

## Some questions on subgroups of 3-dimensional Poincaré duality groups

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### ABSTRACT

*We state a number of open questions on 3-dimensional Poincaré duality groups and their subgroups, motivated by considerations from 3-manifold topology.*

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The notion of Poincaré duality group of dimension  $n$  (or  $PD_n$ -group, for short) is essentially an algebraic analogue of the notion of aspherical  $n$ -manifold. When  $n = 1$  or  $2$  the analogy is precise; the only such groups are the fundamental groups of the circle and the aspherical closed surfaces, and two such manifolds are homeomorphic if and only if their groups are isomorphic [EM80, EL81]. As is well known, the fundamental group also plays a central role in 3-dimensional topology. Every closed 3-manifold admits an essentially unique decomposition with prime factors which either have finite fundamental group, or are  $S^1 \times S^2$  or are aspherical. The work of Perelman implies that every homotopy equivalence between irreducible aspherical 3-manifolds is homotopic to a homeomorphism. It is natural to ask also whether every  $PD_3$ -group is the fundamental group of some 3-manifold. This is so if the group has sufficiently nice subgroups. An affirmative answer in general would suggest that a large part of the study of 3-manifolds may be reduced to algebra. It should be noted also that  $PD_3$ -groups may occur in other contexts, where the geometry does not immediately provide a corresponding 3-manifold (cf. [H]).

Prompted by the main result of [KK05], we define an *open*  $PD_n$ -group to be a countable group  $G$  of cohomological dimension  $\leq n - 1$  such that every nontrivial  $FP$  subgroup  $H$  with  $H^s(H; \mathbb{Z}[H]) = 0$  for  $s < n - 1$  is the ambient group of a  $PD_n$ -pair  $(H, \mathcal{T})$ , for some set of monomorphisms  $\mathcal{T}$ .

Every subgroup of infinite index in a  $PD_3$ -group  $G$  is an open  $PD_3$ -group in our sense, by Theorem 1.3 of [KK05]. (The analogies are precise if  $n = 2$ , but these definitions are too broad when  $n \geq 4$ . We shall consider only the cases  $n = 2$  or  $3$ .)

Here we shall present a number of questions on subgroups of  $PD_3$ -groups, motivated by results conjectured or already established geometrically for 3-manifold groups. The corresponding questions for subgroups of open  $PD_3$ -groups should be considered with these. (See also the lists of problems in [W], [GS] and [NR]). Any group with a finite 2-dimensional Eilenberg – Mac Lane complex is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to  $D^4$ . (Conjecturally such groups are exactly the finitely presentable groups of cohomological dimension 2). On applying the reflection group trick of Davis to the boundary we see that each such group embeds in a  $PD_4$ -group [Da83]. Thus the case considered here is critical. (If we assume the  $PD_3$ -group is coherent and has a finite  $K(G, 1)$ -complex, as is the case for all 3-manifold groups, a number of these questions have clear answers. On the other hand, assuming that  $G$  is virtually representable onto  $Z$  appears of limited use beyond simplifying the characterization of Seifert 3-manifold groups.)

We shall assume throughout that  $G$  is an orientable  $PD_3$ -group. The normalizer and centralizer of a subgroup  $H$  of  $G$  shall be denoted by  $N_G(H)$  and  $C_G(H)$ , respectively. We shall also let  $\zeta G = C_G(G)$ ,  $G'$  and  $G^{(\omega)} = \cap G^{(n)}$  denote the centre, the commutator subgroup and the intersection of the terms of the derived series of  $G$ , respectively. A group has a given property *virtually* if it has a subgroup of finite index with that property. Since our interest is in  $PD_3$ -groups, we shall use “3-manifold group” to mean fundamental group of an *aspherical* closed 3-manifold.

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## The group

The central question is whether every  $PD_3$ -group is the fundamental group of some aspherical closed 3-manifold. (See the final section.) The following

questions represent possibly simpler consequences.

To begin with, the augmentation  $\mathbb{Z}[G]$ -module  $Z$  has a finite projective resolution, and there is a 3-dimensional  $K(G, 1)$  complex. Hence  $G$  is almost finitely presentable ( $FP_2$ ). The  $K(G, 1)$ -complex is finitely dominated, and hence a Poincaré complex in the sense of [Wl67], if and only if  $G$  is finitely presentable. (For every  $n \geq 4$  there are  $PD_n$ -groups which are not finitely presentable [Da98].) For each  $g \in G$  with infinite conjugacy class  $[G : C_G(\langle g \rangle)] = \infty$ , so  $c.d.C_G(\langle g \rangle) \leq 2$  [St77] and hence  $C_G(\langle g \rangle)/\langle g \rangle$  is locally virtually free, by Theorem 8.4 of [B]. Therefore  $G$  must satisfy the Strong Bass Conjecture, by [Ec86]. (This observation is due to Eckmann and Sykiotis.)

If  $G$  is an  $FP_2$  group such that  $H^2(G; \mathbb{Z}[G]) \cong Z$  then  $G$  is virtually a  $PD_2$ -group [Bo04].

If  $M = K(G, 1)$  is a closed 3-manifold we may assume it has one 0-cell and one 3-cell, and equal numbers of 1- and 2-cells. Hence  $G$  has a finite presentation of deficiency 0; this is clearly best possible, since  $\beta_1(G; \mathbb{F}_2) = \beta_2(G; \mathbb{F}_2)$ . Moreover  $G$  is  $FF$ , i.e., the augmentation module  $Z$  has a finite free  $\mathbb{Z}[G]$ -resolution, while  $\tilde{K}_0(\mathbb{Z}[G]) = Wh(G) = 0$  and  $\tilde{M} \cong R^3$ , so  $G$  is 1-connected at  $\infty$ .

- (1) Is  $G$  finitely presentable? If so, does it have a presentation of deficiency 0?
- (2) Is  $G$  of type  $FF$ ?
- (3) Is  $\tilde{K}_0(\mathbb{Z}[G]) = 0$ ? Is  $Wh(G) = 0$ ?
- (4) Is  $G$  1-connected at  $\infty$ ? Does it have a boundary in the sense of [Be96]?
- (5) Is  $K(G, 1)$  homotopy equivalent to a finite complex?
- (6) If  $G$  is an  $FP_3$  group such that  $H^3(G; \mathbb{Z}[G]) \cong Z$  is  $G$  virtually a  $PD_3$ -group?

### Subgroups in general

Since  $G$  has cohomological dimension 3 it has no nontrivial finite subgroups. Any nontrivial element  $g$  generates an infinite cyclic subgroup  $\langle g \rangle$ ; it is not known whether there need be any other proper subgroups. If a subgroup  $H$  of  $G$  has finite index then it is also a  $PD_3$ -group. The cases when  $[G : H]$  is infinite are of more interest, and then either  $c.d.H = 2$  or  $H$  is free, by [St77] and [S]. If there is a finitely generated (respectively,  $FP_2$ ) subgroup of cohomological dimension 2 there is one such which has one end (i.e., which

is indecomposable with respect to free product). A solvable subgroup  $S$  of Hirsch length  $h(S) \geq 2$  must be finitely presentable, since either  $[G : S]$  is finite or  $c.d.S = 2 = h(S)$  [Gi79]. (In particular, abelian subgroups of rank  $> 1$  are finitely generated.)

3-manifold groups are coherent: finitely generated subgroups are finitely presentable. In fact something stronger is true: if  $H$  is a finitely generated subgroup it is the fundamental group of a compact 3-manifold (possibly with boundary) [Sc73]. If  $\pi$  is the fundamental group of a graph manifold then the group ring  $\mathbb{Z}[\pi]$  is coherent. (The corresponding result for lattices in  $PSL(2, \mathbb{C})$  is apparently not known.)

If  $G$  is a  $PD_3$ -group with a one-ended  $FP_2$  subgroup  $H$  then there is a system of monomorphisms  $\sigma$  such that  $(H, \sigma)$  is a  $PD_3$ -pair [KK05]. In particular,  $\chi(H) \leq 0$ . If  $\mathbb{Z}[G]$  is coherent as a ring then every finitely generated subgroup of  $G$  is  $FP_2$ .

In general  $PD_n$ -groups with  $n \geq 4$  are not coherent. If  $\pi$  is a surface group with  $\chi(\pi) < 0$  then it has a noncyclic free subgroup  $F$  and so  $F \times F$  is a subgroup of  $\pi \times \pi$ . It is well known that such groups  $F \times F$  have finitely generated subgroups which are not finitely related (cf. Section 8.2 of [B]). However  $F \times F$  is not a subgroup of any  $PD_3$ -group [KR89].

Let  $M$  be a closed orientable 3-manifold. Then  $M$  is Haken, Seifert fibred or hyperbolic, by the Geometrization Theorem. With [KM10] it follows that if  $\pi_1(M)$  is infinite then it has a  $PD_2$ -subgroup. A transversality argument implies that every element of  $H_2(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong [M; S^1]$  is represented by an embedded submanifold. If  $M$  is aspherical it follows that  $H_2(\pi_1(M); \mathbb{Z})$  is generated by elements represented by surface subgroups of  $\pi_1(M)$ . If  $G/G'$  is infinite then  $G$  is an HNN extension with finitely generated base and associated subgroups [BS78], and so has a finitely generated subgroup of cohomological dimension 2.

- (7) Is there a noncyclic proper subgroup? If so, is there one of cohomological dimension 2? and finitely generated?
- (8) Is there a subgroup which is a surface group? Is every element of  $H_2(G; \mathbb{Z})$  represented by such a subgroup?
- (9) Is  $G$  coherent? Is  $\mathbb{Z}[G]$  coherent as a ring?
- (10) Does every (finitely presentable) subgroup of cohomological dimension 2

- have a (finite) 2-dimensional Eilenberg-Mac Lane complex (with  $\chi \leq 0$ )?
- (11) If  $H$  is a finitely generated subgroup with one end, is there a system of monomorphisms  $\sigma$  such that  $(H, \sigma)$  is a  $PD_3$ -pair? In particular, does every such subgroup of infinite index have infinite abelianization? contain a surface group?

### Ascendant subgroups

If  $N$  is an  $FP_2$  ascendant subgroup of  $G$  and  $c.d.N = 2$  then it is a surface group and  $G$  has a subgroup of finite index which is a surface bundle group. If  $c.d.N = 1$  then  $N \cong Z$  and either  $G$  is virtually poly- $Z$  or  $N$  is normal in  $G$  and  $[G : C_G(N)] \leq 2$  [BH91, Hi06]. In the latter case  $G$  is the group of a Seifert fibred 3-manifold [Bo04]. It is easy to find examples among normal subgroups of 3-manifold groups to show that finite generation of  $N$  is necessary for these results.

If  $N$  is finitely generated, normal and  $[G : N] = \infty$  then  $H^1(G/N; \mathbb{Z}[G/N])$  is isomorphic to  $H^1(G; \mathbb{Z}[G/N])$  and hence to  $H_2(G; \mathbb{Z}[G/N]) \cong H_2(N; \mathbb{Z})$ , by Poincaré duality. In particular,  $G/N$  has two ends if and only if  $H_2(N; \mathbb{Z}) \cong Z$ . In the latter case Shapiro's lemma and Poincaré duality together imply that  $H^2(N; \lim_{\rightarrow} M_i)$  is 0 for any direct system  $M_i$  with limit 0. Hence  $N$  is  $FP_2$  by Brown's criterion [Br75] and so is a surface group by the above result.

