \mathbb{Q}/\mathbb{Z}) \cong H^4(\pi; \mathbb{Z})$  be the Bockstein isomorphism. Then  $c_{X*}[X]$  and  $\kappa_2(X)$  generate isomorphic cyclic groups, and are paired by the equation  $\beta^{-1}(\kappa_2(X))(c_{X*}[X]) = \frac{1}{|\pi|}$ .

When  $\pi$  is infinite,  $\pi_2(X) \cong \overline{H^1(\pi; \mathbb{Z}[\pi])}$  and  $X$  is determined by the triple  $(\pi, w, \kappa_1(X))$ , where  $\kappa_1(X) \in H^3(\pi; \pi_2(X))$  is now the first nontrivial  $k$ -invariant. In this case the connection between the homological and Postnikov invariants is not so clear.

The work of Turaev has been extended to the case of  $PD_3$ -pairs with aspherical boundary components by Bleile [11]. The relative version of the Realization Theorem proven there requires also that the boundary components be  $\pi_1$ -injective. The Loop Theorem of Crisp [21] should also be noted here.

The homotopy type of a higher dimensional  $PD_n$ -complex  $X$  is determined by the triple  $(P_{n-2}(X), w, f_{X*}[X])$ , where  $f_X : X \rightarrow P_{n-2}(X)$  is the Postnikov  $(n-2)$ -stage and  $w = w_1(X)$  [2]. If  $\tilde{X}$  is  $(n-2)$ -connected then  $P_{n-2}(X) \simeq K(\pi, 1)$ , so this triple is a direct analogue of Hendriks’ invariant.

#### 4. REDUCTION TO INDECOMPOSABLES

In his foundational 1967 paper Wall asked whether  $PD_3$ -complexes behaved like 3-manifolds with regards to connected sum [92]. Consider the following conditions

- (1)  $X$  is a non-trivial connected sum;
- (2)  $\pi = \pi_1(X)$  is a nontrivial free product;
- (3) either  $\pi$  has infinitely many ends, or  $\pi \cong D_\infty = Z/2Z * Z/2Z$ .

Clearly (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). Wall asked whether either of these implications could be reversed. Turaev used his Realization Theorem to show that (1)  $\Leftrightarrow$  (2) (the “Splitting Theorem”).

**Theorem.** [91] *A  $PD_3$ -complex  $X$  is indecomposable with respect to connected sum if and only if  $\pi = \pi_1(X)$  is indecomposable with respect to free product.*  $\square$

The further analysis (for  $\pi$  infinite) is based on the following three observations.

- (1)  $\pi_2(X) \cong \overline{H^1(\pi; \Lambda)}$ , by Poincaré duality;
- (2) Since  $\pi$  is  $FP_2$  we have  $\pi \cong \pi\mathcal{G}$ , where  $(\mathcal{G}, \Gamma)$  is a finite graph of finitely generated groups in which each vertex group has at most one end and each edge group is finite, by Theorem VI.6.3 of [23]. Hence  $H^1(\pi; \Lambda)$  has a “Chiswell-Mayer-Vietoris” presentation as a right  $\Lambda$ -module:

$$0 \rightarrow \bigoplus_{v \in V_f} \mathbb{Z}[G_v \setminus \pi] \xrightarrow{\Delta} \bigoplus_{e \in E} \mathbb{Z}[G_e \setminus \pi] \rightarrow H^1(\pi; \Lambda) \rightarrow 0.$$

Here  $V_f$  is the set of vertices with finite vertex groups,  $E$  is the set of edges, and the image of a coset  $G_v g$  in  $\pi$  under  $\Delta$  is

$$\Delta(G_v g) = \sum_{o(e)=v} (\sum_{G_e h \subset G_v} G_e h g) - \sum_{t(e)=v} (\sum_{G_e h \subset G_v} G_e h g),$$

with outer sums over edges  $e$  and inner sums over cosets of  $G_e$  in  $G_v$ .

- (3) Since  $\tilde{X}$  has only one nontrivial homology group in positive degrees,

$$H_i(C; \pi_2(X)) \cong H_{i+3}(C; \mathbb{Z}),$$

for any subgroup  $C$  of  $\pi$  and all  $i \geq 1$ , by a simple devissage.

These were first used together to show that if  $\pi$  is infinite and has a nontrivial finite normal subgroup then  $X \simeq S^1 \times \mathbb{R}P^2$  [44].

Crisp added an ingenious combinatorial argument to give a substantial partial answer to the second part of Wall’s question.

**Theorem.** [20] *Let  $X$  be an indecomposable orientable  $PD_3$ -complex. Then either  $X$  is aspherical or  $\pi = \pi_1(X)$  is virtually free.*  $\square$

His arguments strengthened the result on finite normal subgroups.

**Theorem.** *If  $g \neq 1 \in \pi$  has finite order then either  $w(g) = 1$  and the centralizer  $C_\pi(g)$  is finite or  $g^2 = 1$ ,  $w(g) = -1$  and  $C_\pi(g)$  has two ends.*  $\square$

If  $C_\pi(g)$  has two ends it is in fact  $\langle g \rangle \times \mathbb{Z}$ , by Corollary 7 of [51].

The Realization Theorem (in the form given earlier), Crisp’s result on centralizers and the “Normalizer Condition” (the fact that proper subgroups of finite nilpotent groups are properly contained in their normalizers) lead to an almost complete characterization of the class of indecomposable, virtually free groups which are fundamental groups of orientable  $PD_3$ -complexes.

**Theorem.** [50] *If a finitely generated virtually free group  $\pi$  is the fundamental group of an indecomposable orientable  $PD_3$ -complex  $X$  then  $\pi = \pi\mathcal{G}$  where  $(\mathcal{G}, \Gamma)$  is a finite graph of finite groups such that*

- (1) *the underlying graph  $\Gamma$  is a linear tree;*
- (2) *all vertex groups have cohomological period dividing 4, and at most one is not dihedral;*
- (3) *all edge groups are nontrivial, and at most one has order  $> 2$ .*

*If an edge group has order  $> 2$  then it is  $\mathbb{Z}/6\mathbb{Z}$ , and one of the adjacent vertex groups is  $B \times \mathbb{Z}/d\mathbb{Z}$  with  $B = T_1^*$  or  $I^*$ . Every such group  $\pi$  with all edge groups  $\mathbb{Z}/2\mathbb{Z}$  is the fundamental group of such a complex.*  $\square$

The Realization Theorem is used to exclude subgroups of period  $> 4$ , as well as to support the final assertion. The first infinite example [47] is the group  $S_3 *_{\mathbb{Z}/2\mathbb{Z}} S_3$  with presentation

$$\langle a, b, c \mid cac = a^2, cbc = b^2, c^2 = 1 \rangle.$$

We shall give an indication of the style of arguments in the following lemma.

