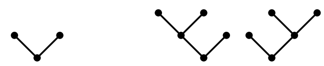
Full Binary Trees

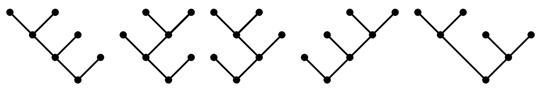
A full binary tree is a rooted tree in which each internal vertex has exactly two children. Thus, a full binary tree with n internal vertices has $2n$ edges. Since a tree has one more vertex than it has edges, a full binary tree with n internal vertices has $2n+1$ vertices, $2n$ edges and $n+1$ leaves.

How many full binary trees are there with n internal vertices?

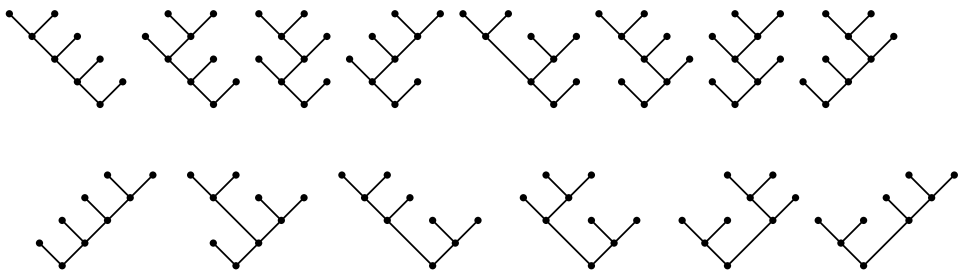
For $n = 1$, there is 1 full binary tree and for $n = 2$, there are 2 full binary trees.



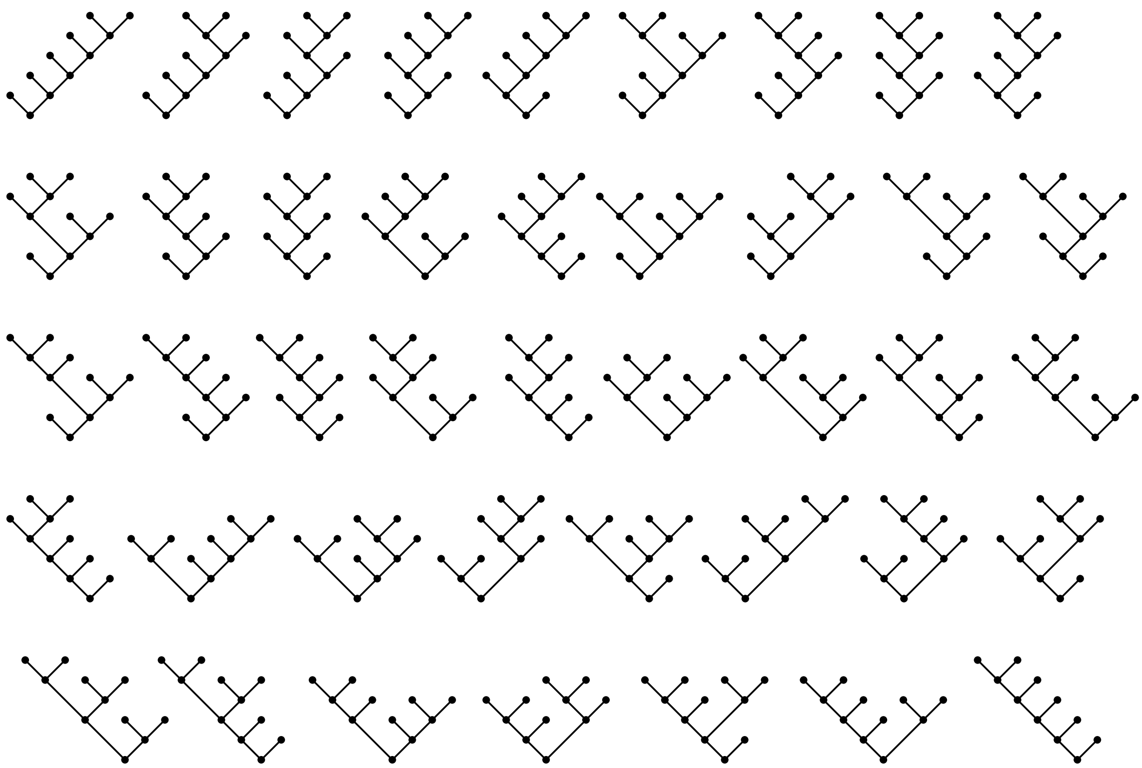
For $n = 3$, there are 5 such full binary trees.



For $n = 4$, there are 14 such full binary trees.



For $n = 5$, there are 42 full binary trees.



In fact, the number of full binary trees with n internal vertices is the Catalan number c_n.