- (12) Is there a simple  $PD_3$ -group?
- (13) Is  $G$  virtually representable onto  $Z$ ?
- (14) Must a finitely generated *normal* subgroup  $N$  be finitely presentable?
- (15) Suppose  $N \leq U$  are subgroups of  $G$  with  $U$  finitely generated and indecomposable,  $[G : U]$  infinite,  $N$  subnormal in  $G$  and  $N$  not cyclic. Is  $[G : N_G(U)] < \infty$ ? (Cf. [El84].)

### Centralizers and normalizers

If  $G$  is a  $PD_3$ -group with nontrivial centre then  $\zeta G$  is finitely generated and  $G$  is the fundamental group of an aspherical Seifert fibred 3-manifold [Bo04]. (See also [Hi85].) Since an elementary amenable group of finite cohomological dimension is virtually solvable [HL92], it follows also that either  $G$  is virtually poly- $Z$  or its maximal elementary amenable normal subgroup is cyclic.

Every strictly increasing sequence of centralizers  $C_0 < C_1 < \dots < C_n = G$  in a  $PD_3$ -group  $G$  has length  $n$  at most 4 [Hi06]. (This was known earlier,

with the bound 11, for  $G$  the group of an aspherical 3-manifold [Kr90a]. The finiteness of such sequences in any  $PD_3$ -group is due to Castel [Ca07].)

An element  $g$  is a root of  $x$  if  $x = g^n$  for some  $n$ . Clearly all roots of  $x$  are in  $C_G(x)$ . If  $C_G(x)$  is finitely generated then  $x$  is not infinitely divisible. For if  $c.d.C_G(x) = 1$  then  $C_G(x) \cong Z$ ; if  $c.d.C_G(x) = 2$  then  $C_G(x)/\langle x \rangle$  is virtually free, by Theorem 8.4 of [B]; and if  $c.d.C_G(x) = 3$  then  $C_G(x)/\langle x \rangle$  is virtually a  $PD_2$ -group [Bo04]. Conversely, if  $x$  is not infinitely divisible then  $C_G(x)$  is finitely generated [Ca07].

If  $x$  is a nontrivial element of  $G$  then  $\langle x \rangle$  is infinite cyclic and so  $[N_G(\langle x \rangle) : C_G(x)] \leq 2$ . If  $F$  is a finitely generated nonabelian free subgroup of  $G$  then  $N_G(F)$  is finitely generated and  $N_G(F)/F$  is finite or virtually  $Z$  [Hi06]. (See [Sc76] for another argument in the 3-manifold case.) If  $H$  is an  $FP_2$  subgroup which is a nontrivial free product but is not free then  $[N_G(H) : H] < \infty$  and  $C_G(H) = 1$  [Hi06].

If  $H$  is a one-ended  $FP_2$  subgroup of infinite index in  $G$  then either  $[G : N_G(K)]$  or  $[N_G(K) : K]$  is finite. (See Lemma 2.15 of [H]). More precisely, define an increasing sequence of subgroups  $\{H_i | i \geq 0\}$  by  $H_0 = H$  and  $H_i = N_G(H_{i-1})$  for  $i > 0$ . Then  $\hat{H} = \cup H_i$  is  $FP_2$  and either  $c.d.\hat{H} = 2$ ,  $\hat{H}$  has one end and  $N_G(\hat{H}) = \hat{H}$ , or  $\hat{H}$  is a  $PD_3$ -group and  $G$  is virtually the group of a surface bundle, by Theorem 2.17 of [H]. In particular, if  $G$  has a subgroup  $H$  which is a surface group with  $\chi(H) = 0$  (respectively,  $< 0$ ) then either it has such a subgroup which is its own normalizer in  $G$  or  $G$  is virtually the group of a surface bundle.

If  $C_G(\langle x \rangle)$  is nonabelian then it is  $FP_2$ , and is either of bounded Seifert type or has finite index in  $G$  [Ca07]. In the latter case either  $[G : C_G(\langle x \rangle)] \leq 2$  or  $G$  is virtually  $Z^3$ , by Theorem 2 of [Hi06].

If the sequence of centralizers  $C_1 \cong Z < C_2 < C_3 < C_4 = G$  is strictly increasing then  $C_3$  must be nonabelian. (See [Hi06].) Hence it is  $FP_2$  [Ca07], and so either  $G$  is Seifert or  $c.d.C_3 = 2$ . In all cases it follows that  $C_2 \cong Z^2$ . Equivalently, if  $G$  has a maximal abelian subgroup  $A$  which is not finitely generated then  $1 < A < G$  is the only sequence of centralizers containing  $A$ .

The *commensurator* in  $G$  of a subgroup  $H$  is the subgroup  $Comm_G(H) = \{g \in G \mid [H : H \cap gHg^{-1}] < \infty \text{ and } [H : H \cap g^{-1}Hg] < \infty\}$ . It clearly contains  $N_G(H)$ .

If  $x \neq 1$  in  $G$  then the Baumslag-Solitar relation  $tx^pt^{-1} = x^q$  implies that  $p = \pm q$  [KK05] and it follows easily that  $Comm_G(\langle x \rangle) = \cup N_G(\langle x^{n!} \rangle)$ . Since the chain of centralizers  $C_G(\langle x^{n!} \rangle)$  is increasing and  $[N_G(\langle x^k \rangle) : C_G(\langle x^k \rangle)] \leq 2$  for any  $k$  it follows that  $Comm_G(\langle x \rangle) = N_G(\langle x^{n!} \rangle)$  for some  $n \geq 1$ .