**Lemma.** *Let  $X$  and  $\pi$  be as in the theorem. If  $S \leq \pi$  is a  $p$ -group, for some prime  $p$ , then  $S$  is cyclic or  $S \cong Q(2^k)$  for some  $k \geq 3$ .*

*Proof.* Since  $\pi$  is virtually free it has a free normal subgroup  $F$  of finite index. Since  $FS$  has finite index in  $\pi$ , the corresponding covering space  $X_{FS}$  is again an orientable  $PD_3$ -complex. After replacing  $X$  by an indecomposable factor of  $X_{FS}$ , if necessary, we may assume that  $\pi = \pi\mathcal{G}$ , where  $(\mathcal{G}, \Gamma)$  is a finite graph of finite  $p$ -groups. We may also assume that if an edge  $e$  has distinct vertices  $v, w$  then  $G_e \neq G_v$  or  $G_w$ . But then  $N_\pi(G_e)$  is infinite, by the Normalizer Condition and basic facts about free products with amalgamation. Since  $G_e$  is finite,  $C_\pi(G_e)$  is also infinite, which contradicts Crisp's result on centralizers. Hence there is only one vertex. Similarly, there are no edges, and so  $\pi = S$  is finite. Hence  $\tilde{X} \simeq S^3$  and  $S$  is as described.  $\square$

Is there an example with edge group  $Z/6Z$ ? If so, there is one with group  $(D_{2m} \times Z/3Z) *_{Z/6Z} T_1^*$ , for some  $m$  with  $(m, 6) = 1$  [50]. (Such groups have presentations  $\langle w, x, z, \mid z^2x = xzxz, x^2 = z^3, wzw = z, w^m = 1 \rangle$ . Thus far, we have not been able to apply the criterion of the Realization Theorem effectively.)

Are there examples which "arise naturally", perhaps as infinite cyclic covers of closed 4-manifolds? This is so for the generalized quaternionic group  $Q(24, 13, 1)$  and for certain other finite groups which have cohomological period 4 but are not 3-manifold groups [38]. [Note also that if  $M$  is a closed 4-manifold with  $\chi(M) = 0$  and  $f : \pi_1(M) \rightarrow \mathbb{Z}$  is an epimorphism with finitely generated kernel  $\kappa$  then the associated infinite cyclic covering space  $M_\kappa$  is a  $PD_3$ -space, by Theorem 4.5 of [41].]

In the non-orientable case we have the following result.

**Theorem.** [51] *Let  $P$  be an indecomposable non-orientable  $PD_3$ -complex. Then  $\text{Ker}(w_1(P))$  is torsion-free. If it is free then it has rank 1.*  $\square$

In particular, if  $\pi_1(P)$  is not virtually free then  $\pi_1(P) \cong \pi\mathcal{G}$ , where each vertex group of  $(\mathcal{G}, \Gamma)$  has one end, and each edge group has order 2. The orientation-preserving subgroups of the vertex groups are then  $PD_3$ -groups. Such 3-manifolds are unions of quotients of punctured aspherical 3-manifolds by free involutions. However, it is not known whether the vertices of  $\Gamma$  must all have even valence, nor whether an indecomposable  $PD_3$ -complex with orientation cover homotopy equivalent to a 3-manifold must be homotopy equivalent to a 3-manifold. On the other hand, if  $\pi_1(P)$  is virtually free then  $P \simeq RP^2 \times S^1$  or  $S^2 \tilde{\times} S^1$  [50].

The above arguments apply with little change to the study of  $PD_n$ -complexes with  $(n - 2)$ -connected universal cover. When  $n$  is odd, the results are similar. However when  $n$  is even it is not known whether the group must be virtually torsion-free. On the other hand, if  $\pi$  is indecomposable, virtually free and has no dihedral subgroups of order  $> 2$  then either  $\pi$  has order  $\leq 2$  or it has two ends and its maximal finite subgroups have cohomology of period dividing  $n$ . (See [12].)

## 5. ASPHERICAL CASE

The work of Perelman implies that every homotopy equivalence between aspherical 3-manifolds is homotopic to a homeomorphism. It is natural to ask also whether every  $PD_3$ -group is the fundamental group of an aspherical closed 3-manifold. An affirmative answer in general would suggest that a large part of the study of 3-manifolds may be reduced to algebra.

If  $G$  is a  $PD_3$ -group which has a subgroup isomorphic to  $\pi_1(M)$  where  $M$  is an aspherical 3-manifold then  $G$  is itself a 3-manifold group. For  $M$  is either Haken, Seifert fibred or hyperbolic, by the Geometrization Theorem of Perelman and Thurston, and so we may apply [99], Section 63 of [98] or Mostow rigidity, respectively. Thus it is no loss of generality to assume that  $G$  is orientable.

It may also be convenient to assume also that  $G$  is coherent, and, in particular, finitely presentable. (No  $PD_3$ -group has  $F(2) \times F(2)$  as a subgroup [69]; this is evidence that  $PD_3$ -groups might be coherent.)

The key approaches to this question seem to be through

- (1) splitting over proper subgroups – geometric group theory; or
- (2) homological algebra; or
- (3) topology.

Of course, there are overlaps between these. The fact that  $PD_2$ -groups are surface groups is one common ingredient.

(1). This approach has been most studied, particularly in the form of the Cannon Conjecture, and there is a good exposition based on JSJ decompositions of (finitely presentable)  $PD_3$ -groups and pairs of groups (as in [25]) in [93]. If one takes this approach it is natural to consider also the question of realizing  $PD_3$ -pairs of groups.

Splitting of  $PD_3$ -groups over proper subgroups was first considered by Thomas [89]. Kropholler showed that  $PD_n$ -groups with *Max-c* (the maximum condition on centralizers) have canonical splittings along codimension-1 poly- $Z$  subgroups [64]. (When  $n = 3$  such subgroups are  $\mathbb{Z}^2$  or the Klein bottle group  $\mathbb{Z} \rtimes \mathbb{Z}$ .) Castel showed that all  $PD_3$ -groups have *Max-c*, and used [81] to give a JSJ decomposition for arbitrary  $PD_3$ -groups (i.e., not assuming finite presentability) [19].

[Kropholler and Roller have considered splittings of a  $PD_n$ -group  $G$  over subgroups which are  $PD_{n-1}$ -groups. If  $S$  is such a subgroup let  $\overline{\mathbb{F}_2[S]} = \text{Hom}(\mathbb{F}_2[S], \mathbb{F}_2)$ . Then Poincaré duality (for each of  $G$  and  $S$ ) and Shapiro's Lemma together give  $H^1(G; \overline{\mathbb{F}_2[S]} \otimes_{\mathbb{F}_2[S]} \mathbb{F}_2[G]) = \mathbb{F}_2$ , and  $G$  splits over a subgroup commensurable with  $S$  if and only if the restriction to  $H^1(S; \overline{\mathbb{F}_2[S]} \otimes_{\mathbb{F}_2[S]} \mathbb{F}_2[G])$  is 0. See [67, 68, 69].]

In the simplest cases,  $G$  is either solvable, of Seifert type or atoroidal (i.e., has no abelian subgroup of rank  $> 1$ ). The solvable case is easy, and the Seifert case was settled by Bowditch [14]. (If  $G/G'$  is infinite, this case follows from the earlier work of Eckmann, Müller and Linnell [42].) The most studied aspect of the atoroidal case is the **Cannon Conjecture**, that an atoroidal, Gromov hyperbolic  $PD_3$ -group should be a cocompact lattice in  $PSL(2, \mathbb{C})$ . (In [5] it is shown that a Gromov hyperbolic  $PD_3$ -group has boundary  $S^2$ , and in [58] it is shown that an atoroidal  $PD_3$ -group which acts geometrically on a locally compact  $CAT(0)$  space is Gromov hyperbolic.)

(2). The homological approach perhaps has the least prospect of success, as it starts from the bare definition of a  $PD_3$ -group, and needs something else, to connect with topology. However it has proven useful in the subsidiary task of finding purely algebraic proofs for algebraic properties of 3-manifold groups, an activity that I have pursued for some time. One can also show that if  $G$  has sufficiently nice subgroups then it is a 3-manifold group. For instance, if  $G$  has a nontrivial  $FP_2$  ascendant subgroup of infinite index, then either  $G$  is the group of a Seifert fibred 3-manifold or it is virtually the group of a mapping torus [49]. In particular,  $G$  is

the fundamental group of a  $Sol^3$ -manifold or of a Seifert fibred 3-manifold if and only if it has a nontrivial locally-nilpotent normal subgroup.

This strategy seems to work best when there subgroups which are surface groups. One relatively new ingredient is the Algebraic Core Theorem of Kapovich and Kleiner [57], which ensures that this is so if  $G$  has an  $FP_2$  subgroup with one end. If  $H$  is a surface subgroup then either  $H$  has finite index in its commensurator  $Comm_G(H)$  or  $H$  has a subgroup  $K$  of finite index such that  $[G : N_G(K)]$  is finite (and then  $Comm_G(H) = Comm_G(K) \geq N_G(K)$  has finite index in  $G$ ). More generally, if  $H$  is a  $FP_2$  subgroup then either  $G$  is virtually the group of a mapping torus or  $H$  has finite index in a subgroup  $\hat{H}$  which is its own normalizer in  $G$ . Does  $G$  then split over  $H$ ?

However it remains possible that there may be  $PD_3$ -groups which are simple groups, or even Tarski monsters, whose only proper subgroups are infinite cyclic. It is then not at all clear what to do. [Once again, the Davis construction may be used to give  $PD_n$ -groups containing Tarski monsters, for all  $n > 3$  [78].]

(3). To the best of my knowledge, no-one has explored the third option in any detail. It has the advantage of direction connection with topology, but needs  $G$  to be finitely presentable. Here one starts from the fact that if  $X$  is an orientable  $PD_3$ -complex then there is a degree-1 map  $f : M \rightarrow X$  with domain a closed orientable 3-manifold. (This is not hard to see, but is a consequence of working in a low dimension.) Since  $\pi_1(f)$  is surjective and  $\pi_1(X)$  is finitely presentable,  $\text{Ker}(\pi_1(f))$  is normally generated by finitely many elements of  $\pi_1(M)$ , which may be represented by a link  $L \subset M$ . (The link  $L$  is far from being unique!) We might hope to modify  $M$  by Dehn surgery on  $L$  to render the kernel trivial. This *is* possible if  $X$  is homotopy equivalent to a closed orientable 3-manifold  $N$ , for  $M$  may then be obtained from  $N$  by Dehn surgery on a link whose components are null-homotopic in  $N$  [32]. However, Gadgil's argument appears to use the topology of the target space in an essential way. Moreover, there are  $PD_3$ -complexes which are not homotopy equivalent to manifolds, and so this cannot be carried through in general. (All known counterexamples have finite covers which are homotopy equivalent to manifolds.)

Let  $L = \amalg_{i \leq m} L_i$  be a link in a 3-manifold  $M$  and let  $n(L) = \amalg_{i \leq m} n(L_i)$  be an open regular neighbourhood of  $L$  in  $M$ . We shall say that  $L$  admits a *drastic* surgery if there is a family of slopes  $\gamma_i \subset \partial n(L_i)$  such that the normal closure of  $\{[\gamma_1], \dots, [\gamma_n]\}$  in  $\pi_1(M - n(L))$  meets the image of each peripheral subgroup  $\pi_1(\partial n(L_i))$  in a subgroup of finite index.

If  $X$  is an aspherical orientable  $PD_3$ -complex and  $f : M \rightarrow X$  is a degree-1 map such that  $\text{Ker}(f_*)$  is represented by a link  $L$  which admits a drastic surgery then after the surgery we may assume that  $\text{Ker}(f_*)$  is normally generated by finitely many elements of finite order. Let  $M = \#_{i=1}^r M_i$  be the decomposition into irreducibles. Since  $X$  is aspherical the map  $f$  extends to a map from  $f_\vee : \vee_{i=1}^r M_i \rightarrow X$ . Elementary considerations then show that  $f_\vee$  restricts to a homotopy equivalence from one of the aspherical summands of  $M$  to  $X$ .

Unfortunately there are knots which do not admit drastic surgeries, but we do have considerable latitude in our choice of link  $L$  representing  $\text{Ker}(f_*)$ . In particular, we may modify  $L$  by a link homotopy, and so the key question may be:

is every knot  $K \subset M$  *homotopic* to one which admits a drastic surgery?



The existence of  $PD_3$ -complexes which are not homotopy equivalent to 3-manifolds shows that we cannot expect a stronger result, in which “meets the image . . . finite index” is replaced by “contains . . .  $\pi_1(\overline{\partial n(L_i)})$ ” in the definition of drastic surgery. Can we combine Dehn surgery with passage to finite covers and varying  $L$  by link-homotopy?

[There is a parallel issue in the  $PD_2$ -case. Here the strategy can be justified *ex post facto*: if a degree-1 map  $f : M \rightarrow N$  of closed orientable surfaces is not a homotopy equivalence then there is a non-separating simple closed curve  $\gamma \subset M$  with image in the kernel of  $\pi_1(f)$  [29]. Surgery on  $\gamma$  replaces  $f$  by a new degree-1 map  $f' : M' \rightarrow N$ , where  $\chi(M') = \chi(M) + 2$ . After finitely many iterations we obtain a degree-1 map  $\hat{f} : \hat{M} \rightarrow N$ , with  $\chi(\hat{M}) = \chi(N)$ . Such a map must be a homotopy equivalence. However it seems that Edmonds’ argument requires the codomain  $N$  to also be a 2-manifold, which is what we want to prove! Can we avoid a vicious circle?]

On a more speculative level, can we use stabilization with products to bring the methods of high dimensional topology (as in the Novikov Conjecture) to bear? Is  $G \times \mathbb{Z}^r$  realizable by an aspherical  $(r + 3)$ -manifold for some  $r > 0$ ?

Among the most promising new ideas for studying  $PD_3$ -groups since Wall’s survey are the JSJ decomposition for arbitrary  $PD_3$ -groups [19], based on the work of Scott and Swarup, the Algebraic Core Theorem, based on coarse geometry [57], and the use of profinite and pro- $p$  completions, particular in connection with the Tits alternative, as in [13] and [61].

## 6. QUESTIONS ON $PD_3$ -GROUPS AND THEIR SUBGROUPS: PREAMBLE

In the following sections we shall present a number of questions on subgroups of  $PD_3$ -groups, motivated by results conjectured or already established geometrically for 3-manifold groups. The underlying question is whether every  $PD_3$ -group  $G$  is the fundamental group of some aspherical closed 3-manifold, and has been discussed above. The following questions represent possibly simpler consequences. (If we assume  $G$  is coherent and has a finite  $K(G, 1)$ -complex, as is the case for all 3-manifold groups, a number of these questions have clear answers.)

Prompted by the main result of [57], we define an *open*  $PD_n$ -group to be a countable group  $G$  of cohomological dimension  $\leq n - 1$  such that every nontrivial  $FP$  subgroup  $H$  with  $H^s(H; \mathbb{Z}[H]) = 0$  for  $s < n - 1$  is the ambient group of a  $PD_n$ -pair  $(H, \mathcal{T})$ , for some set of monomorphisms  $\mathcal{T}$ . Every subgroup of infinite index in a  $PD_3$ -group  $G$  is an open  $PD_3$ -group in our sense, by Theorem 1.3 of [57]. (The analogies are precise if  $n = 2$ , but these definitions are too broad when  $n \geq 4$ . We shall consider only the case  $n = 3$ .)

The corresponding questions for subgroups of open  $PD_3$ -groups should be considered with these. Any group with a finite 2-dimensional Eilenberg – Mac Lane complex is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to  $D^4$ . (Conjecturally such groups are exactly the finitely presentable groups of cohomological dimension 2). On applying the reflection group trick of Davis to the boundary we see that each such group embeds in a  $PD_4$ -group [22]. Thus the case considered here is critical.

We assume throughout that  $G$  is an orientable  $PD_3$ -group. The normalizer and centralizer of a subgroup  $H$  of  $G$  shall be denoted by  $N_G(H)$  and  $C_G(H)$ , respectively. We shall also let  $\zeta G = C_G(G)$ ,  $G'$  and  $G^{(\omega)} = \bigcap G^{(n)}$  denote the

centre, the commutator subgroup and the intersection of the terms of the derived series of  $G$ , respectively. A group has a given property *virtually* if it has a subgroup of finite index with that property.

Since we are interested in  $PD_3$ -groups, we shall use *3-manifold group* henceforth to mean *fundamental group of an aspherical closed 3-manifold*.

## 7. THE GROUP

If  $M = K(G, 1)$  is a closed 3-manifold we may assume it has one 0-cell and one 3-cell, and equal numbers of 1- and 2-cells. Hence  $G$  has a finite presentation of deficiency 0; this is clearly best possible, since  $\beta_1(G; \mathbb{F}_2) = \beta_2(G; \mathbb{F}_2)$ . Moreover  $G$  is  $FF$ , i.e., the augmentation module  $\mathbb{Z}$  has a finite *free*  $\mathbb{Z}[G]$ -resolution, while  $\tilde{K}_0(\mathbb{Z}[G]) = Wh(G) = 0$  and  $\tilde{M} \cong \mathbb{R}^3$ , so  $G$  is 1-connected at  $\infty$ .

In general, the augmentation  $\mathbb{Z}[G]$ -module  $\mathbb{Z}$  has a finite projective resolution, so  $G$  is almost finitely presentable ( $FP_2$ ), and there is a 3-dimensional  $K(G, 1)$  complex. The  $K(G, 1)$ -complex is finitely dominated, and hence a Poincaré complex in the sense of [92], if and only if  $G$  is finitely presentable.

For each  $g \in G$  with infinite conjugacy class  $[G : C_G(\langle g \rangle)] = \infty$ , so  $c.d.C_G(\langle g \rangle) \leq 2$  [83]. Hence  $C_G(\langle g \rangle)/\langle g \rangle$  is locally virtually free, by Theorem 8.4 of [6]. Therefore  $G$  must satisfy the Strong Bass Conjecture, by [26].

An  $FP_2$  group  $S$  such that  $H^2(S; \mathbb{Z}[S]) \cong \mathbb{Z}$  is virtually a  $PD_2$ -group [14].

- (1) Is  $G$  finitely presentable?
- (2) If  $G$  is finitely presentable does it have deficiency 0?
- (3) Is  $G$  of type  $FF$ ?
- (4) Is  $\tilde{K}_0(\mathbb{Z}[G]) = 0$ ? Is  $Wh(G) = 0$ ?
- (5) Is  $G$  1-connected at  $\infty$ ?
- (6) Is  $K(G, 1)$  homotopy equivalent to a finite complex?
- (7) If a group  $S$  is  $FP_3$  and  $H^3(S; \mathbb{Z}[S]) \cong \mathbb{Z}$  is  $S$  virtually a  $PD_3$ -group?

If (7) is true then centres of 2-knot groups are finitely generated.

## 8. SUBGROUPS IN GENERAL

Since  $G$  has cohomological dimension 3 it has no nontrivial finite subgroups. Any nontrivial element  $g$  generates an infinite cyclic subgroup  $\langle g \rangle$ ; it is not known whether there need be any other proper subgroups. If a subgroup  $H$  of  $G$  has finite index then it is also a  $PD_3$ -group. The cases when  $[G : H]$  is infinite are of more interest, and then either  $c.d.H = 2$  or  $H$  is free, by [83] and [84]. If there is a finitely generated (respectively,  $FP_2$ ) subgroup of cohomological dimension 2 there is one such which has one end (i.e., which is indecomposable with respect to free product). A solvable subgroup  $S$  of Hirsch length  $h(S) \geq 2$  must be finitely presentable, since either  $[G : S]$  is finite or  $c.d.S = 2 = h(S)$  [34]. (In particular, abelian subgroups of rank  $> 1$  are finitely generated.)

3-manifold groups are *coherent*: finitely generated subgroups are finitely presentable. In fact something stronger is true: if  $H$  is a finitely generated subgroup it is the fundamental group of a compact 3-manifold (possibly with boundary) [79]. We shall say that a group  $G$  is *almost coherent* if every finitely generated subgroup of  $G$  is  $FP_2$ . This usually suffices for homological arguments, and is implied by either coherence of the group or coherence of the group ring. (If  $\pi$  is the fundamental

group of a graph manifold then the group ring  $\mathbb{Z}[\pi]$  is coherent. The corresponding result for lattices in  $PSL(2, \mathbb{C})$  is apparently not known.)

If  $G$  is a  $PD_3$ -group with a one-ended  $FP_2$  subgroup  $H$  then there is a system of monomorphisms  $\sigma$  such that  $(H, \sigma)$  is a  $PD_3$ -pair [57]. Hence  $\chi(H) \leq 0$ . In particular, no  $PD_3$ -group has a subgroup  $F \times F$  with  $F$  a noncyclic free group. (This was first proven in [69].) As such groups  $F \times F$  have finitely generated subgroups which are not finitely related (cf. Section 8.2 of [6]), this may be regarded as weak evidence for coherence. (On the other hand, every surface group  $\sigma$  with  $\chi(\sigma) < 0$  has such a subgroup  $F$  and so  $F \times F$  is a subgroup of  $\sigma \times \sigma$ . Thus  $PD_n$ -groups with  $n \geq 4$  need not be coherent.)

