1. Let \( n \) be the integer 228618946967762521. Explain how 3-torsion elements in \( \mathbb{Z}/n\mathbb{Z}^* \) can be used to factor \( n \), and demonstrate this with \( x = 90208952368431523 \).

2. a. Find the discrete logarithm \( x \) of 2 with respect to the base 3 in \( \mathbb{F}_p^* \), where \( p = 1234621183 \). Use the Pollig-Hellman reduction, noting that \( p - 1 = 2 \cdot 3 \cdot 83 \cdot 383 \cdot 6473 \), and give the values you determine for \( x \mod 2 \), \( x \mod 3 \), etc.

   b. Now determine the discrete logarithm \( \log_3(2) \) in \( \mathbb{F}_p^* \), where \( p = 65537 \), expressing the result in base 2.

3. Verify that the ring \( \mathbb{Z}[\tau]/(13) \), where \( \tau^3 - \tau + 1 = 0 \) is a field, that 61 divides the order of \( \mathbb{Z}[\tau]/(13)^* \), and that \( x = \tau + 6 \) and \( y = \tau + 10 \) have exact order 61.

   a. Partition \( \mathbb{F}_{13^3} \) into disjoint sets

   \[
   \begin{align*}
   S_1 &= \{a + b\tau + c\tau^2 \in \mathbb{F}_{13^3} : 0 \leq a \leq 4\}, \\
   S_2 &= \{a + b\tau + c\tau^2 \in \mathbb{F}_{13^3} : 5 \leq a \leq 8\}, \\
   S_3 &= \{a + b\tau + c\tau^2 \in \mathbb{F}_{13^3} : 9 \leq a\},
   \end{align*}
   \]

   and use these to determine four cycles and tails in the Pollard \( \rho \) method beginning with an initial value of the form \( x^n y^m \). Give both the elements \( x^n y^m \) and the exponents \((n_i, m_i)\) in the sequence. Use your cycles to determine the discrete logarithm \( \log_x(y) \).

   b. Find the complete set of relations between the elements

   \[-1, \tau, 2, 3, \tau^2 + 1, \tau^2 + \tau + 1, -2\tau - 1, x, y\]

   of \( \mathbb{F}_{13^3} \), and demonstrate how to use these to determine \( \log_x(y) \).