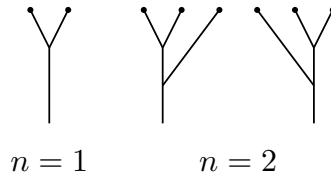


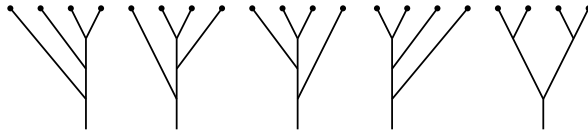
River Systems

Consider a river system with $(n+1)$ sources which eventually merge to form a single stream. Assuming that no more than two streams merge at any point. In how many ways that the mergers can take place.

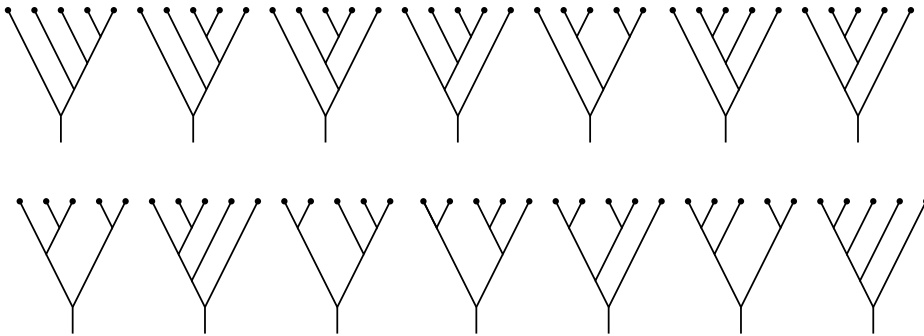
For $n = 1$, there is clearly only 1 way and for $n = 2$, there are 2 ways:



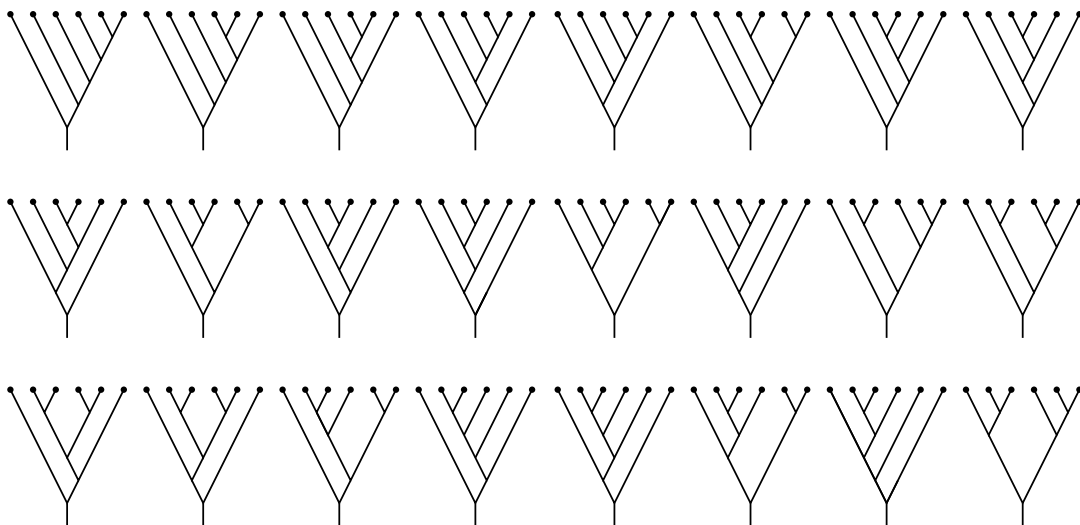
For $n = 3$, there are 5 ways:

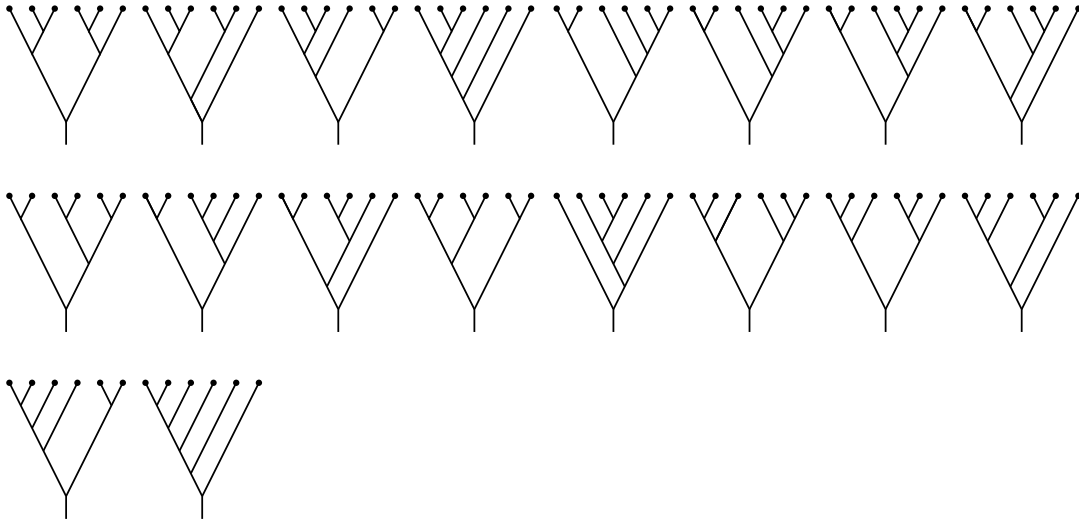


For $n = 4$, there are 14 ways:



For $n = 5$, there are 42 ways.





For a river system with $n + 1$ sources, the number of ways that the mergers can take place is the Catalan number c_n .

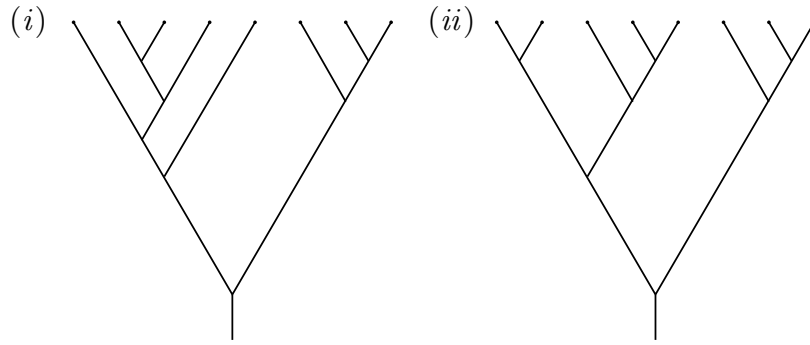
Connection with the second bracketing problem

Given a river system with $n + 1$ sources which eventually merge to form a single stream and no more than two streams merge at a point, we obtain a well-parenthesized product of $n + 1$ numbers x_0, x_1, \dots, x_n as follows: Let the $n + 1$ river sources be x_0, x_1, \dots, x_n . Then the river system recursively defines a well-parenthesized product of x_0, x_1, \dots, x_n by the following rule:

Labelling rule: When two streams A and B merge at a point, we label the new stream with (AB) .

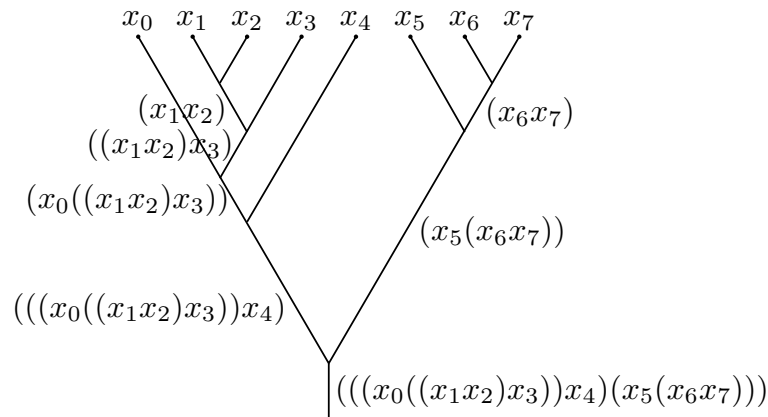
Given a well-parenthesized product of $n + 1$ numbers x_0, x_1, \dots, x_n , we obtain a river system as follows. A labeled river system is determined by first labeling the last stream with the well-parenthesized product, then moving upwards by adding two streams labeled A and B to each stream with label (AB) . The sources of the river will be labeled with the numbers x_0, x_1, \dots, x_n .

1. Write down the well-parenthesized products corresponding to the following river systems.



Solution.

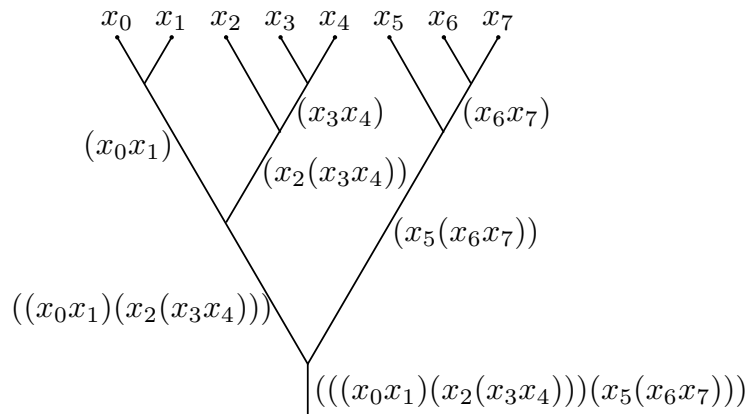
- (i) The well-parenthesized product can be obtained as follows:



Hence the well-parenthesized product is

$$(((x_0((x_1x_2)x_3))x_4)(x_5(x_6x_7))).$$

- (ii) The well-parenthesized product can be obtained as follows:



Hence the well-parenthesized product is

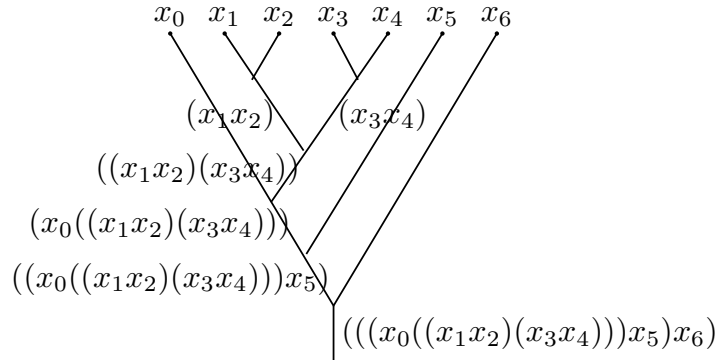
$$(((x_0x_1)(x_2(x_3x_4)))(x_5(x_6x_7))).$$

2. Draw and label the river systems defined by the following well-parenthesized products:

$$(i) \quad (((x_0((x_1x_2)(x_3x_4)))x_5)x_6) \quad (ii) \quad (((x_0x_1)x_2)((x_3(x_4x_5))x_6))$$

Solution.

(i) The corresponding river system can be obtained as follows:



(ii) The corresponding river system can be obtained as follows:

