Commutative Algebra Semester 1, 2006

This assignment is due on Friday 5 May and will count for 10% of the final mark.

Throughout, let $A$ be a commutative ring with one and let $M', M, M'', N', N$ and $N''$ be $A$-modules.

1. Suppose that $f : N' \rightarrow N$ and $g : N \rightarrow N''$ are both $A$-module homomorphisms.
   (a) Prove that $0 \rightarrow N' \rightarrow N \rightarrow N''$ is exact if and only if for all $A$-modules $M$ the induced sequence 
   $$0 \rightarrow \text{Hom}_A(M, N') \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N'')$$
   is exact.
   (b) Find an exact sequence $N \rightarrow N'' \rightarrow 0$ of $A$-modules and an $A$-module $M$ such that 
   $$\text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N'') \rightarrow 0$$
   is not exact.

2. Suppose that 
   $$
   \begin{array}{cccccc}
   0 & \rightarrow & M' & \alpha \rightarrow & M & \beta \rightarrow & M'' & \rightarrow & 0 \\
   f' \downarrow & & \downarrow f & & \downarrow f'' & & \\
   0 & \rightarrow & N' & \rightarrow & N & \rightarrow & N'' & \rightarrow & 0
   \end{array}
   $$
   is a commutative diagram of $A$-modules. By the snake lemma, there exists an induced long exact sequence of the form 
   $$0 \rightarrow \ker f' \rightarrow \ker f \rightarrow \ker f'' \rightarrow \text{coker} f' \rightarrow \text{coker} f \rightarrow \text{coker} f'' \rightarrow 0.$$
   (a) Prove exactness at $\text{coker} f$ and at $\text{coker} f''$.
   (b) Find an example where each of the modules $\ker f'$, $\ker f$, $\ker f''$, $\text{coker} f'$, $\text{coker} f$ and $\text{coker} f''$ are nonzero and each of the induced maps is also nonzero.

3. Suppose that $p_1, p_2, \ldots, p_n$ are prime ideals of a ring $A$.
   (a) Show that $S = A \setminus (p_1 \cup p_2 \cup \cdots \cup p_n)$ is a multiplicatively closed set.
   (b) Describe the set of ideals of $S^{-1}A$ in terms of the set of ideals of $A$.

4. If $M$ is an $A$-module let $\mathcal{P}(M)$ be the set of prime ideals $p$ of $A$ such that $M_p \neq 0$.
   (a) Show that $M \neq 0$ if and only if $\mathcal{P}(M)$ is nonempty.
   (b) Suppose that $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence. Prove that 
   $$\mathcal{P}(M) = \mathcal{P}(M') \cup \mathcal{P}(M'').$$