If  $H$  is a  $PD_2$ -group then Theorem 1.3 and Proposition 4.4 of [KR89a] imply that either  $[Comm_G(H) : H] < \infty$  or  $H$  is commensurable with a subgroup  $K$  such that  $[G : N_G(K)] < \infty$ .

- (16) Is every abelian subgroup of  $G$  finitely generated?

### The derived series and perfect subgroups

The intersection  $P = \cap G^{(\alpha)}$  of the terms of the transfinite derived series for  $G$  is a perfect normal subgroup of  $G$  and is the maximal perfect subgroup. Either  $G/P$  is finite (and is a solvable group with cohomological period dividing 4) or  $c.d.P = 2$  or  $P = 1$ . If  $c.d.P = 2$  then  $P$  cannot be  $FP_2$ , for otherwise it would be a surface group [Hi87]. Note that  $P \subseteq G^{(\omega)}$ , and if  $c.d.P = 2$  then  $c.d.G^{(\omega)} = 2$  also. If  $[G : P]$  is infinite and  $\zeta G \neq 1$  then  $P = 1$ .

If  $G$  is not virtually representable onto  $Z$  then  $G/G^{(\omega)}$  is either a finite solvable group with cohomological period dividing 4 (and  $G^{(\omega)}$  is a perfect  $PD_3$ -group) or is a finitely generated infinite residually finite-solvable group with one or infinitely many ends.

If  $G$  is a  $PD_3$ -group and  $H$  is a nontrivial  $FP_2$  subgroup such that  $H^1(H; \mathbb{Z}) = 0$  then  $[G : H]$  is finite. (Use [KK05]. See [Ho82] for 3-manifold groups.) However if  $G$  is the group of the (aspherical) 3-manifold obtained by 0-framed surgery on a nontrivial knot with  $\Delta = 1$  then  $G'$  is a perfect normal subgroup which is not finitely generated. Note also that there is a superperfect group with a finite 2-dimensional Eilenberg-Mac Lane complex [DV73].

- (17) Can a nontrivial finitely generated normal subgroup of infinite index be perfect?
- (18) If a finitely generated, infinite residually solvable group has infinitely many ends must it be virtually representable onto  $Z$ ?
- (19) If  $P = 1$  is  $G$  residually solvable (i.e., is  $G^{(\omega)} = 1$  also)?

### The Tits alternative

Let  $N$  be the subgroup generated by all the normal subgroups which have no nonabelian free subgroup. Then  $N$  is the maximal such subgroup, and

clearly it contains the maximal elementary amenable normal subgroup of  $G$ . If  $N$  is nontrivial then either  $N \cong Z$ ,  $c.d.N = 2$  or  $N = G$ . If  $N$  is a rank 1 abelian subgroup then  $N \cong Z$ . (For otherwise  $N \leq G'$  and  $G' \leq C_G(N)$ , so either  $[G : C_G(N)]$  is finite, which can be excluded by [Bo04], or  $G'$  is abelian, by Theorem 8.8 of [Bi], in which case  $G$  is solvable and hence virtually poly- $Z$ , and  $N$  must again be finitely generated.) If  $c.d.N = 2$  then  $N$  cannot be  $FP_2$ , for otherwise it would be a surface group and  $G$  would be virtually the group of a surface bundle [Hi87]. Since  $N$  has no nonabelian free subgroup this would imply that  $N$  and hence  $G$  are virtually poly- $Z$ , and so  $N = G$ . Similarly, if  $N = G$  and  $G/G'$  has rank at least 2 then there is an epimorphism  $\phi : G \rightarrow Z$  with finitely generated kernel [BNS87]. Hence  $\text{Ker}(\phi)$  is a surface group and so  $G$  is poly- $Z$ .

Subgroups which are coherent, locally virtually indicable and contain no nonabelian free subgroups are virtually solvable [Hi03], and hence (using [Gi79] and Corollary 1.4 of [KK05]) virtually abelian or virtually poly- $Z$ . If  $G$  is the fundamental group of a Haken 3-manifold then subgroups which contain no noncyclic free group are virtually poly- $Z$  [EJ73].

- (20) Is  $N$  the maximal elementary amenable normal subgroup?
- (21) If  $H$  is a finitely generated subgroup which has no nonabelian free subgroup must it be virtually poly- $Z$ ?

### Atoroidal groups

We shall say that  $G$  is *atoroidal* if all of its finitely generated abelian subgroups are cyclic. Two-generator subgroups of atoroidal 3-manifold groups are either free or of finite index [JS79]. If  $G$  is the group of an atoroidal virtually Haken 3-manifold then it can be embedded as a discrete uniform subgroup of  $PSL(2, \mathbb{C})$  [Th82, CS83]. Question 18 of [Th82] asks whether every closed hyperbolic 3-manifold has a finite covering space which fibres over the circle.

If an atoroidal  $PD_3$ -group acts geometrically on a locally compact  $CAT(0)$  space then it is Gromov hyperbolic [KK07].

- (22) Is every atoroidal  $PD_3$ -group *automatic* [Ep92]? *negatively curved* [AS91]? isomorphic to a discrete uniform subgroup of  $PSL(2, \mathbb{C})$ ?
- (23) Does every atoroidal  $PD_3$ -group have a nontrivial finitely generated subnormal subgroup of infinite index? (Cf. question 18 of [Th82].)

## Splitting

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of splitting theorems for  $PD_3$ -groups. This issue was raised in [Th84], the first paper on  $PD_3$ -groups. Since then the most substantial results on splittings of Poincaré duality groups are to be found in the papers [Du89], [Kr90,93] and [KR88,88a,89,89a].

If  $G$  is an ascending HNN extension with  $FP_2$  base  $H$  then  $H$  is a  $PD_2$ -group and is normal in  $G$ , and so  $G$  is the group of a surface bundle. (This follows from Lemma 3.4 of [BG85].) If  $G$  has no noncyclic free subgroup and  $G/G'$  is infinite then  $G$  is an ascending HNN extension with finitely generated base and associated subgroups. If  $G$  is residually finite and has a subgroup isomorphic to  $Z^2$  then either  $G$  is virtually poly- $Z$  or it has subgroups of finite index with abelianization of arbitrarily large rank. (A residually finite  $PD_3$ -group which has a subgroup  $H \cong Z^2$  is virtually split over a subgroup commensurate with  $H$  [KR88], so we may suppose that  $G$  splits over  $Z^2$ , and then we may use the argument of [Ko87], which is essentially algebraic.)

If  $G$  has a subgroup  $H \cong Z^2$  then either  $G$  splits over a subgroup commensurate with  $H$  or it has a nontrivial abelian normal subgroup [Kr90]. (Kropholler assumes that  $G$  has *max-c*, since proven in [Ca07].) In the latter case it is a 3-manifold group [Bo04].

- (24) If  $G$  is a nontrivial free product with amalgamation or HNN extension does it split over a  $PD_2$  group?
- (25) If  $G$  is a nontrivial free product with amalgamation is it virtually representable onto  $Z$ ?
- (26) Can  $G$  be a properly ascending HNN extension (with base not  $FP_2$ )?
- (27) If  $G$  has an  $FP_2$ , one ended subgroup  $H$  such that  $c.d.H = 2$  and  $N_G(H) = H$  does  $G$  have a subgroup of finite index which splits over a subgroup commensurate with  $H$ ? In particular is this so if  $H$  is a  $PD_2$ -group? if  $H \cong Z^2$ ?
- (28) Suppose  $G$  is not virtually poly- $Z$  and that  $G/G'$  is infinite. Does  $G$  have subgroups of finite index whose abelianization has rank  $\geq 2$ ?
- (29) Suppose that  $G$  is an HNN extension with stable letter  $t$ , base  $H$  and associated subgroup  $F \subset H$ . Is  $\mu(G) = \cap t^k F t^{-k}$  finitely generated? (See [Ka89] for a related result on knot groups, and also [So91].)

### Residual finiteness, hopficity, cohopficity, Baumslag-Solitar groups

Let  $K_n = \cap\{H \subset G \mid [G : H] \text{ divides } n!\}$ . Then  $[G : K_n]$  is finite, for all  $n \geq 1$ , and  $G$  is residually finite if and only if  $\cap K_n = 1$ . If  $G$  is not virtually representable onto  $Z$  this intersection is also the intersection of the terms in the more rapidly descending series given by  $K_n^{(n)}$ , and is contained in  $G^{(\omega)}$ .

If  $G$  is not virtually simple then  $[G : \cap K_n] = \infty$ . If moreover  $G$  is a 3-manifold group then either  $G$  is solvable or there is a prime  $p$  such that  $G$  has subgroups  $H$  of finite index with  $\beta_1(H; \mathbb{F}_p)$  arbitrarily large [Me91]. Hence either some such  $H$  maps onto  $Z$  or the pro- $p$  completion of any such subgroup with  $\beta_1(H; \mathbb{F}_p) > 1$  is a pro- $p$   $PD_3$ -group [KZ08].

The groups of 3-manifolds are residually finite, by [He87] and the Geometrization Theorem. Hence they are hopfian, i.e., onto endomorphisms of such groups are automorphisms. The Baumslag-Solitar groups  $\langle x, t \mid tx^pt^{-1} = x^q \rangle$  embed in  $PD_4$ -groups. Since these groups are not hopfian, there are  $PD_4$ -groups which are not residually finite [Me90]. A  $PD_3$ -group  $G$  has no Baumslag-Solitar subgroups with  $p \neq \pm q$  [KK05]. (This was shown earlier for 3-manifold groups [Kr90a].)

An injective endomorphism of a  $PD_3$ -group must have image of finite index, by Strebel's theorem. A 3-manifold group satisfies the *volume condition* (isomorphic subgroups of finite index have the same index) if and only if it is not solvable and is not virtually a product [WW94, WY99]. In particular, such 3-manifold groups are cohopfian, i.e., injective endomorphisms are automorphisms. (It is not hard to show that the volume condition is a property of commensurability classes; this is not so for cohopficity.)

Let  $\mathcal{X}$  be the class of groups of cohomological dimension 2 which have an infinite cyclic subgroup which is commensurate with all of its conjugates. The finitely generated groups in this class are the fundamental groups of finite graphs of groups in which each vertex group and each edge group is infinite cyclic. This class includes the torus knot groups and the Baumslag-Solitar groups. If  $G$  is a  $PD_3$ -group with no nontrivial abelian normal subgroup and which contains a subgroup isomorphic to  $Z^2$  then  $G$  splits over an  $\mathcal{X}$ -group [Kr93]. (See also [Kr90b].)

(30) Are all  $PD_3$ -groups residually finite?

(31) Which finitely generated  $\mathcal{X}$ -groups are subgroups of  $PD_3$ -groups?

- (32) Let  $\widehat{G}$  be a pro- $p$   $PD_3$ -group. Is  $G$  virtually representable onto  $\widehat{Z}_p$ ?
- (33) Do all  $PD_3$ -groups satisfy the volume condition?

### Other questions

We conclude with some somewhat more topological questions.

If  $G$  is a  $PD_3$ -group which has a subgroup isomorphic to  $\pi_1(M)$  where  $M$  is an aspherical 3-manifold then  $G$  is itself a 3-manifold group. For  $M$  is either Haken, Seifert fibred or hyperbolic, by the Geometrization Theorem, and so we may apply [Zi82], Section 63 of [Z] or Mostow rigidity, respectively.

Every  $PD_3$ -complex  $X$  is a connected sum of  $PD_3$ -complexes whose fundamental groups are indecomposable with respect to free product [Tu89]. The indecomposable summands are either aspherical or have virtually free fundamental group [Cr00]. However there are two such homotopy types with fundamental group  $S_3 *_Z/2Z S_3$  [Hi04']. The bordism Hurewicz homomorphism from  $\Omega_n(X)$  to  $H_n(X; \mathbb{Z})$  is an epimorphism in degrees  $n \leq 4$ . (See [HV] for the corresponding result for possibly nonorientable  $PD_n$ -complexes, using  $w_1$ -twisted bordism and homology.) Thus there is a degree-1 map  $f : M \rightarrow X$ , where  $M$  is a closed 3-manifold. Is it possible to modify  $f$  by Dehn surgery (and passing to finite covers) to obtain a homotopy equivalence?

If  $P$  is a  $PD_3$ -complex with fundamental group  $\pi$  then  $\pi_2(P) \cong \overline{H^1(\pi; \mathbb{Z}[\pi])}$  as a left  $\mathbb{Z}[\pi]$ -module. If  $\pi$  is torsion free but not free then  $H^2(P; \pi_2(P)) = 0$ ,  $\pi_2(P)$  is a projective  $\mathbb{Z}[\pi]$ -module and any two of the conditions “ $\pi$  is  $FF$ ”, “ $P$  is homotopy equivalent to a finite complex” and “ $\pi_2(P)$  is stably free” imply the third. Moreover if  $\pi$  is a nontrivial free group then  $\pi_2(P)$  has projective dimension 1 and  $H^2(P; \pi_2(P)) \cong Z$ . (See [H].) (If  $\pi$  is not torsion free then the projective dimension of  $\pi_2(P)$  is infinite.) The cohomology group  $H^2(P; \pi_2(P))$  arises in studying homotopy classes of self homotopy equivalences of  $P$ . If  $N$  is a  $P^2$ -irreducible 3-manifold and  $\pi_1(N)$  is virtually free then  $H^2(N; \pi_2(N)) \cong Z$ , and otherwise  $H^2(N; \pi_2(N)) = 0$  [HL74].

See [Hi04] and the references there for work on maps of nonzero degree between  $PD_3$ -groups.

If every  $PD_3$ -group is a 3-manifold group classical knot groups can be characterized in terms of orientable  $PD_3$ -pairs  $(G, Z^2)$  of weight 1.

- (34) Does every  $PD_3$ -complex have a finite cover which is homotopy equivalent

to a closed 3-manifold? Equivalently, is every  $PD_3$ -group virtually a 3-manifold group?

- (35) Let  $X$  be an orientable  $PD_3$ -complex such that  $\pi_1(X)$  is not virtually free. Is  $H^2(X; \pi_2(X)) = 0$ ? If  $\pi_1(X)$  is torsion free, is  $\pi_2(X)$  a free  $\pi_1(X)$ -module?
- (36) Is there a purely algebraic analogue of orbifold hyperbolization which may be used to show that every  $FP$  group of cohomological dimension  $k$  is a subgroup of a  $PD_{2k}$ -group?
- (37) Is a  $PD_3$ -group of subexponential growth virtually nilpotent?
- (38) Let  $X$  be a  $PD_3$ -complex. Is  $X \times S^1$  or  $X \times S^1 \times S^1$  homotopy equivalent to a closed manifold?
- (39) Let  $G$  be a  $PD_3$ -group such that  $G'$  is free. Is  $G$  a semidirect product  $K \rtimes Z$  with  $K$  a  $PD_2$ -group?

### References

- [AS91] Alonso, J.M., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H. Notes on negatively curved groups, in *Group Theory from a Geometric Viewpoint* (edited by E.Ghys, A.Haefliger and A.Verjovsky), World Scientific, Singapore (1991), 3-63.
- [BS89] Baumslag, G. and Shalen, P.B. Groups whose three-generator subgroups are free, *Bull. Austral. Math. Soc.* 40 (1989), 163-174.
- [Be96] Bestvina, M. Local homological properties of boundaries of groups, *Michigan Math. J.* 43 (1996), 123-139.
- [BB97] Bestvina, M. and Brady, N. Morse theory and finiteness properties of groups, *Invent. Math.* 129 (1997), 445-470.
- [B] Bieri, R. *Homological Dimensions of Discrete Groups*, Queen Mary College Mathematical Notes, London (1976).
- [BE78] Bieri, R. and Eckmann, B. Relative homology and Poincaré duality for group pairs, *J. Pure Appl. Alg.* 13 (1978), 277-319.
- [BH91] Bieri, R. and Hillman, J.A. Subnormal subgroups in 3-dimensional Poincaré duality groups, *Math. Z.* 206 (1991), 67-69.
- [BNS87] Bieri, R., Neumann, W.D. and Strebel, R. A geometric invariant of discrete groups, *Inventiones Math.* 90 (1987), 451-477.
- [BS78] Bieri, R. and Strebel, R. Almost finitely presentable soluble groups, *Commentarii Math. Helvetici* 53 (1978), 258-278.
- [Bo04] Bowditch, B.H. Planar groups and the Seifert conjecture, *J. Reine u. Angew. Math.* 576 (2004), 11-62.
- [Br75] Brown, K.S. A homological criterion for finiteness, *Comment. Math. Helvetici* 50 (1975), 129-135.

- [Br85] Brown, K.S. and Geoghegan, R. Cohomology with free coefficients of the fundamental group of a graph of groups, *Comment. Math. Helv.* 60 (1985), 31–45.
- [Ca07] Castel, F. Centralisateurs d'éléments dans les  $PD_3$ -paires, *Commentarii Math. Helvetici* 82 (2007), 499–517.
- [Co95] Cornick, J. On the homology of group graded algebras, *J.Algebra* 74 (1995), 999–1023.
- [Cr00] Crisp, J.S. The decomposition of Poincaré duality complexes, *Commentarii Math. Helvetici* 75 (2000), 232–246.
- [CS83] Culler, M. and Shalen, P.B. Varieties of group representations and splittings of 3-manifolds, *Ann. Math.* 117 (1983), 109-146.
- [Da83] Davis, M. Groups generated by reflections and aspherical manifolds not covered by Euclidean space, *Ann. Math.* 117 (1983), 293-325.
- [Da98] Davis, M. The cohomology of a Coxeter group with group ring coefficients, *Duke Math. J.* 91 (1998), 397-314.
- [Da02] Davis, M.W. Exotic aspherical manifolds, in *Topology of High-Dimensional Manifolds, Trieste (2001)*, ICTP Lecture Notes 9, Trieste (2002).
- [DD] Dicks, W. and Dunwoody, M.J. *Groups acting on Graphs*, Cambridge University Press (1989).
- [Du89] Dunwoody, M.J. Bounding the decomposition of a Poincaré duality group, *Bull. London Math. Soc.* 21 (1989), 466-468.
- [DV73] Dyer, E. and Vasquez, A.T. Some small aspherical spaces, *J. Austral. Math. Soc.* 16 (1973), 332-352.
- [DV76] Dyer, E. and Vasquez, A.T. Some properties of two-dimensional Poincaré duality groups, in *Algebra, Topology and Category Theory (a collection of papers in honour of Samuel Eilenberg)*, Academic Press, New York (1976), 45-54.
- [Ec86] Eckmann, B. Cyclic homology of groups and the Bass conjecture, *Commentarii Math. Helvetici* 61 (1986), 193-202.
- [EL81] Eckmann, B. and Linnell, P.A. Poincaré duality groups of dimension 2, II, *Commentarii Math. Helvetici* 58 (1981), 111-114.
- [EM80] Eckmann, B. and Müller, H. Poincaré duality groups of dimension 2, *Commentarii Math. Helvetici* 55 (1980), 510-520.
- [El84] Elkalla, H.S. Subnormal groups in 3-manifold groups, *J. London Math. Soc.* 30 (1984), 342-360.
- [Ep92] Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S. and Thurston, W.P. *Word Processing and Group Theory*, Jones and Bartlett, Boston - London (1992).
- [EJ73] Evans, B. and Jaco, W. Varieties of groups and 3-manifolds, *Topology* 12 (1973), 83-97.

- [GS] Gersten, S.M. and Stallings, J.R. (editors) *Combinatorial Group Theory and Topology*, Annals of Mathematics Study 111, Princeton University Press, Princeton (1987).
- [GS79] Gildenhuys, D. Classification of soluble groups of cohomological dimension two, *Math. Z.* 166 (1979), 21-25.
- [GS81] Gildenhuys, D. and Strebel, R. On the cohomological dimension of soluble groups, *Canadian Math. Bull.* 24 (1981), 385-392.
- [GW] González-Acuña, F. and Whitten, W. *Imbeddings of Three-manifold Groups*, American Mathematical Society Memoir 474 (1992).
- [GW94] González-Acuña, F. and Whitten, W. Cohopficity of 3-manifold groups, *Top. Appl* 56 (1994), 87-97.
- [Ha82] Hausmann, J.-C. Fundamental group problems related to Poincaré duality, in *Canadian Mathematical Society Conference Proceedings vol. 2, part 2* (1982), 327-336.
- [He87] Hempel, J. Residual finiteness for 3-manifolds, in [GS], 379-396.
- [HL74] Hendriks, H. and Laudendach, F. Scindement d'une équivalence d'homotopie en dimension 3, *Ann. E.N.S. Ser. (4)* 7 (1974), 203-218.
- [H] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs, vol. 5, Geometry and Topology Publications (2002).
- [Hi85] Hillman, J.A. Seifert fibre spaces and Poincaré duality groups, *Math. Z.* 190 (1985), 365-369.
- [Hi87] Hillman, J.A. Three-dimensional Poincaré duality groups which are extensions, *Math. Z.* 195 (1987), 89-92.
- [Hi03] Hillman, J.A. Tits alternatives and low dimensional topology, *J. Math. Soc. Japan* 55 (2003), 365-383.
- [Hi04] Hillman, J.A. Homomorphisms of nonzero degree between  $PD_n$ -groups, *J. Austral. Math. Soc.* 77 (2004), 335-348.
- [Hi04'] Hillman, J.A. An indecomposable  $PD_3$ -complex: II. *Alg. Geom. Top.* 4 (2004), 1103-1109.
- [Hi06] Hillman, J.A. Centralizers and normalizers of subgroups of  $PD_3$ -groups and group pairs, *J. Pure Appl. Alg.* 204 (2006), 244-257.
- [HL92] Hillman, J.A. and Linnell, P.A. Elementary amenable groups of finite Hirsch length are locally finite by virtually solvable, *J. Austral. Math. Soc.* 52 (1992), 237-241.
- [Ho82] Howie, J. On locally indicable groups, *Math. Z.* 180 (1982), 445-461.
- [JS79] Jaco, W. and Shalen, P.B. Seifert fibred spaces in 3-manifolds, *American Mathematical Society Memoir* 220 (1979).
- [KM10] Kahn, J. and Markovic, V. Immersing almost geodesic surfaces in a closed hyperbolic three manifold, arXiv:0910.5501v3 [math.GT]
- [Ka89] Kakimizu, O. On maximal fibred submanifolds of a knot exterior,

