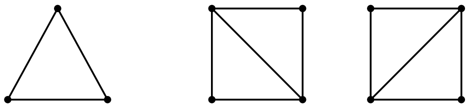
Triangulations of Convex Polygon

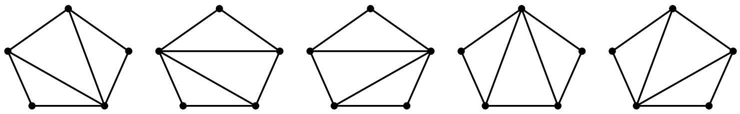
Problem : In how many ways can a convex polygon with $n+1$ sides (labelled 0, 1, 2, ... , n) be divided into triangles by non-intersecting diagonals?

For $n = 2$, there is only 1 such dissection and for $n = 3$, there are 2 such dissections.

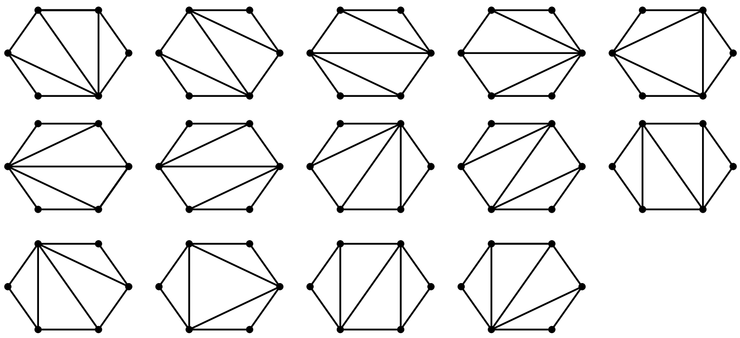


$n = 2$ $\qquad$ $n = 3$

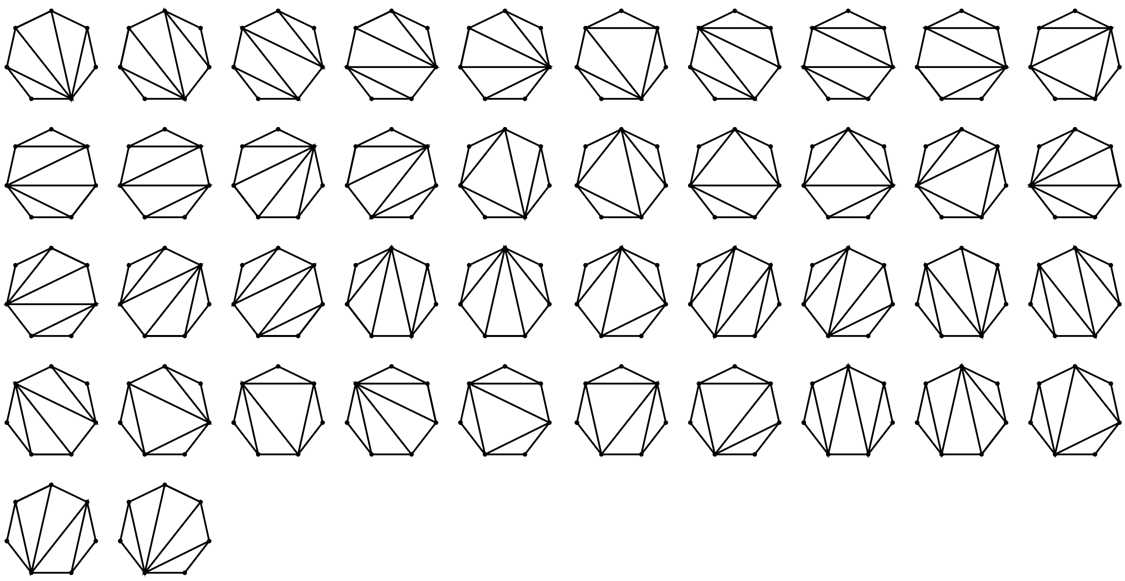
For $n = 4$, there are 5 such dissections.



For $n = 5$, there are 14 such dissections.



For $n = 6$, there are 42 such dissections.



In fact, for any convex polygon with $n+1$ sides (labelled 0, 1, 2, ... , n), the number of ways to divide it into triangles by non-intersecting diagonals, is c_{n-1} .

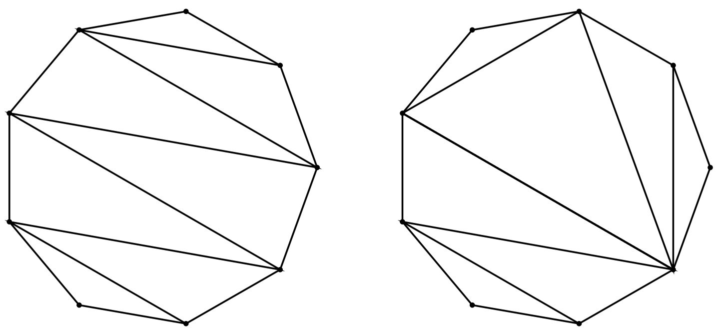
Connection with the second bracketing problem

A connection with the second bracketing problem is described as follows.

For $i = 0, 1, \ldots, n-1$, label side i with x_i. For each triangle, if two sides are labelled, then label the third side with a bracket containing the product of the other two sides. Then we obtain a well-parenthesized product of $n+1$ numbers $x_0, x_1, \ldots, x_n$.

Given a well-parenthesized product of $n+1$ numbers $x_0, x_1, \ldots, x_n$, we obtain a triangulation as follows. For $i = 0, 1, \ldots, n-1$, label side i with x_i. If $(x_i x_{i+1})$ appears in the well-parenthesized product, then we join a line to make x_i, x_{i+1} and $(x_i x_{i+1})$ into a triangle, and so on.

1. Write down the well-parenthesized products corresponding to the following triangulations.

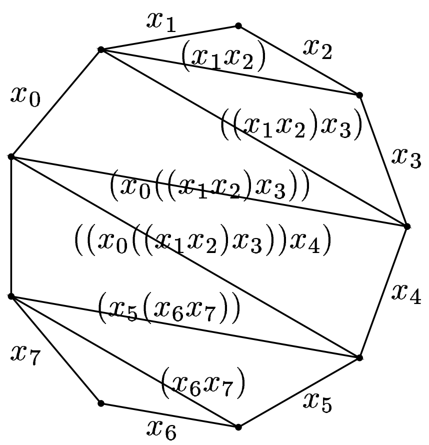


(*i*) (*ii*)

Solution.

(*i*) The well-parenthesized product can be obtained as follows:

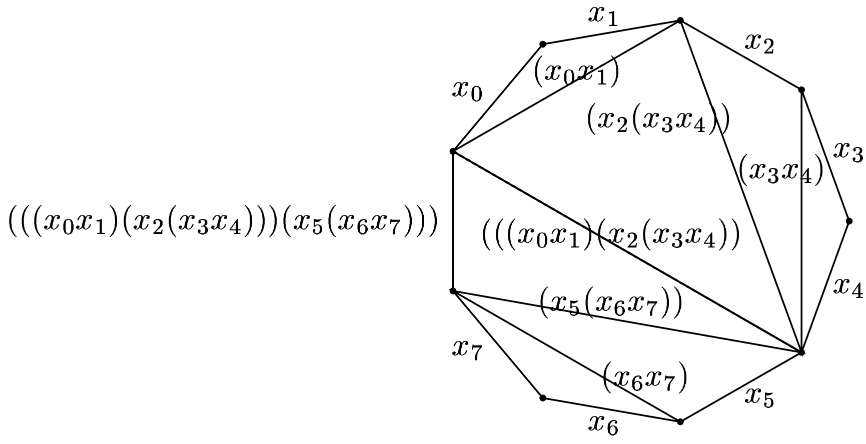
$(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7)))$



Hence the well-parenthesized product is

$$(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7)))$$

(*ii*) The well-parenthesized product can be obtained as follows:



Hence the well-parenthesized product is

$$(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7)))$$

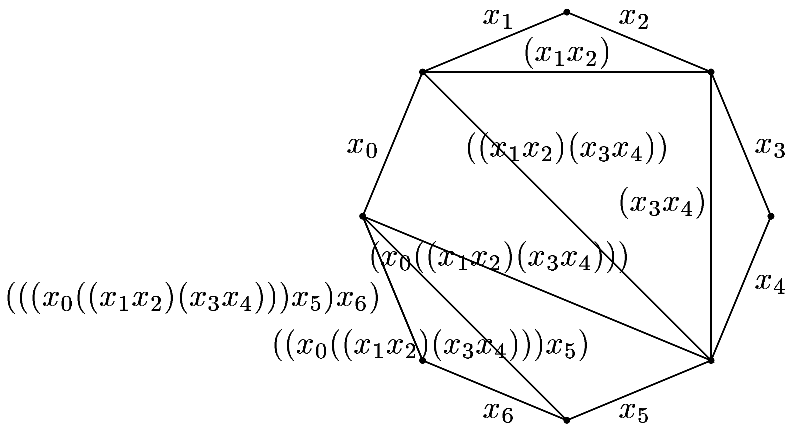
2. Draw and label the triangulation defined by the following well-parenthesized products

(i) $(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)$

(ii) $(((x_0x_1)x_2)((x_3(x_4x_5))x_6))$

Solution.

(i) The corresponding triangulation can be obtained as follows:



(ii) The corresponding triangulation can be obtained as follows: