Second Bracketing Problem – Well-Parenthesized Products

Problem 1: Suppose you have numbers x_0, x_1, x_2, ... , x_n and you want to multiply them together. In how many ways can you insert brackets into the string $x_0x_1x_2\ldots x_n$ so that the order of multiplication is completely specified? Each pair of brackets should contain just two terms.

Problem 2: Suppose you have numbers x_0, x_1, x_2, ... , x_n and you want to add them together. In how many ways can you insert brackets into the string $x_0x_1x_2\ldots x_n$ so that the order of addition is completely specified? Each pair of brackets should contain just two terms.

For $n = 1$, there is clearly only 1 way: (x_0x_1), and for $n = 2$, there are two ways:

$$(x_0(x_1x_2)) \quad \text{and} \quad ((x_0x_1)x_2).$$

For $n = 3$, there are five ways:

$$(x_0(x_1(x_2x_3))); \; (x_0((x_1x_2)x_3)); \; ((x_0(x_1x_2))x_3); \; ((x_0x_1)(x_2x_3)), \; (((x_0x_1)x_2)x_3).$$

For $n = 4$, there are 14 ways:

$$(x_0(x_1(x_2(x_3x_4)))), \; (x_0(x_1((x_2x_3)x_4))), \; (x_0((x_1(x_2x_3))x_4)), \; ((x_0(x_1(x_2x_3)))x_4),$$

$$(x_0((x_1x_2)(x_3x_4))), \; (x_0(((x_1x_2)x_3)x_4)), \; ((x_0((x_1x_2)x_3))x_4), \; ((x_0(x_1x_2))(x_3x_4)),$$

$$(((x_0(x_1x_2))x_3)x_4), \; ((x_0x_1)(x_2(x_3x_4))), \; ((x_0x_1)((x_2x_3)x_4)), \; (((x_0x_1)(x_2x_3))x_4),$$

$$(((x_0x_1)x_2)(x_3x_4)), \; ((((x_0x_1)x_2)x_3)x_4).$$

For $n = 5$, there are 42 ways:

(1) $(x_0(x_1(x_2(x_3(x_4x_5)))))$	(2) $(x_0(x_1(x_2((x_3x_4)x_5))))$
(3) $(x_0(x_1((x_2(x_3x_4))x_5)))$	(4) $(x_0((x_1(x_2(x_3x_4)))x_5))$
(5) $((x_0(x_1(x_2(x_3x_4))))x_5)$	(6) $(x_0(x_1((x_2x_3)(x_4x_5))))$
(7) $(x_0(x_1(((x_2x_3)x_4)x_5)))$	(8) $(x_0((x_1((x_2x_3)x_4))x_5))$
(9) $((x_0(x_1((x_2x_3)x_4)))x_5)$	(10) $(x_0((x_1(x_2x_3))(x_4x_5)))$
(11) $(x_0(((x_1(x_2x_3))x_4)x_5))$	(12) $((x_0((x_1(x_2x_3))x_4))x_5)$
(13) $((x_0(x_1(x_2x_3)))(x_4x_5))$	(14) $(((x_0(x_1(x_2x_3)))x_4)x_5)$
(15) $(x_0((x_1x_2)(x_3(x_4x_5))))$	(16) $(x_0((x_1x_2)((x_3x_4)x_5)))$
(17) $(x_0(((x_1x_2)(x_3x_4))x_5))$	(18) $((x_0((x_1x_2)(x_3x_4)))x_5)$
(19) $(x_0(((x_1x_2)x_3)(x_4x_5)))$	(20) $(x_0((((x_1x_2)x_3)x_4)x_5))$
(21) $((x_0(((x_1x_2)x_3)x_4))x_5)$	(22) $((x_0((x_1x_2)x_3))(x_4x_5))$
(23) $(((x_0((x_1x_2)x_3))x_4)x_5)$	(24) $((x_0(x_1x_2))(x_3(x_4x_5)))$
(25) $((x_0(x_1x_2))((x_3x_4)x_5))$	(26) $(((x_0(x_1x_2))(x_3x_4))x_5)$
(27) $(((x_0(x_1x_2))x_3)(x_4x_5))$	(28) $((((x_0(x_1x_2))x_3)x_4)x_5)$

(29) $((x_0x_1)(x_2(x_3(x_4x_5))))$ (30) $((x_0x_1)(x_2((x_3x_4)x_5)))$
(31) $((x_0x_1)((x_2(x_3x_4))x_5))$ (32) $(((x_0x_1)(x_2(x_3x_4)))x_5)$
(33) $((x_0x_1)((x_2x_3)(x_4x_5)))$ (34) $((x_0x_1)(((x_2x_3)x_4)x_5))$
(35) $(((x_0x_1)((x_2x_3)x_4))x_5)$ (36) $(((x_0x_1)(x_2x_3))(x_4x_5))$
(37) $((((x_0x_1)(x_2x_3))x_4)x_5)$ (38) $(((x_0x_1)x_2)(x_3(x_4x_5)))$
(39) $(((x_0x_1)x_2)((x_3x_4)x_5))$ (40) $((((x_0x_1)x_2)(x_3x_4))x_5)$
(41) $((((x_0x_1)x_2)x_3)(x_4x_5))$ (42) $(((((x_0x_1)x_2)x_3)x_4)x_5)$

For $n = 6$, there are 132 ways:

(1) $(x_0(x_1(x_2(x_3(x_4(x_5x_6))))))$ (2) $(x_0(x_1(x_2(x_3((x_4x_5)x_6)))))$
(3) $(x_0(x_1(x_2((x_3(x_4x_5))x_6))))$ (4) $(x_0(x_1((x_2(x_3(x_4x_5)))x_6)))$
(5) $(x_0((x_1(x_2(x_3(x_4x_5))))x_6))$ (6) $((x_0(x_1(x_2(x_3(x_4x_5)))))x_6)$
(7) $(x_0(x_1(x_2((x_3x_4)(x_5x_6)))))$ (8) $(x_0(x_1(x_2(((x_3x_4)x_5)x_6))))$
(9) $(x_0(x_1((x_2((x_3x_4)x_5))x_6)))$ (10) $(x_0((x_1(x_2((x_3x_4)x_5)))x_6))$
(11) $((x_0(x_1(x_2((x_3x_4)x_5))))x_6)$ (12) $(x_0(x_1((x_2(x_3x_4))(x_5x_6))))$
(13) $(x_0(x_1(((x_2(x_3x_4))x_5)x_6)))$ (14) $(x_0((x_1((x_2(x_3x_4))x_5))x_6))$
(15) $((x_0(x_1((x_2(x_3x_4))x_5)))x_6)$ (16) $(x_0((x_1(x_2(x_3x_4)))(x_5x_6)))$
(17) $(x_0(((x_1(x_2(x_3x_4)))x_5)x_6))$ (18) $((x_0((x_1(x_2(x_3x_4)))x_5))x_6)$
(19) $((x_0(x_1(x_2(x_3x_4))))(x_5x_6))$ (20) $(((x_0(x_1(x_2(x_3x_4))))x_5)x_6)$
(21) $(x_0(x_1((x_2x_3)(x_4(x_5x_6)))))$ (22) $(x_0(x_1((x_2x_3)((x_4x_5)x_6))))$
(23) $(x_0(x_1(((x_2x_3)(x_4x_5))x_6)))$ (24) $(x_0((x_1((x_2x_3)(x_4x_5)))x_6))$
(25) $((x_0(x_1((x_2x_3)(x_4x_5))))x_6)$ (26) $(x_0(x_1(((x_2x_3)x_4)(x_5x_6))))$
(27) $(x_0(x_1((((x_2x_3)x_4)x_5)x_6)))$ (28) $(x_0((x_1(((x_2x_3)x_4)x_5))x_6))$
(29) $((x_0(x_1(((x_2x_3)x_4)x_5)))x_6)$ (30) $(x_0((x_1((x_2x_3)x_4))(x_5x_6)))$
(31) $(x_0(((x_1((x_2x_3)x_4))x_5)x_6))$ (32) $((x_0((x_1((x_2x_3)x_4))x_5))x_6)$
(33) $((x_0(x_1((x_2x_3)x_4)))(x_5x_6))$ (34) $(((x_0(x_1((x_2x_3)x_4)))x_5)x_6)$
(35) $(x_0((x_1(x_2x_3))(x_4(x_5x_6))))$ (36) $(x_0((x_1(x_2x_3))((x_4x_5)x_6)))$
(37) $(x_0(((x_1(x_2x_3))(x_4x_5))x_6))$ (38) $((x_0((x_1(x_2x_3))(x_4x_5)))x_6)$
(39) $(x_0(((x_1(x_2x_3))x_4)(x_5x_6)))$ (40) $(x_0((((x_1(x_2x_3))x_4)x_5)x_6))$
(41) $((x_0(((x_1(x_2x_3))x_4)x_5))x_6)$ (42) $((x_0((x_1(x_2x_3))x_4))(x_5x_6))$
(43) $(((x_0((x_1(x_2x_3))x_4))x_5)x_6)$ (44) $((x_0(x_1(x_2x_3)))(x_4(x_5x_6)))$
(45) $((x_0(x_1(x_2x_3)))((x_4x_5)x_6))$ (46) $(((x_0(x_1(x_2x_3)))(x_4x_5))x_6)$
(47) $(((x_0(x_1(x_2x_3)))x_4)(x_5x_6))$ (48) $((((x_0(x_1(x_2x_3)))x_4)x_5)x_6)$
(49) $(x_0((x_1x_2)(x_3(x_4(x_5x_6)))))$ (50) $(x_0((x_1x_2)(x_3((x_4x_5)x_6))))$
(51) $(x_0((x_1x_2)((x_3(x_4x_5))x_6)))$ (52) $(x_0(((x_1x_2)(x_3(x_4x_5)))x_6))$
(53) $((x_0((x_1x_2)(x_3(x_4x_5))))x_6)$ (54) $(x_0((x_1x_2)((x_3x_4)(x_5x_6))))$
(55) $(x_0((x_1x_2)(((x_3x_4)x_5)x_6)))$ (56) $(x_0(((x_1x_2)((x_3x_4)x_5))x_6))$
(57) $((x_0((x_1x_2)((x_3x_4)x_5)))x_6)$ (58) $(x_0(((x_1x_2)(x_3x_4))(x_5x_6)))$
(59) $(x_0((((x_1x_2)(x_3x_4))x_5)x_6))$ (60) $((x_0(((x_1x_2)(x_3x_4))x_5))x_6)$

(61)	$((x_0((x_1x_2)(x_3x_4)))(x_5x_6))$	(62)	$(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)$
(63)	$(x_0(((x_1x_2)x_3)(x_4(x_5x_6))))$	(64)	$(x_0(((x_1x_2)x_3)((x_4x_5)x_6)))$
(65)	$(x_0((((x_1x_2)x_3)(x_4x_5))x_6))$	(66)	$((x_0(((x_1x_2)x_3)(x_4x_5)))x_6)$
(67)	$(x_0((((x_1x_2)x_3)x_4)(x_5x_6)))$	(68)	$(x_0(((((x_1x_2)x_3)x_4)x_5)x_6))$
(69)	$((x_0((((x_1x_2)x_3)x_4)x_5))x_6)$	(70)	$((x_0(((x_1x_2)x_3)x_4))(x_5x_6))$
(71)	$(((x_0(((x_1x_2)x_3)x_4))x_5)x_6)$	(72)	$((x_0((x_1x_2)x_3))(x_4(x_5x_6)))$
(73)	$((x_0((x_1x_2)x_3))((x_4x_5)x_6))$	(74)	$(((x_0((x_1x_2)x_3))(x_4x_5))x_6)$
(75)	$(((x_0((x_1x_2)x_3))x_4)(x_5x_6))$	(76)	$((((x_0((x_1x_2)x_3))x_4)x_5)x_6)$
(77)	$((x_0(x_1x_2))(x_3(x_4(x_5x_6))))$	(78)	$((x_0(x_1x_2))(x_3((x_4x_5)x_6)))$
(79)	$((x_0(x_1x_2))((x_3(x_4x_5))x_6))$	(80)	$(((x_0(x_1x_2))(x_3(x_4x_5)))x_6)$
(81)	$((x_0(x_1x_2))((x_3x_4)(x_5x_6)))$	(82)	$((x_0(x_1x_2))(((x_3x_4)x_5)x_6))$
(83)	$(((x_0(x_1x_2))((x_3x_4)x_5))x_6)$	(84)	$(((x_0(x_1x_2))(x_3x_4))(x_5x_6))$
(85)	$((((x_0(x_1x_2))(x_3x_4))x_5)x_6)$	(86)	$(((x_0(x_1x_2))x_3)(x_4(x_5x_6)))$
(87)	$(((x_0(x_1x_2))x_3)((x_4x_5)x_6))$	(88)	$((((x_0(x_1x_2))x_3)(x_4x_5))x_6)$
(89)	$((((x_0(x_1x_2))x_3)x_4)(x_5x_6))$	(90)	$(((((x_0(x_1x_2))x_3)x_4)x_5)x_6)$
(91)	$((x_0x_1)(x_2(x_3(x_4(x_5x_6)))))$	(92)	$((x_0x_1)(x_2(x_3((x_4x_5)x_6))))$
(93)	$((x_0x_1)(x_2((x_3(x_4x_5))x_6)))$	(94)	$((x_0x_1)((x_2(x_3(x_4x_5)))x_6))$
(95)	$(((x_0x_1)(x_2(x_3(x_4x_5))))x_6)$	(96)	$((x_0x_1)(x_2((x_3x_4)(x_5x_6))))$
(97)	$((x_0x_1)(x_2(((x_3x_4)x_5)x_6)))$	(98)	$((x_0x_1)((x_2((x_3x_4)x_5))x_6))$
(99)	$(((x_0x_1)(x_2((x_3x_4)x_5)))x_6)$	(100)	$((x_0x_1)((x_2(x_3x_4))(x_5x_6)))$
(101)	$((x_0x_1)(((x_2(x_3x_4))x_5)x_6))$	(102)	$(((x_0x_1)((x_2(x_3x_4))x_5))x_6)$
(103)	$(((x_0x_1)(x_2(x_3x_4)))(x_5x_6))$	(104)	$((((x_0x_1)(x_2(x_3x_4)))x_5)x_6)$
(105)	$((x_0x_1)((x_2x_3)(x_4(x_5x_6))))$	(106)	$((x_0x_1)((x_2x_3)((x_4x_5)x_6)))$
(107)	$((x_0x_1)(((x_2x_3)(x_4x_5))x_6))$	(108)	$(((x_0x_1)((x_2x_3)(x_4x_5)))x_6)$
(109)	$((x_0x_1)(((x_2x_3)x_4)(x_5x_6)))$	(110)	$((x_0x_1)((((x_2x_3)x_4)x_5)x_6))$
(111)	$(((x_0x_1)(((x_2x_3)x_4)x_5))x_6)$	(112)	$(((x_0x_1)((x_2x_3)x_4))(x_5x_6))$
(113)	$((((x_0x_1)((x_2x_3)x_4))x_5)x_6)$	(114)	$(((x_0x_1)(x_2x_3))(x_4(x_5x_6)))$
(115)	$(((x_0x_1)(x_2x_3))((x_4x_5)x_6))$	(116)	$((((x_0x_1)(x_2x_3))(x_4x_5))x_6)$
(117)	$((((x_0x_1)(x_2x_3))x_4)(x_5x_6))$	(118)	$(((((x_0x_1)(x_2x_3))x_4)x_5)x_6)$
(119)	$(((x_0x_1)x_2)(x_3(x_4(x_5x_6))))$	(120)	$(((x_0x_1)x_2)(x_3((x_4x_5)x_6)))$
(121)	$(((x_0x_1)x_2)((x_3(x_4x_5))x_6))$	(122)	$((((x_0x_1)x_2)(x_3(x_4x_5)))x_6)$
(123)	$(((x_0x_1)x_2)((x_3x_4)(x_5x_6)))$	(124)	$(((x_0x_1)x_2)(((x_3x_4)x_5)x_6))$
(125)	$((((x_0x_1)x_2)((x_3x_4)x_5))x_6)$	(126)	$((((x_0x_1)x_2)(x_3x_4))(x_5x_6))$
(127)	$(((((x_0x_1)x_2)(x_3x_4))x_5)x_6)$	(128)	$((((x_0x_1)x_2)x_3)(x_4(x_5x_6)))$
(129)	$((((x_0x_1)x_2)x_3)((x_4x_5)x_6))$	(130)	$(((((x_0x_1)x_2)x_3)(x_4x_5))x_6)$
(131)	$(((((x_0x_1)x_2)x_3)x_4)(x_5x_6))$	(132)	$((((((x_0x_1)x_2)x_3)x_4)x_5)x_6)$

In fact, the number of well-parenthesized products of $n+1$ variables is the Catalan number c_n.

Connection with the first bracketing problem

Let v be a balanced string of n left brackets and n right brackets. Note that v always starts with a left '(' and ends with a right ')'. We put an extra left '(' at the start of the string v and consider the string (v. Then we use the following instructions to obtain a well-parenthesized products of $n+1$ variables.

Reading from left to right, replace the left brackets '(' by $x_0, x_1, x_2, \ldots, x_n$. Then again reading from left to right, each time we encounter a right bracket we place a left bracket to the left of the preceding two terms.

Given a well-parenthesized products of $n+1$ variables $x_0, x_1, x_2, \ldots, x_n$, we obatin a balanced string of n left brackets and n right brackets as follows. We reversed the above construction by removing all the left brackets, removing x_0, then replacing each of the $, x_1, x_2, \ldots, x_n$. by a left bracket '(' .

1. For each of the following balanced strings of brackets, construct the corresponding well-parenthesized product.

(*i*) (())()(()())

(*ii*) (()())(())()()

(*iii*) (()()())()(())(()())

Solution.

We use the above instructions to obtain the well-parenthesized products as follows.

(*i*)

$$\begin{aligned}
(v = ((())()(()()) &\to x_0x_1x_2))x_3)x_4x_5)x_6)) \\
&\to x_0(x_1x_2))x_3)x_4x_5)x_6)) \\
&\to (x_0(x_1x_2))x_3)x_4x_5)x_6)) \\
&\to ((x_0(x_1x_2))x_3)x_4x_5)x_6)) \\
&\to ((x_0(x_1x_2))x_3)(x_4x_5)x_6)) \\
&\to ((x_0(x_1x_2))x_3)((x_4x_5)x_6)) \\
&\to (((x_0(x_1x_2))x_3)((x_4x_5)x_6))
\end{aligned}$$

(*ii*)

$$\begin{aligned}
(v = ((()())(())()() &\to x_0x_1x_2)x_3))x_4x_5))x_6)x_7) \\
&\to x_0(x_1x_2)x_3))x_4x_5))x_6)x_7) \\
&\to x_0((x_1x_2)x_3))x_4x_5))x_6)x_7) \\
&\to (x_0((x_1x_2)x_3))x_4x_5))x_6)x_7) \\
&\to (x_0((x_1x_2)x_3))(x_4x_5))x_6)x_7) \\
&\to ((x_0((x_1x_2)x_3))(x_4x_5))x_6)x_7) \\
&\to (((x_0((x_1x_2)x_3))(x_4x_5))x_6)x_7) \\
&\to ((((x_0((x_1x_2)x_3))(x_4x_5))x_6)x_7)
\end{aligned}$$

$$
\begin{aligned}
(iii)\quad (v = ((()()())()(())(()()) &\rightarrow x_0x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow x_0(x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow x_0((x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow x_0(((x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow (x_0(((x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow ((x_0(((x_1x_2)x_3)x_4))x_5)x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow ((x_0(((x_1x_2)x_3)x_4))x_5)(x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow (((x_0(((x_1x_2)x_3)x_4))x_5)(x_6x_7))x_8x_9)x_{10}))\\
&\rightarrow (((x_0(((x_1x_2)x_3)x_4))x_5)(x_6x_7))(x_8x_9)x_{10}))\\
&\rightarrow (((x_0(((x_1x_2)x_3)x_4))x_5)(x_6x_7))((x_8x_9)x_{10}))\\
&\rightarrow ((((x_0(((x_1x_2)x_3)x_4))x_5)(x_6x_7))((x_8x_9)x_{10}))
\end{aligned}
$$

2. Construct balanced strings of brackets corresponding to the following well-parenthesized products

- (i) $(x_0((x_1((x_2x_3)x_4))x_5))$
- (ii) $(((x_0(x_1x_2))((x_3x_4)x_5))x_6)$
- (iii) $(((x_0(x_1(x_2x_3)))x_4)(x_5x_6))$
- (iv) $(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)$
- (v) $(((x_0x_1)x_2)((x_3(x_4x_5))x_6))$
- (vi) $(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7)))$
- (vii) $(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7)))$

Solution.

We use the above construction to obtain the corresponding strings of brackets as follows.

(i)
$$
\begin{aligned}
(x_0((x_1((x_2x_3)x_4))x_5)) &\rightarrow x_1x_2x_3)x_4))x_5))\\
&\rightarrow ((()())())
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(((x_0(x_1x_2))((x_3x_4)x_5))x_6) &\rightarrow x_1x_2))x_3x_4)x_5))x_6)\\
&\rightarrow (())(()())()
\end{aligned}
$$

(iii)
$$
\begin{aligned}
(((x_0(x_1(x_2x_3)))x_4)(x_5x_6)) &\rightarrow x_1x_2x_3)))x_4)x_5x_6))\\
&\rightarrow ((()))()(())
\end{aligned}
$$

(iv)
$$
\begin{aligned}
(((x_0((x_1x_2)(x_3x_4)))x_5)x_6) &\rightarrow x_1x_2)x_3x_4)))x_5)x_6)\\
&\rightarrow (()(()))()()
\end{aligned}
$$

(v)
$$
\begin{aligned}
(((x_0x_1)x_2)((x_3(x_4x_5))x_6)) &\rightarrow x_1)x_2)x_3x_4x_5))x_6))\\
&\rightarrow ()()((())())
\end{aligned}
$$

(vi)
$$
\begin{aligned}
(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7))) &\rightarrow x_1x_2)x_3))x_4)x_5x_6x_7)))\\
&\rightarrow (()())()((()))
\end{aligned}
$$

(vii)
$$
\begin{aligned}
(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7))) &\rightarrow x_1)x_2x_3x_4)))x_5x_6x_7)))\\
&\rightarrow ()((()))((()))
\end{aligned}
$$