- Math. Ann. 284 (1989), 515-528.
- [KK05] Kapovich, M. and Kleiner, B. Coarse Alexander duality and duality groups, *J. Diff. Geom.* 69 (2005), 279-352.
- [KK07] Kapovich, M. and Kleiner, B. The weak hyperbolization conjecture for 3-dimensional  $CAT(0)$ -groups, *Groups Geom Dyn.* 1 (2007), 61-79.
- [KZ08] Kochloukova, D.H. and Zaleskii, P. Profinite and pro- $p$  completions of Poincaré duality groups of dimension 3, *Trans. Amer. Math. Soc.* 360 (2008), 1927-1949.
- [Ko87] Kojima, S. Finite covers of 3-manifolds containing essential surfaces of Euler characteristic 0, *Proc. Amer. Math. Soc.* 101 (1987), 743-747.
- [Kr90] Kropholler, P.H. An analogue of the torus decomposition theorem for certain Poincaré duality groups, *Proc. London Math. Soc.* 60 (1990), 503-529.
- [Kr90a] Kropholler, P.H. A note on centrality in 3-manifold groups, *Math. Proc. Cambridge Philos. Soc.* 107 (1990), 261-266.
- [Kr90b] Kropholler, P.H. Baumslag-Solitar groups and some other groups of cohomological dimension two, *Comment. Math. Helv.* 65 (1990), 547-558.
- [Kr93] Kropholler, P.H. A group theoretic proof of the torus theorem, in [NR], 138-158.
- [KR88] Kropholler, P.H. and Roller, M.A. Splittings of Poincaré duality groups, *Math. Z.* 197 (1988), 421-438.
- [KR88a] Kropholler, P.H. and Roller, M.A. Splittings of Poincaré duality groups II, *J. London Math. Soc.* 38 (1988), 410-420.
- [KR89] Kropholler, P.H. and Roller, M.A. Splittings of Poincaré duality groups III, *J. London Math. Soc.* 39 (1989), 271-284.
- [KR89a] Kropholler, P.H. and Roller, M.A. Relative ends and duality groups. *J. Pure Appl. Alg.* 61 (1989), 197-210.
- [LR75] Lee, R. and Raymond, F. Manifolds covered by Euclidean space, *Topology* 14 (1975), 49-57.
- [Me90] Mess, G. Examples of Poincaré duality groups, *Proc. Amer. Math. Soc.* 110 (1990), 1144-5.
- [Me90'] Mess, G. Finite covers of 3-manifolds and a theorem of Lubotzky, preprint, I.H.E.S. (1990).
- [NR] Niblo, G.A. and Roller, M.A. (editors) *Geometric Group Theory 1*, London Mathematical Society Lecture Note Series 181, Cambridge University Press, Cambridge (1993).
- [Pa92] Parry, W. A sharper Tits alternative for 3-manifold groups, *Israel J. Math.* 77 (1992), 265-271.
- [Sc73] Scott, G.P. Compact submanifolds of 3-manifolds, *J. London Math. Soc.* 1 (1973), 437-440.
- [Sc76] Scott, G.P. Normal subgroups in 3-manifold groups, *J. London Math. Soc.* 13 (1976), 5-12.
- [So91] Soma, T. Virtual fibre groups in 3-manifold groups,

- J. London Math. Soc. 43 (1991), 337-354.
- [St77] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups, *Commentarii Math. Helvetici* 52 (1977), 317-324.
- [S] Stallings, J.R. *Group Theory and 3-Dimensional Manifolds*, Yale Mathematical Monograph 4, Yale University Press, New Haven - London (1971).
- [Th84] Thomas, C.B. Splitting theorems for certain  $PD^3$ -groups, *Math. Z.* 186 (1984), 201-209.
- [Th96] Thomas, C.B. 3-Manifolds and  $PD(3)$ -groups, in *Novikov Conjectures, Index Theorems and Rigidity*, vol. 2, London Math. Soc. Lecture Series 227 (1996).
- [Th82] Thurston, W.P. Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* 6 (1982), 357-381.
- [Tu89] Turaev, V.G. Three-dimensional Poincaré complexes: classification and splitting, *Math. Sbornik* 180 (1989), 809-830.
- [Wa78] Waldhausen, F. Algebraic  $K$ -theory of generalized free products, *Ann. Math.* 108 (1978), 135-256.
- [W] Wall, C.T.C. (editor) *Homological Group Theory*, London Mathematical Society Lecture Notes Series 36, Cambridge University Press (1979).
- [Wl67] Wall, C.T.C. Poincaré complexes: I, *Ann. Math.* 86 (1967), 213-245.
- [Wn91] Wang, S. The existence of maps of nonzero degree between aspherical 3-manifolds, *Math. Z.* 208 (1991), 147-160.
- [Wn93] Wang, S. The  $\pi_1$ -injectivity of self-maps of nonzero degree on 3-manifolds, *Math. Ann.* 297 (1993), 171-189.
- [WW93] Wang, S. and Wu, Y. Any knot complement covers at most one knot complement, *Pac. J. Math.* 158 (1993), 387-395.
- [WW94] Wang, S. and Wu, Y.Q. Covering invariants and co-hopficity of 3-manifold groups, *Proc. London Math. Soc.* 68 (1994), 203-224.
- [WY99] Wang, S. and Yu, F. Covering degrees are determined by graph manifolds involved, *Comment. Math. Helv.* 74 (1999), 238-247.
- [Z] Zieschang, H. *Finite Groups of Mapping Classes of Surfaces*, Lecture Notes in Mathematics 875, Springer-Verlag, Berlin - Heidelberg - New York (1981).
- [Zi82] Zimmermann, B. Das Nielsensche Realisierungsproblem für hinreichend grosse 3-Mannigfaltigkeiten, *Math. Z.* 180 (1982), 349-359.