

*Standard Young tableaux of shape  $(n, n)$*

*Problem 1:* Given two rows of boxes with  $n$  boxes in each row:

		...	
		...	

In how many ways can you place the numbers  $1, 2, \dots, 2n$  in the boxes so that the numbers increase from left to right and so that each number in the bottom row is larger than the number in the box above it?

*Problem 2:* Given  $2n$  people of different heights, in how many ways can these  $2n$  people be lined up in increasing heights, in two rows of length  $n$  each, so that everyone in the second row is taller than the corresponding person in the first row?

For  $n = 1$ , there is only one such arrangement and for  $n = 2$ , there are 2 such arrangements:

1	1 2	1 3
2	3 4	2 4

For  $n = 3$ , there are 5 such arrangements:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5 6	3 5 6	3 4 6	2 5 6	2 4 6

For  $n = 4$ , there are 14 such arrangements:

1 2 3 4	1 2 3 5	1 2 3 6	1 2 3 7	1 2 4 5
5 6 7 8	4 6 7 8	4 5 7 8	4 5 6 8	3 6 7 8
1 2 4 6	1 2 4 7	1 2 5 6	1 2 5 7	1 3 4 5
3 5 7 8	3 5 6 8	3 4 7 8	3 4 6 8	2 6 7 8
1 3 4 6	1 3 4 7	1 3 5 6	1 3 5 7	
2 5 7 8	2 5 6 8	2 4 7 8	2 4 6 8	

For  $n = 5$ , there are 42 such arrangements:

1 2 3 4 5	1 2 3 4 6	1 2 3 4 7	1 2 3 4 8	1 2 3 4 9
6 7 8 9 10	5 7 8 9 10	5 6 8 9 10	5 6 7 9 10	5 6 7 8 10
1 2 3 5 6	1 2 3 5 7	1 2 3 5 8	1 2 3 5 9	1 2 3 6 7
4 7 8 9 10	4 6 8 9 10	4 6 7 9 10	4 6 7 8 10	4 5 8 9 10
1 2 3 6 8	1 2 3 6 9	1 2 3 7 8	1 2 3 7 9	1 2 4 5 6
4 5 7 9 10	4 5 7 8 10	4 5 6 9 10	4 5 6 8 10	3 7 8 9 10

1 2 4 5 7 3 6 8 9 10	1 2 4 5 8 3 6 7 9 10	1 2 4 5 9 3 6 7 8 10	1 2 4 6 7 3 5 8 9 10	1 2 4 6 8 3 5 7 9 10
1 2 4 6 9 3 5 7 8 10	1 2 4 7 8 3 5 6 9 10	1 2 4 7 9 3 5 8 8 10	1 2 5 6 7 3 4 8 9 10	1 2 5 6 8 3 4 7 9 10
1 2 5 6 9 3 4 7 8 10	1 2 5 7 8 3 4 6 9 10	1 2 5 7 9 3 4 6 8 10	1 3 4 5 6 2 7 8 9 10	1 3 4 5 7 2 6 8 9 10
1 3 4 5 8 2 6 7 9 10	1 3 4 5 9 2 6 7 8 10	1 3 4 6 7 2 5 8 9 10	1 3 4 6 8 2 5 7 9 10	1 3 4 6 9 2 5 7 8 10
1 3 4 7 8 2 5 6 9 10	1 3 4 7 9 2 5 6 8 10	1 3 5 6 7 2 4 8 9 10	1 3 5 6 8 2 4 7 9 10	1 3 5 6 9 2 4 7 8 10
1 3 5 7 8 2 4 6 9 10	1 3 5 7 9 2 4 6 8 10			

For  $n = 6$ , there are 132 such tableau, the top row of which are given by: ( $a = 10$ ,  $b = 11$ ,  $c = 12$ )

123456, 123457, 123458, 123459, 12345a, 12345b,  
 123467, 123468, 123469, 12346a, 12346b,  
 123478, 123479, 12347a, 12347b,  
 123489, 12348a, 12348b, 12349a, 12349b,  
 123567, 123568, 123569, 12356a, 12356b,  
 123578, 123579, 12357a, 12357b,  
 123589, 12358a, 12358b, 12359a, 12359b,  
 123678, 123679, 12367a, 12367b,  
 123689, 12368a, 12368b, 12369a, 12369b,  
 123789, 12378a, 12378b, 12379a, 12379b,  
 124567, 124568, 124569, 12456a, 12456b,  
 124578, 124579, 12457a, 12457b,  
 124589, 12458a, 12458b, 12459a, 12459b,  
 124678, 124679, 12467a, 12467b,  
 124689, 12468a, 12468b, 12469a, 12469b,  
 124789, 12478a, 12478b, 12479a, 12479b,  
 125678, 125679, 12567a, 12567b,  
 125689, 12568a, 12568b, 12569a, 12569b,  
 125789, 12578a, 12578b, 12579a, 12579b,  
 134567, 134568, 134569, 13456a, 13456b,  
 134578, 134579, 13457a, 13457b,

134589, 13458a, 13458b, 13459a, 13459b,  
134678, 134679, 13467a, 13467b,  
134689, 13468a, 13468b, 13469a, 13469b,  
134789, 13478a, 13478b, 13479a, 13479b,  
135678, 135679, 13567a, 13567b,  
135689, 13568a, 13568b, 13569a, 13569b,  
135789, 13578a, 13578b, 13579a, 13579b,

In fact, for any  $n$ , the number of such possible  $(n, n)$  tableau is the Catalan number  $c_n$ .

*Connection with the first bracketing problem*

Given an  $(n, n)$  tableau such that the numbers increase from left to right and such that each number in the bottom row is larger than the number in the box above it, we obtain a corresponding balanced string of brackets as follows: To each integer  $i$  in the top row, we associate a left bracket to the  $i$ -th position; and to each integer  $j$  in the bottom row, we associate a right bracket to the  $j$ -th position.

Given a balanced string of brackets, we first label the numbers under the brackets, in ascending order, and write down the integers under the left brackets into the top row and the integers under the right brackets into the bottom row of the  $2n$  boxes. Then we obtain the corresponding  $(n, n)$  tableau so that the numbers increase from left to right and so that each number in the bottom row is larger than the number in the box above it.

1. Construct balanced strings of brackets corresponding to the following  $(n, n)$  tableau.

1	2	3	5	8
4	6	7	9	10

(i)

1	2	5	6	8	11
3	4	7	9	10	12

(ii)

1	2	3	7	9	10
4	5	6	8	11	12

(iii)

*Solution.*

The corresponding balanced strings of brackets are:

(i)  $((())())$ (ii)  $(())(())()$ (iii)  $((()))()()$ .

2. For each of the following balanced strings of brackets, construct the corresponding  $(n, n)$  tableau.

(i)  $(())()()$ (ii)  $(())(())()$ (iii)  $(())()()()$ 

*Solution.*

The corresponding  $(n, n)$  tableau are:

1	2	5	7	8	10
3	4	6	9	11	12

(i)

1	2	4	7	8	11	13
3	5	6	9	10	12	14

(ii)

1	2	4	6	9	11	12	15	16	18
3	5	7	8	10	13	14	17	19	20

(iii)