River Systems

Consider a river system with \((n+1)\) sources which eventually merge to form a single stream. Assuming that no more than two streams merge at any point. In how many ways that the mergers can take place.

For \(n = 1\), there is clearly only 1 way and for \(n = 2\), there are 2 ways:

\[
\begin{align*}
  & Y \\
  & Y \\
  & Y
\end{align*}
\]

\(n = 1 \quad n = 2\)

For \(n = 3\), there are 5 ways:

\[
\begin{align*}
  & Y \quad Y \quad Y \\
  & Y \quad Y \quad Y \\
  & Y \quad Y \quad Y
\end{align*}
\]

For \(n = 4\), there are 14 ways:

\[
\begin{align*}
  & Y \quad Y \quad Y \quad Y \\
  & Y \quad Y \quad Y \quad Y \\
  & Y \quad Y \quad Y \quad Y \\
  & Y \quad Y \quad Y \quad Y
\end{align*}
\]

For \(n = 5\), there are 42 ways.
For a river system with $n + 1$ sources, the number of ways that the mergers can take place is the Catalan number $c_n$.

Connection with the second bracketing problem

Given a river system with $n + 1$ sources which eventually merge to form a single stream and no more than two streams merge at a point, we obtain a well-parenthesized product of $n + 1$ numbers $x_0, x_1, \ldots, x_n$ as follows: Let the $n + 1$ river sources be $x_0, x_1, \ldots, x_n$. Then the river system recursively defines a well-parenthesized product of $x_0, x_1, \ldots, x_n$ by the following rule:

Labelling rule: When two streams $A$ and $B$ merge at a point, we label the new stream with $(AB)$.

Given a well-parenthesized product of $n + 1$ numbers $x_0, x_1, \ldots, x_n$, we obtain a river system as follows. A labeled river system is determined by first labeling the last stream with the well-parenthesized product, then moving upwards by adding two streams labeled $A$ and $B$ to each stream with label $(AB)$. The sources of the river will be labeled with the numbers $x_0, x_1, \ldots, x_n$. 
1. Write down the well-parenthesized products corresponding to the following river systems.

\[(i)\]
\[
\begin{array}{c}
\text{x}_0 \\
\text{x}_1 \\
\text{x}_2 \\
\text{x}_3 \\
\text{x}_4 \\
\text{x}_5 \\
\text{x}_6 \\
\text{x}_7 \\
\end{array}
\]

\[(ii)\]
\[
\begin{array}{c}
\text{x}_0 \\
\text{x}_1 \\
\text{x}_2 \\
\text{x}_3 \\
\text{x}_4 \\
\text{x}_5 \\
\text{x}_6 \\
\text{x}_7 \\
\end{array}
\]

**Solution.**

(i) The well-parenthesized product can be obtained as follows:

\[
\begin{array}{c}
\text{x}_0 \\
\text{x}_1 \\
\text{x}_2 \\
\text{x}_3 \\
\text{x}_4 \\
\text{x}_5 \\
\text{x}_6 \\
\text{x}_7 \\
\end{array}
\]

\[
\begin{array}{c}
\text{(x}_1\text{x}_2) \\
\text{((x}_1\text{x}_2)\text{x}_3) \\
\text{(x}_0\text{((x}_1\text{x}_2)\text{x}_3)) \\
\text{(((x}_0\text{((x}_1\text{x}_2)\text{x}_3)x}_4) \\
\text{(((x}_0\text{((x}_1\text{x}_2)\text{x}_3)x}_4)(\text{x}_5\text{x}_6\text{x}_7))
\end{array}
\]

Hence the well-parenthesized product is

\[((((x}_0\text{((x}_1\text{x}_2)\text{x}_3))x}_4)(\text{x}_5\text{x}_6\text{x}_7))).

(ii) The well-parenthesized product can be obtained as follows:

\[
\begin{array}{c}
\text{x}_0 \\
\text{x}_1 \\
\text{x}_2 \\
\text{x}_3 \\
\text{x}_4 \\
\text{x}_5 \\
\text{x}_6 \\
\text{x}_7 \\
\end{array}
\]

\[
\begin{array}{c}
\text{(x}_0\text{x}_1) \\
\text{((x}_3\text{x}_4) \\
\text{(x}_2\text{(x}_3\text{x}_4)) \\
\text{(x}_5\text{x}_6\text{x}_7)) \\
\text{(((x}_0\text{x}_1)(\text{x}_2\text{x}_3\text{x}_4)) \\
\text{(((x}_0\text{x}_1)(\text{x}_2\text{x}_3\text{x}_4))(\text{x}_5\text{x}_6\text{x}_7))
\end{array}
\]

Hence the well-parenthesized product is

\[((((x}_0\text{x}_1)(\text{x}_2\text{x}_3\text{x}_4))(\text{x}_5\text{x}_6\text{x}_7))).
2. Draw and label the river systems defined by the following well-parenthesized products:

(i) \(((x_0((x_1x_2)(x_3x_4)))x_5)x_6)  
(ii) \(((x_0x_1)x_2)((x_3(x_4x_5))x_6))

Solution.

(i) The corresponding river system can be obtained as follows:

(ii) The corresponding river system can be obtained as follows: