Triangulations of Convex Polygon

Problem: In how many ways can a convex polygon with \( n + 1 \) sides (labelled 0, 1, 2, \ldots, n) be divided into triangles by non-intersecting diagonals?

For \( n = 2 \), there is only 1 such dissection and for \( n = 3 \), there are 2 such dissections.

For \( n = 4 \), there are 5 such dissections.

For \( n = 5 \), there are 14 such dissections.

For \( n = 6 \), there are 42 such dissections.

In fact, for any convex polygon with \( n + 1 \) sides (labelled 0, 1, 2, \ldots, n), the number of ways to divide it into triangles by non-intersecting diagonals, is \( c_{n-1} \).
Connection with the second bracketing problem

A connection with the second bracketing problem is described as follows.

For \( i = 0, 1, \ldots, n - 1 \), label side \( i \) with \( x_i \). For each triangle, if two sides are labelled, then label the third side with a bracket containing the product of the other two sides. Then we obtain a well-parenthesized product of \( n + 1 \) numbers \( x_0, x_1, \ldots, x_n \).

Given a well-parenthesized product of \( n + 1 \) numbers \( x_0, x_1, \ldots, x_n \), we obtain a triangulation as follows. For \( i = 0, 1, \ldots, n - 1 \), label side \( i \) with \( x_i \). If \((x_i x_{i+1})\) appears in the well-parenthesized product, then we join a line to make \( x_i, x_{i+1} \) and \((x_i x_{i+1})\) into a triangle, and so on.

1. Write down the well-parenthesized products corresponding to the following triangulations.

\[
\begin{array}{cc}
(i) & (ii)
\end{array}
\]

Solution.

(i) The well-parenthesized product can be obtained as follows:

\[
(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7)))
\]

Hence the well-parenthesized product is

\[
(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7)))
\]

(ii) The well-parenthesized product can be obtained as follows:
Hence the well-parenthesized product is

$$(((x_0 x_1) (x_2 (x_3 x_4))) (x_5 (x_6 x_7)))$$

2. Draw and label the triangulation defined by the following well-parenthesized products

(i) $$(((x_0 (x_1 x_2) (x_3 x_4)) x_5) x_6)$$

(ii) $$(((x_0 x_1) x_2) ((x_3 (x_4 x_5)) x_6))$$

Solution.

(i) The corresponding triangulation can be obtained as follows:

(ii) The corresponding triangulation can be obtained as follows: