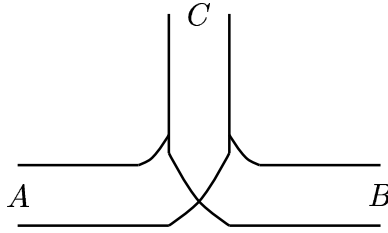


Railway wagon problem or 231-avoiding permutations

Problem 1: There are n railway wagons at A on the track above. The wagons are moved from A to B through the central track. It is assumed that the central track can accommodate all n of them and that they travel only from left to right (i.e., they may not be moved from C back to A .) How many arrangements are possible at B ?

Problem 2: How many one-to-one functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ are there such that there does not exist $i < j < k$ with $a_k < a_i < a_j$?

Problem 3: How many permutations $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$ are there such that there does not exist $i < j < k$ with $a_k < a_i < a_j$? [Such permutation is called *231-avoiding permutation*.]

Problem 4: How many n -tuples (a_1, a_2, \dots, a_n) of distinct integers, where $1 \leq a_i \leq n$, are there such that there does not exist $i < j < k$ with $a_k < a_i < a_j$?

Let $A = \{1, 2, 3, \dots, n\}$. For $n = 1$, $A = \{1\}$, there is only 1 way, by moving the 1 from A up to the central track C and then down to B ; that is, UD.

For $n = 2$, $A = \{1, 2\}$, there are 2 ways given by UUDD and UDUD; the first one gives 21 at B and the second gives 12 at B .

For $n = 3$, $A = \{1, 2, 3\}$, there are 5 ways given by

UUUDDD, UUDUDD, UUUDDUD, UDUUDD, UDUDUD.

These give rise to the following permutations in B :

321, 312, 132, 213, 123.

For $n = 4$, $A = \{1, 2, 3, 4\}$, there are 14 ways given by

UUUUDDDD, UUDUDDDD, UUUDDUDD, UUUDDDUD, UUDUDDDD,
 UUDUDUDD, UUDUDDUD, UUUUUDD, UUUDDUDUD, UDUUUDDDD,
 UDUUDUDD, UDUUDDUD, UDUDUDD, UDUDUDUD.

These give rise to the following permutations in B .

4321, 4312, 4132, 1432, 4213, 4123, 1423,
 2143, 1243, 3214, 3124, 1324, 2134, 1234.

For $n = 5$, there are 42 ways given by

UUUUUDDDDDD, UUUUDUDDDD, UUUUDDUDDD, UUUUDDDUDD
 UUUUDDDDUD, UUUDUDDDD, UUUDUDUDDD, UUUDUDDUDD
 UUUDUDDDDUD, UUUDDUUDDD, UUUDUDDUDD, UUUDUDDUD
 UUUDDDUDD, UUUDDDUDUD, UUDUUUDDDD, UUDUUUDDDD
 UUDUUUDDUDD, UUDUUUDDUD, UUDUDUDDDD, UUDUDUDDUDD
 UUDUDUDDUD, UUDUDDUDD, UUDUDDUDUD, UUDUDDUDD
 UUDUDDUDD, UUDUDDUDDUD, UUDUDDUDD, UUDUDDUDDUD
 UDUUUUDDDD, UDUUUUDDDD, UDUUUUDDUDD, UDUUUUDDUD
 UDUUUUDDDD, UDUUUUDDUDD, UDUUUUDDUD, UDUUUUDDUDD
 UDUUUUDDUD, UDUUUUDDDD, UDUUUUDDUDD, UDUUUUDDUD
 UDUUUUDDUDD, UDUUUUDDUD.

These give rise to the following permutations in B .

54321, 54312, 54132, 51432, 15432, 54213, 54123, 51423, 15423,
 52143, 51243, 15243, 21543, 12543, 53214, 53124, 51324, 15324,
 52134, 51234, 15234, 21534, 12534, 32154, 31254, 13254, 21354,
 12354, 43215, 43125, 41325, 14325, 42135, 41235, 14235, 21435,
 12435, 32145, 31245, 13245, 21345, 12345.

For $n = 6$, there are 132 ways given by:

UUUUUUDDDDDD, UUUUUUDDDDDD, UUUUUUDDUDDDD,
 UUUUUUDDDUDD, UUUUUUDDDDUDD, UUUUUUDDDDUD,
 UUUUDUDDDDDD, UUUUDUDDUDDDD, UUUUDUDDUDDDD,
 UUUUDUDDDDUDD, UUUUDUDDDDUD, UUUUDDUDDDD,
 UUUUDDUDDUDD, UUUUDDUDDUDD, UUUUDDUDDUD,
 UUUUDDDUUDD, UUUUDDDDUDUDD, UUUUDDDDUDDUD,
 UUUDUUDUDDDD, UUUDUUDUDDDD, UUUDUUDUDDUDD,
 UUUDUUDUDDUD, UUUDUDUDDDD, UUUDUDUDDUDD,
 UUUDUDUDDUDD, UUUDUDUDDUD, UUUDUDDUDD,
 UUUDUDDUDDUD, UUUDUDDUDDUD, UUUDUDDUDD,
 UUUDUDDUDDUD, UUUDUDDUDDUD, UUUDUDDUDD,
 UUUDUDDUDDUD, UUUDUDDUDDUD, UUUDUDDUDD,
 UUUDUDDUDDUD, UUUDUDDUDDUD, UUUDUDDUDD,
 UUUDUDDUDDUD, UUUDUDDUDDUD, UUUDUDDUDD.

Connection with the first bracketing problem

Given a balanced string of brackets, we obtain the corresponding arrangement of wagons as follows. We replace the left bracket (by U and the right bracket) by D.

Given an arrangement of wagons, we obtain the corresponding balanced string of brackets as follows. First convert the arrangement using U and D, and then replace U by the left bracket (and D by the right bracket).

1. For each of the following arrangements of wagons, write down the corresponding balanced strings of brackets:

- (i) 162534
- (ii) 142365
- (iii) 213654.

Solution.

The arrangements correspond to

UUUDUDDUDDUD, UUDDUUDUDDUD, UUUDDDUDUDD

and so the corresponding balanced strings of brackets are:

- (i) ((()())())()
- (ii) (()())()()()
- (iii) (((()))()()).

2. For each of the following balanced strings of brackets, write down the corresponding arrangements of wagons:

- (i) ()()()()
- (ii) ()()()()()
- (iii) ()()()()()()()()

Solution.

The arrangements of wagons are given by

UDDUDUUDUDD, UUDUDDUDDUDUD

and

UUDUDUDDUDUDDUUDUDD

and so the corresponding arrangements of wagons are:

- (i) 312465
- (ii) 1243756
- (iii) 312546(10)789.