Let  $M$  be a closed orientable 3-manifold. Then  $M$  is Haken, Seifert fibred or hyperbolic, by the Geometrization Theorem. With [55] it follows that if  $\pi_1(M)$  is infinite then it has a  $PD_2$ -subgroup. A transversality argument implies that every element of  $H_2(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong [M; S^1]$  is represented by an embedded submanifold. If  $M$  is aspherical it follows that  $H_2(\pi_1(M); \mathbb{Z})$  is generated by elements represented by surface subgroups of  $\pi_1(M)$ .

If  $G/G'$  is infinite then  $G$  is an HNN extension with finitely generated base and associated subgroups [10], and so has a finitely generated subgroup of cohomological dimension 2. If, moreover,  $G$  is almost coherent then it has a  $PD_2$ -subgroup [57].

If  $H$  is a subgroup of  $G$  which is a  $PD_2$ -group then  $H$  has finite index in a maximal such subgroup. This is clear if  $\chi(H) < 0$ , by the multiplicativity of  $\chi$  in the passage to subgroups of finite index. If  $\chi(H) = 0$  we argue instead that an infinite increasing union of copies of  $\mathbb{Z}^2$  must have cohomological dimension 3.

- (8) Is there a noncyclic proper subgroup? If so, is there one of cohomological dimension 2? and finitely generated?
- (9) Is there a subgroup which is a surface group?
- (10) Is every element of  $H_2(G; \mathbb{Z})$  represented by a  $PD_2$  subgroup?
- (11) Is  $G$  (almost) coherent?
- (12) Is  $\mathbb{Z}[G]$  coherent as a ring?
- (13) Does every (finitely presentable) subgroup of cohomological dimension 2 have a (finite) 2-dimensional Eilenberg-Mac Lane complex (with  $\chi \leq 0$ )?
- (14) Let  $H$  be a finitely generated subgroup with one end and of infinite index in  $G$ . Does  $H$  have infinite abelianization? contain a surface group?

## 9. ASCENDANT SUBGROUPS

If  $M$  is a closed aspherical 3-manifold which is not a graph manifold then  $M$  has a finite covering space which fibres over the circle [1, 77]. Hence indecomposable finitely generated subgroups of infinite index in such groups are (finitely presentable) semidirect products  $F \rtimes \mathbb{Z}$ , with  $F$  a free group. Such groups are HNN extensions with finitely generated free base, and associated subgroups free factors of the base [31].

If  $N$  is an  $FP_2$  ascendant subgroup of  $G$  and  $c.d.N = 2$  then it is a surface group and  $G$  has a subgroup of finite index which is a surface bundle group. If  $c.d.N = 1$  then  $N \cong \mathbb{Z}$  and either  $G$  is virtually poly- $\mathbb{Z}$  or  $N$  is normal in  $G$  and  $[G : C_G(N)] \leq 2$  [8, 49]. In the latter case  $G$  is the group of a Seifert fibred 3-manifold [14]. It is easy to find examples among normal subgroups of 3-manifold groups to show that finite generation of  $N$  is necessary for these results.

If  $N$  is finitely generated, normal and  $[G : N] = \infty$  then  $H^1(G/N; \mathbb{Z}[G/N])$  is isomorphic to  $H^1(G; \mathbb{Z}[G/N])$  and hence to  $H_2(G; \mathbb{Z}[G/N]) \cong H_2(N; \mathbb{Z})$ , by Poincaré duality. If  $G/N$  has two ends, then after passing to a subgroup of finite index, we may assume that  $G/N \cong \mathbb{Z}$ . Shapiro's lemma and Poincaré duality (for each of  $G$  and  $G/N$ ) together imply that  $\lim_{\rightarrow} H^2(N; M_i)$  is 0 for any direct system  $M_i$  with limit 0. (See Theorem 1.19 of [41].) Hence  $N$  is  $FP_2$  by Brown's criterion [17] and so is a surface group by the above result.

- (15) Is there a simple  $PD_3$ -group?
- (16) Is  $G$  virtually representable onto  $\mathbb{Z}$ ?
- (17) Must a finitely generated *normal* subgroup  $N$  be finitely presentable?
- (18) Suppose  $N \leq U$  are subgroups of  $G$  with  $U$  finitely generated and indecomposable,  $[G : U]$  infinite,  $N$  subnormal in  $G$  and  $N$  not cyclic. Is  $[G : N_G(U)] < \infty$ ? (Cf. [30].)
- (19) Let  $G$  be a  $PD_3$ -group such that  $G'$  is free. Is  $G$  a semidirect product  $K \rtimes \mathbb{Z}$  with  $K$  a  $PD_2$ -group?

#### 10. CENTRALIZERS, NORMALIZERS AND COMMENSURATORS

If  $G$  is a  $PD_3$ -group with nontrivial centre then  $\zeta G$  is finitely generated and  $G$  is the fundamental group of an aspherical Seifert fibred 3-manifold [14]. (See also [42].) Since an elementary amenable group of finite cohomological dimension is virtually solvable [52], it follows also that either  $G$  is virtually poly- $\mathbb{Z}$  or its maximal elementary amenable normal subgroup is cyclic.

Every strictly increasing sequence of centralizers  $C_0 < C_1 < \dots < C_n = G$  in a  $PD_3$ -group  $G$  has length  $n$  at most 4 [49]. (The finiteness of such sequences in any  $PD_3$ -group is due to Castel [19].) On the other hand, the 1-relator group with presentation  $\langle t, x \mid tx^2t^{-1} = x^3 \rangle$  has an infinite chain of centralizers, and hence so does the  $PD_4$ -group obtained from it by the Davis construction [66].

If the sequence of centralizers  $C_1 \cong \mathbb{Z} < C_2 < C_3 < C_4 = G$  is strictly increasing then  $C_3$  must be nonabelian. (See [49].) Hence it is  $FP_2$  [19], and so either  $G$  is Seifert or  $c.d.C_3 = 2$ . In all cases it follows that  $C_2 \cong \mathbb{Z}^2$ . Equivalently, if  $G$  has a maximal abelian subgroup  $A$  which is not finitely generated then  $1 < A < G$  is the only sequence of centralizers containing  $A$ .

If every abelian subgroup of  $G$  is finitely generated then the centralizer  $C_G(x)$  of any  $x \in G$  is finitely generated [19]. It then follows that every centralizer is either  $\mathbb{Z}$ , finitely generated and of cohomological dimension 2 or of index  $\leq 2$  in  $G$  [49]. (Applying the Davis construction to the group with presentation  $\langle t, x \mid txt^{-1} = x^2 \rangle$  gives a  $PD_4$ -group with an abelian subgroup which is not finitely generated [73].)

An element  $g$  is a *root* of  $x$  if  $x = g^n$  for some  $n$ . All roots of  $x$  are in  $C_G(x)$ . If  $C_G(x)$  is finitely generated then  $x$  is not infinitely divisible. For if  $c.d.C_G(x) = 1$  then  $C_G(x) \cong \mathbb{Z}$ ; if  $c.d.C_G(x) = 2$  then  $C_G(x)/\langle x \rangle$  is virtually free, by Theorem 8.4 of [6]; and if  $c.d.C_G(x) = 3$  then  $C_G(x)/\langle x \rangle$  is virtually a  $PD_2$ -group [14]. Conversely, if  $x$  is not infinitely divisible then  $C_G(x)$  is finitely generated [19].

If  $C_G(x)$  is nonabelian then it is  $FP_2$ , and is either of bounded Seifert type or has finite index in  $G$  [19]. In the latter case either  $[G : C_G(x)] \leq 2$  or  $G$  is virtually  $\mathbb{Z}^3$ , by Theorem 2 of [49].

If  $x$  is a nontrivial element of  $G$  then  $[N_G(\langle x \rangle) : C_G(x)] \leq 2$  (since  $\langle x \rangle \cong \mathbb{Z}$ ). If  $F$  is a finitely generated nonabelian free subgroup of  $G$  then  $N_G(F)$  is finitely generated and  $N_G(F)/F$  is finite or virtually  $\mathbb{Z}$  [49]. (See [80] for another argument

in the 3-manifold case.) If  $H$  is an  $FP_2$  subgroup which is a nontrivial free product but is not free then  $[N_G(H) : H] < \infty$  and  $C_G(H) = 1$  [49].

If  $H$  is a one-ended  $FP_2$  subgroup of infinite index in  $G$  then either  $[G : N_G(H)]$  or  $[N_G(H) : H]$  is finite. (See Lemma 2.15 of [41]). More precisely, define an increasing sequence of subgroups  $\{H_i | i \geq 0\}$  by  $H_0 = H$  and  $H_i = N_G(H_{i-1})$  for  $i > 0$ . Then  $\hat{H} = \cup H_i$  is  $FP_2$  and either  $c.d.\hat{H} = 2$ ,  $\hat{H}$  has one end and  $N_G(\hat{H}) = \hat{H}$ , or  $\hat{H}$  is a  $PD_3$ -group and  $G$  is virtually the group of a surface bundle, by Theorem 2.17 of [41]. In particular, if  $G$  has a subgroup  $H$  which is a surface group with  $\chi(H) = 0$  (respectively,  $< 0$ ) then either it has such a subgroup which is its own normalizer in  $G$  or  $G$  is virtually the group of a surface bundle.

The *commensurator* in  $G$  of a subgroup  $H$  is the subgroup

$$Comm_G(H) = \{g \in G \mid [H : H \cap gHg^{-1}] < \infty \text{ and } [H : H \cap g^{-1}Hg] < \infty\}.$$

It clearly contains  $N_G(H)$ .

If  $x \neq 1$  in  $G$  then the Baumslag-Solitar relation  $tx^pt^{-1} = x^q$  implies that  $p = \pm q$  [19]. It follows easily that  $Comm_G(\langle x \rangle) = \cup N_G(\langle x^{n!} \rangle)$ . Since the chain of centralizers  $C_G(\langle x^{n!} \rangle)$  is increasing and  $[N_G(\langle x^k \rangle) : C_G(\langle x^k \rangle)] \leq 2$  for any  $k$  it follows that  $Comm_G(\langle x \rangle) = N_G(\langle x^{n!} \rangle)$  for some  $n \geq 1$ .

If  $H$  is a  $PD_2$ -group then Theorem 1.3 and Proposition 4.4 of [70] imply that either  $[Comm_G(H) : H] < \infty$  or  $H$  is commensurable with a subgroup  $K$  such that  $[G : N_G(K)] < \infty$ , and so  $[G : Comm_G(H)] < \infty$ . This dichotomy is similar to the one for normalizers of  $FP_2$  subgroups cited above. It can be shown that if  $H \cong \mathbb{Z}^2$  then either  $Comm_G(H) = N_G(H)$  or  $G$  is virtually  $\mathbb{Z}^3$ . However, the exceptional cases do occur. If  $G = B_1$  is the flat 3-manifold group with presentation  $\langle t, x, y \mid txt^{-1} = x^{-1}, ty = yt, xy = yx \rangle$  and  $A$  is the subgroup generated by  $\{t, y\}$  then  $N_G(A) = A$  but  $Comm_G(A) = G$ .

(20) Is every abelian subgroup of  $G$  finitely generated?

(21) If  $G$  is not virtually abelian and  $H$  is an  $FP_2$  subgroup such that  $N_G(H) = H$  is  $[Comm_G(H) : H]$  finite?

## 11. THE DERIVED SERIES AND PERFECT SUBGROUPS

Let  $G^{(\omega)} = \cap G^{(n)}$  be the intersection of the terms of the derived series for  $G$ . If  $G^{(\omega)} = G^{(n)}$  for some finite  $n$  then  $n \leq 3$ , and  $G/G^{(\omega)}$  is either a finite solvable group with cohomological period dividing 4, or has two ends and is  $\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  or  $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ , or has one end and is a solvable  $PD_3$ -group. (The argument given in [45] for orientable 3-manifold groups also applies here.) There is a similar result for the lower central series. If  $G$  is orientable and  $G_{[\omega]} = G_{[n]}$  for some finite  $n$  then  $n \leq 3$ , and  $G/G_{[\omega]}$  is finite,  $\mathbb{Z}$  or a nilpotent  $PD_3$ -group [87].

If  $G$  is not virtually representable onto  $\mathbb{Z}$  then  $G/G^{(\omega)}$  is either a finite solvable group with cohomological period dividing 4 (and  $G^{(\omega)}$  is a perfect  $PD_3$ -group) or is a finitely generated, infinite, residually finite-solvable group with one or infinitely many ends. Let  $M$  be the (aspherical) 3-manifold obtained by 0-framed surgery on a nontrivial knot  $K$  with Alexander polynomial  $\Delta_K \doteq 1$ , and let  $G = \pi_1(M)$ . Then  $G'$  is a perfect normal subgroup which is not finitely generated. (In this case  $G_{[\omega]} = G^{(\omega)} = G'$ , and  $G/G^{(\omega)} \cong \mathbb{Z}$ .) Replacing a suitable solid torus in  $RP^3 \# RP^3$  by the exterior of such a knot  $K$  gives an example with  $G/G^{(\omega)} \cong D_\infty$ .

Let  $\kappa$  be a perfect normal subgroup of the fundamental group  $\pi$  of a  $PD_3$ -complex  $X$ . Then  $\rho = \pi/\kappa$  is  $FP_2$ , since  $\pi$  is  $FP_2$  and  $H_1(\kappa; \mathbb{Z}) = 0$ . The arguments of [20]

give  $\rho \cong (*_{i=1}^r G_i) * V$ , where each factor  $G_i$  has one end and  $V$  is virtually free. Moreover, if  $\rho$  is infinite and has a nontrivial finite normal subgroup then  $\rho$  has two ends. (However, the further analysis of [50] does not apply, since there is no analogue of the Splitting Theorem of Turaev.) We also have  $H_2(\kappa; \mathbb{Z}) \cong H^1(\rho; \mathbb{Z}[\rho])$  as an abelian group. In particular, if  $\kappa$  is acyclic then  $\rho$  is a  $PD_3$ -group.

