1. Let $M$ be an $A$-module with composition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = (0).$$

(a) Let $N_1 \supset N_2$ be submodules of $M$. Prove that $N_1/N_2$ is simple if and only if

$$((N_1 \cap M_{j-1}) + M_j)/(N_2 \cap M_{j-1}) + M_j) \cong M_{j-1}/M_j$$

for exactly one value of $j$.

(b) Show that $M$ is simple only if $M_p$ is simple for every prime $p$. Find a counterexample to the converse statement.

2. Let $G$ be a finite group of automorphisms of a ring $A$ and let $A^G$ be the subring of elements fixed by $G$ (i.e. $\sigma(x) = x$ for all $\sigma \in G$).

(a) Prove that $A$ is integral over $A^G$.

(b) Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(A^G)$, and let $\varphi : X \rightarrow Y$ be the pullback map with respect to inclusion. Show that $|\varphi^{-1}(p)|$ is finite and divides $|G|$.

(c) Let $B$ be a finite $A$-algebra, and let $\varphi : X = \text{Spec}(B) \rightarrow Y = \text{Spec}(A)$ be the pullback map. Show that $\varphi$ has finite fibers $\varphi^{-1}(p)$.

3. A topological space $X$ is said to be Noetherian if the open subsets of $X$ satisfy the ACC.

(a) Show that the ACC on open subsets is equivalent to the DCC on closed subsets.

(b) Show that the following are equivalent:

- $X$ is Noetherian.
- Every open subspace of $X$ is compact.
- Every subspace of $X$ is compact.

(c) Show that $X = \text{Spec}(A)$ is a Noetherian topological space if $A$ is a Noetherian ring.