Connection with the second bracketing problem

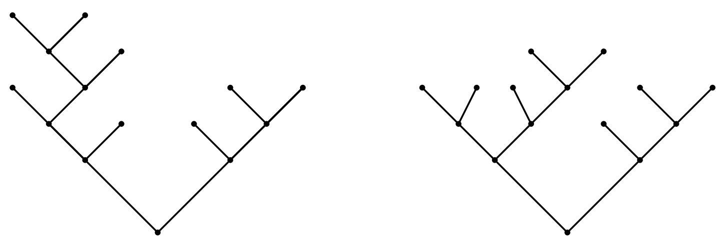
Given a full binary tree with n internal vertices and $n+1$ leaves, we obtain a well-parenthesized product of $n+1$ numbers $x_0, x_1, \dots, x_n$ as follows. We label the leaves of the tree as they are encountered along a transversal with $x_0, x_1, \dots, x_n$. Then the tree recursively defines a well-parenthesized product of $x_0, x_1, \dots, x_n$ by the following rule.

Labelling rule: If v is an internal vertex with left child a and right child b, having labels A and B, respectively, then label v with (AB).

The label on the root of the tree will be the well-parenthesized product.

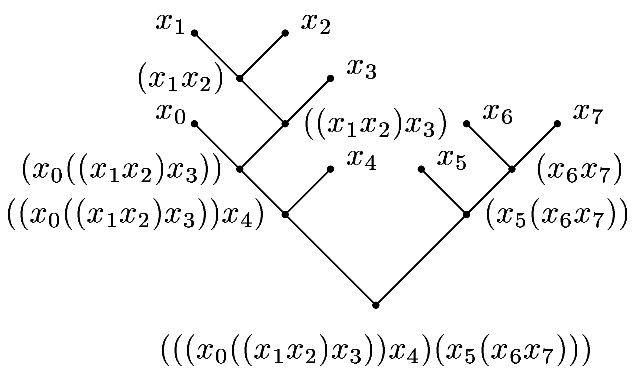
Given a well-parenthesized product of $n+1$ numbers $x_0, x_1, \dots, x_n$, we obtain a full binary tree as follows. A labeled full binary tree is determined by first labeling the root with the well-parenthesized product, then moving from the outer parentheses inward by adding two children labeled A and B to each vertex v with label (AB). The leaves of the tree will be labeled with the numbers $x_0, x_1, \dots, x_n$ in the order encountered by a transversal.

1. Write down the well-parenthesized products corresponding to the following full binary trees.



Solution.

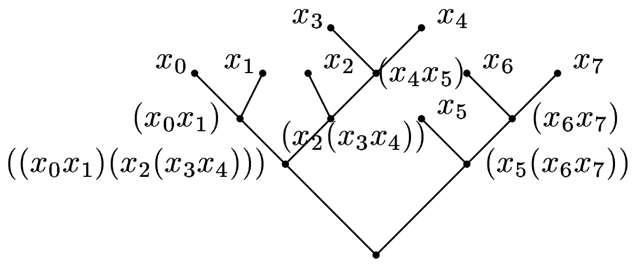
(i) The well-parenthesized product can be obtained as follows::



Hence the well-parenthesized product is

$$(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7))).$$

(*ii*) The well-parenthesized product can be obtained as follows::



$$(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7)))$$

Hence the well-parenthesized product is

$$(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7))).$$

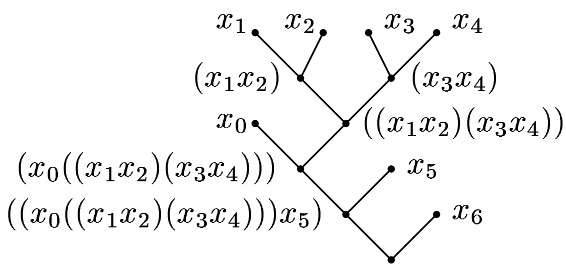
2. Draw and label the full binary tree defined by each of the following well-parenthesized products

(*i*) $(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)$

(*ii*) $(((x_0x_1)x_2)((x_3(x_4x_5))x_6))$

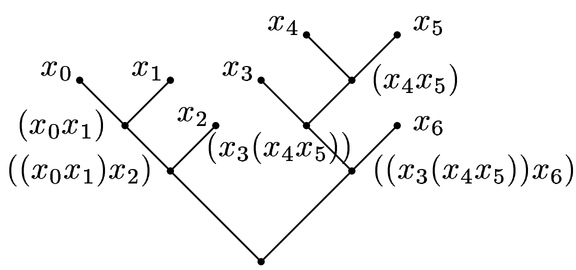
Solution.

(*i*) The corresponding full binary tree is:



$$(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)$$

(*ii*) The corresponding full binary tree is:



$$(((x_0x_1)x_2)((x_3(x_4x_5))x_6))$$