The intersection  $P = \cap G^{(\alpha)}$  of the terms of the transfinite derived series for  $G$  is the maximal perfect subgroup of  $G$ , and is normal in  $G$ . The quotient  $G/P$  is  $FP_2$ . If  $P \neq 1$  and  $[G : P]$  is infinite then  $c.d.P = 2$ , but  $P$  cannot be  $FP_2$ , for otherwise it would be a surface group [43]. Note that  $P \subseteq G^{(\omega)}$ , and if  $c.d.P = 2$  then  $c.d.G^{(\omega)} = 2$  also. If  $[G : P]$  is infinite and  $\zeta G \neq 1$  then  $P = 1$ .

If  $G$  is a  $PD_3$ -group and  $H$  is a nontrivial  $FP_2$  subgroup such that  $H^1(H; \mathbb{Z}) = 0$  then  $[G : H]$  is finite. (Use [57]. See [53] for 3-manifold groups.)

- (22) Can a nontrivial finitely generated normal subgroup of infinite index be perfect? acyclic?
- (23) If a finitely generated, infinite, residually solvable group has infinitely many ends must it be virtually representable onto  $\mathbb{Z}$ ?
- (24) If  $P = 1$  is  $G$  residually solvable (i.e., is  $G^{(\omega)} = 1$  also)?

## 12. THE TITS ALTERNATIVE

A group *satisfies the Tits alternative* if every finitely generated subgroup is either solvable or contains a non-abelian free group.

Let  $N$  be the subgroup generated by all the normal subgroups which have no nonabelian free subgroup. Then  $N$  is the maximal such subgroup, and clearly it contains the maximal elementary amenable normal subgroup of  $G$ . If  $N$  is nontrivial then either  $N \cong \mathbb{Z}$ ,  $c.d.N = 2$  or  $N = G$ . If  $N$  is a rank 1 abelian subgroup then  $N \cong \mathbb{Z}$ . (For otherwise  $N \leq G'$  and  $G' \leq C_G(N)$ , so either  $[G : C_G(N)]$  is finite, which can be excluded by [14], or  $G'$  is abelian, by Theorem 8.8 of [6], in which case  $G$  is solvable and hence virtually poly- $\mathbb{Z}$ , and  $N$  must again be finitely generated.) If  $c.d.N = 2$  then  $N$  cannot be  $FP_2$ , for otherwise it would be a surface group and  $G$  would be virtually the group of a surface bundle [43]. Since  $N$  has no nonabelian free subgroup this would imply that  $N$  and hence  $G$  are virtually poly- $\mathbb{Z}$ , and so  $N = G$ . Similarly, if  $N = G$  and  $G/G'$  has rank at least 2 then there is an epimorphism  $\phi : G \rightarrow \mathbb{Z}$  with finitely generated kernel [9]. Hence  $\text{Ker}(\phi)$  is a surface group and so  $G$  is poly- $\mathbb{Z}$ .

A finitely generated, torsion-free group is *properly locally cyclic* if every finitely generated subgroup of infinite index is cyclic. If  $G$  is an almost coherent  $PD_3$ -group which is not virtually properly cyclic then every finitely generated subgroup of  $G$  satisfies the Tits alternative [13]. (In fact it suffices for their argument for ‘‘almost coherent’’ to be assumed only for the subgroup, as in [46]). Solvable subgroups are then abelian or virtually poly- $\mathbb{Z}$ , by [34] and Corollary 1.4 of [57].

- (25) Is  $N$  the maximal elementary amenable normal subgroup?
- (26) If  $H$  is a finitely generated subgroup which has no nonabelian free subgroup must it be virtually poly- $\mathbb{Z}$ ?
- (27) In particular, is a  $PD_3$ -group of subexponential growth virtually nilpotent?

## 13. ATOROIDAL GROUPS

We shall say that  $G$  is *atoroidal* if all of its finitely generated abelian subgroups are cyclic. Two-generator subgroups of atoroidal, almost coherent  $PD_3$ -groups are

either free or of finite index, by [3] together with the Algebraic Core Theorem of [57]. 3-Manifolds with atoroidal fundamental group are hyperbolic, by the Geometrization Theorem. Every closed hyperbolic 3-manifold has a finite covering space which fibres over the circle [1, 77].

If  $G$  is atoroidal and acts geometrically on a locally compact  $CAT(0)$  space then it is Gromov hyperbolic [59]. It then has boundary  $S^2$  [5].

- (28) Is every atoroidal  $PD_3$ -group Gromov hyperbolic?
- (29) Does every atoroidal  $PD_3$ -group have a boundary in the sense of [4]?
- (30) The Cannon Conjecture: is every Gromov hyperbolic  $PD_3$ -group isomorphic to a discrete uniform subgroup of  $PSL(2, \mathbb{C})$ ?
- (31) Does every atoroidal  $PD_3$ -group have a nontrivial finitely generated subnormal subgroup of infinite index?

#### 14. SPLITTING

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of splitting theorems for  $PD_3$ -groups. This issue was raised in [89], the first paper on  $PD_3$ -groups. Kropholler and Roller considered splittings of  $PD_n$ -groups over  $PD_{n-1}$  subgroups [67, 68, 69, 70]; see also [24] and [60]. Kropholler gave two different formulations of a torus theorem for  $PD_3$ -groups, one extending to higher dimensions but requiring that the group have *Max-c*, the maximal condition on centralizers [64], and the other with a weaker conclusion [66]. Castel has since shown that every  $PD_3$ -group has *Max-c*, and has given a JSJ-decomposition theorem for  $PD_3$ -groups and group pairs [19].

In particular, if  $G$  has a subgroup  $H \cong \mathbb{Z}^2$  then either  $G$  splits over a subgroup commensurate with  $H$  or it has a nontrivial abelian normal subgroup [64], and so is a 3-manifold group [14]. If  $G$  splits over a  $PD_2$ -group  $H$  then either  $G$  is virtually a semidirect product or  $N_G(H) = H$ . (See §7 of [49].)

If  $G$  is an ascending HNN extension with  $FP_2$  base  $H$  then  $H$  is a  $PD_2$ -group and is normal in  $G$ , and so  $G$  is the group of a surface bundle. (This follows from Lemma 3.4 of [18].) If  $G$  has no noncyclic free subgroup and  $G/G'$  is infinite then  $G$  is an ascending HNN extension with finitely generated base and associated subgroups. If  $G$  is residually finite and has a subgroup isomorphic to  $\mathbb{Z}^2$  then either  $G$  is virtually poly- $\mathbb{Z}$  or it has subgroups of finite index with abelianization of arbitrarily large rank. (A residually finite  $PD_3$ -group which has a subgroup  $H \cong \mathbb{Z}^2$  is virtually split over a subgroup commensurate with  $H$  [67], so we may suppose that  $G$  splits over  $\mathbb{Z}^2$ , and then we may use the argument of [63], which is essentially algebraic.)

- (32) If  $G$  is a nontrivial free product with amalgamation or HNN extension does it split over a  $PD_2$  group?
- (33) If  $G$  is a nontrivial free product with amalgamation is it virtually representable onto  $\mathbb{Z}$ ?
- (34) Can  $G$  be a properly ascending HNN extension (with base not  $FP_2$ )?
- (35) If  $G$  has a subgroup  $H$  which is a  $PD_2$ -group and such that  $N_G(H) = H$  does  $G$  have a subgroup of finite index which splits over  $H$ ? In particular is this so if  $H \cong \mathbb{Z}^2$ ?
- (36) Suppose that  $G$  is an HNN extension with stable letter  $t$ , base  $H$  and associated subgroup  $F \subset H$ . Is  $\mu(G) = \cap t^k F t^{-k}$  finitely generated? (See [56] for a related result on knot groups, and also [82].)

- (37) Suppose  $G$  is not virtually poly- $Z$  and that  $G/G'$  is infinite. Does  $G$  have subgroups of finite index whose abelianization has rank  $\geq 2$ ?

### 15. RESIDUAL FINITENESS, HOPFCITY, COHOPFCITY

Let  $K_n = \cap\{H \subset G \mid [G : H] \text{ divides } n!\}$ . Then  $[G : K_n]$  is finite, for all  $n \geq 1$ , and  $G$  is residually finite if and only if  $\cap K_n = 1$ . If  $G$  is not virtually representable onto  $\mathbb{Z}$  this intersection is also the intersection of the terms in the more rapidly descending series given by  $K_n^{(n)}$ , and is contained in  $G^{(\omega)}$ .

If  $G$  has a maximal finite  $p$ -quotient  $P$  for some prime  $p$  then  $P$  has cohomological period dividing 4, and so is cyclic, if  $p$  is odd, and cyclic or quaternionic, if  $p = 2$ . Hence if  $\beta_1(G; \mathbb{F}_p) > 1$  for some odd prime  $p$ , or if  $\beta_1(G; \mathbb{F}_2) > 2$ , then the pro- $p$  completion of  $G$  is infinite [74].

If  $[G : \cap K_n] = \infty$  and  $G$  is a 3-manifold group then either  $G$  is solvable or there is a prime  $p$  such that  $G$  has subgroups  $H$  of finite index with  $\beta_1(H; \mathbb{F}_p)$  arbitrarily large [74]. Hence either some such  $H$  maps onto  $\mathbb{Z}$  or the pro- $p$  completion of any such subgroup with  $\beta_1(H; \mathbb{F}_p) > 1$  is a pro- $p$   $PD_3$ -group [62]. If  $G$  is almost coherent and  $[G : \cap K_n] = \infty$  then it satisfies the Tits alternative [13].

The groups of 3-manifolds are residually finite, by [39] and the Geometrization Theorem. Hence they are hopfian, i.e., onto endomorphisms of such groups are automorphisms. The Baumslag-Solitar groups  $\langle x, t \mid tx^pt^{-1} = x^q \rangle$  embed in  $PD_4$ -groups. Since these groups are not hopfian, there are  $PD_4$ -groups which are not residually finite [73]. No such Baumslag-Solitar relation with  $|p| \neq |q|$  holds in any  $PD_3$ -group  $G$ ; there is no homomorphism from  $\langle x, t \mid tx^pt^{-1} = x^q \rangle$  to  $G$  [19].

Let  $\mathcal{X}$  be the class of groups of cohomological dimension 2 which have an infinite cyclic subgroup which is commensurate with all of its conjugates. If  $G$  is a  $PD_3$ -group with no nontrivial abelian normal subgroup and which contains a subgroup isomorphic to  $\mathbb{Z}^2$  then  $G$  splits over an  $\mathcal{X}$ -group [66]. (See also [65].) This class includes the Baumslag-Solitar groups and also the fundamental groups of Seifert fibred 3-manifolds with nonempty boundary. If  $H$  is in the latter class and is not virtually  $\mathbb{Z}^2$  then  $\sqrt{H} \cong \mathbb{Z}$  and  $H/\sqrt{H}$  is a free product of cyclic groups. It then follows from the result of [19] on Baumslag-Solitar relations that finitely generated  $\mathcal{X}$ -groups which are subgroups of  $PD_3$ -groups are of this ‘‘Seifert type’’.

An injective endomorphism of a  $PD_3$ -group must have image of finite index, by Strebel’s theorem [83]. A 3-manifold group satisfies the *volume condition* (isomorphic subgroups of finite index have the same index) if and only if it is not solvable and is not virtually a product [96, 97]. In particular, such 3-manifold groups are cohopfian, i.e., injective endomorphisms are automorphisms. The volume condition is a property of commensurability classes; this is not so for cohopficity.

- (38) Does every  $PD_3$ -group have a proper subgroup of finite index?  
 (39) Are all  $PD_3$ -groups residually finite?  
 (40) Let  $\hat{G}$  be a pro- $p$   $PD_3$ -group. Is  $G$  virtually representable onto  $\hat{\mathbb{Z}}_p$ ?  
 (41) Do all  $PD_3$ -groups other than those which are solvable or are virtually products satisfy the volume condition?

### 16. OTHER QUESTIONS

We conclude with some related questions. (Here  $X$  is an indecomposable  $PD_3$ -complex and  $\pi = \pi_1(X)$ .)



- (42) Is  $(D_{2m} \times Z/3Z) *_{Z/6Z} T_1^*$  the group of a  $PD_3$ -complex, for some  $m \geq 5$ ?
- (43) If the orientable double cover of  $X$  is homotopy equivalent to a 3-manifold, is  $X$  itself homotopy equivalent to a 3-manifold?
- (44) If  $\pi_1(X) \cong \pi\mathcal{G}$ , where  $(\mathcal{G}, \Gamma)$  is a graph of groups with vertex groups one-ended and edge groups of order 2, must each vertex of  $\Gamma$  have even valence?
- (45) If  $\mathbb{Z} \oplus Z/2Z \leq \pi$  must  $X \simeq S^1 \times RP^2$ ?
- (46) Is  $X \times S^1$  or  $X \times S^1 \times S^1$  homotopy equivalent to a closed manifold?
- (47) Clarify the connection between  $k_1(X)$  and  $c_{X^*}[X]$  when  $\pi$  is infinite.
- (48) Is there an explicit example of a free action of a generalized quaternionic group  $Q(8a, b, 1)$  (with  $a, b > 1$  and  $(a, b) = 1$ ) on an homology 3-sphere?
- (49) Is there an algebraic analogue of the reflection group trick which may be used to embed  $FP$  groups of cohomological dimension  $k$  in  $PD_{2k}$ -groups?

One way to prove (44) might be to use the Realization Theorem of [11] to show that  $(G_v, \{G_e : o(e) \text{ or } t(e) = v, o(e) \neq t(e)\})$  is the fundamental group system of a  $PD_3$ -pair (with all boundary components  $RP^2$ ), for each vertex  $v \in V(\Gamma)$ .

See [48] and the references there [36, 37, 94, 95] for work on maps of nonzero degree between  $PD_3$ -groups.

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