Divisible sequences

Problem 1: Consider the n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbours; i.e., $a_i|(a_{i-1}+a_{i+1})$, $a_0 = a_{n+1} = 1$. How many such n-tuples are there?

For $n = 1$, there is clearly only one 1-tuple; that is, (2), and the corresponding sequence is 121.

For $n = 2$, there are 2 such sequences; namely 1231 and 1321 and the 2-tuples are $(2,3)$ and $(3,2)$.

For $n = 3$, there are 5 such sequences:

$$12341,\ 12531,\ 13231,\ 13521,\ 14321$$

and the corresponding 3-tuples are

$$(2,3,4),\ (2,5,3),\ (3,2,3),\ (3,5,2),\ (4,3,2).$$

For $n = 4$, there are 14 such sequences:

$$123451,\ 123741,\ 125341,\ 125831,\ 127531,\ 132341,\ 132531,$$
$$135231,\ 135721,\ 138521,\ 143231,\ 143521,\ 147321,\ 154321$$

and the corresponding 4-tuples are

$$(2,3,4,5),\ (2,3,7,4),\ (2,5,3,4),\ (2,5,8,3),\ (2,7,5,3),\ (3,2,3,4),\ (3,2,5,3),$$
$$(3,5,2,3),\ (3,5,7,2),\ (3,8,5,2),\ (4,3,2,3),\ (4,3,5,2),\ (4,7,3,2),\ (5,4,3,2).$$

For $n = 5$, there are 42 such sequences:

1234561, 1234951, 1237451, 1237(11)41, 123(10)741, 1253451,
1253741, 1258341, 1258(11)31, 125(13)831, 1275341, 1275831,
127(12)531, 1297531, 1323451, 1323741, 1325341, 1325831,
1327531, 1352341, 1352531, 1357231, 1357921, 135(12)721,
1385231, 1385721, 138(13)521, 13(11)8521, 1432341, 1432531
1435231, 1435721, 1438521, 1473231, 1473521, 147(10)321
14(11)7321, 1543231, 1543521, 1547321, 1594321, 1654321

and the corresponding 5-tuples are

$$(2,3,4,5,6), (2,3,4,9,5), (2,3,4,7,5), (2,3,7,11,4), (2,3,10,7,4), (2,5,3,4,5),$$
$$(2,5,3,7,4), (2,5,8,3,4), (2,5,8,11,3), (2,5,13,8,3), (2,7,5,3,4), (2,7,5,8,3),$$
$$(2,7,12,5,3), (2,9,7,5,3), (3,2,3,4,5), (3,2,3,7,4), (3,2,5,3,4), (3,2,5,8,3),$$
$$(3,2,7,5,3), (3,5,2,3,4), (3,5,2,5,3), (3,5,7,2,3), (3,5,7,9,2), (3,5,12,7,2),$$
$$(3,8,5,2,3), (3,8,5,7,2), (3,8,13,5,2), (3,11,8,5,2), (4,3,2,3,4), (4,3,2,5,3),$$
$$(4,3,5,2,3), (4,3,5,7,2), (4,3,8,5,2), (4,7,3,2,3), (4,7,3,5,2), (4,7,10,3,2),$$
$$(4,11,7,3,2), (5,4,3,2,3), (5,4,3,5,2), (5,4,7,3,2), (5,9,4,3,2), (6,5,4,3,2).$$

For $n = 6$, there are 132 such sequences:

12345671, 12345(11)61, 12349561, 12349(14)51, 1234(13)951,
12374561, 12374951, 1237(11)451, 1237(11)(15)41, 1237(18)(11)41,
123(10)7451, 123(10)7(11)41, 123(10)(17)741, 123(13)(10)741,
12534561, 12534951, 12537451, 12537(11)41, 1253(10)741,
12583451, 12583741, 1258(11)341, 1258(11)(14)31, 1258(19)(11)341,
125(13)8341, 125(13)8(11)31, 125(13)(21)831, 125(18)(13)831,
12753451, 12753741, 12758341, 12758(11)31, 1275(13)831,
127(12)5341, 127(12)5831, 127(12)(17)531, 127(19)(12)531,
12975341, 12975831, 1297(12)531, 129(16)7531, 12(11)97531,
13234561, 13234951, 13237451, 13237(11)41, 1323(10)741,
13253451, 13253741, 13258341, 13258(11)31, 1325(13)831,
13275341, 13275831, 1327(12)531, 13297531,
13523451, 13523741, 13525341, 13525831, 13527531,
13572341, 13572531, 13579231, 13579(11)21, 1357(16)921,
135(12)7231, 135(12)7921, 135(12)(19)721, 135(17)(12)721,
13852341, 13852531, 13857231, 13857921, 1385(12)721,
138(13)5231, 138(13)5721, 138(13)(18)521, 138(21)(13)521,
13(11)85231, 13(11)85721, 13(11)8(13)521, 13(11)(19)8521, 13(14)(11)8521,
14323451, 14323741, 14325341, 14325831, 14327531,
14352341, 14352531, 14357231, 14357921, 1435(12)721,
14385231, 14385721, 1438(13)521, 143(11)8521,
14732341, 14732531, 14735231, 14735721, 14738521,
147(10)3231, 147(10)3521, 147(10)(13)321, 147(17)(10)321,
14(11)73231, 14(11)73521, 14(11)7(10)321, 14(11)(18)7321, 14(15)(11)7321,
15432341, 15432531, 15435231, 15435721, 15438521,
15473231, 15473521, 1547(10)321, 154(11)7321,
15943231, 15943521, 15947321, 159(13)4321, 15(14)94321,
16543231, 16543521, 16547321, 16594321, 16(11)54321, 17654321.

In fact, for any n, the number of such n-tuples is the Catalan number c_n.

Connection with the first bracket problem

Given a balanced string of n left and n right brackets, we decompose it into the form

$$S_1T_1S_2T_2\cdots S_mT_m$$

where S_i consists of left brackets and T_i consists of right brackets only. Let

$$|S_i| = k_i, \quad |T_i| = \ell_i\,.$$

Then it is easy to see that $\ell_i \le k_i$, $k_1+k_2+\cdots+k_m = n$ and $\ell_1+\ell_2+\cdots+\ell_m = n$. To obtain the corresponding sequence $1a_1a_2\cdots a_n1$ with $a_i|(a_{i-1}+a_{i+1}$, $a_0 = a_{n+1} = 1$, we proceed as follows:

(*i*) Startswith 11.

(*ii*) Draw k_1 vertical lines before 11 : $||\cdots|11$

(*iii*) Replace $|11$ by 121, then $|121$ by 1321, and so on; i.e., replace $|1b_1b_2\cdots 1$ by $1(1+b_1)b_1b_2\cdots 1$. Repeat k_1 times to obtain $1a_1a_2\cdots a_{k_1}1$.

(*iv*) Draw k_2 vertical lines after ℓ_1 numbers; i.e., draw k_2 vertical lines before the number a_{ℓ_1-1} to get

$$1a_1a_2\cdots a_{\ell_1-1}||\cdots|a_{\ell_1}a_{\ell_1+1}\cdots a_k1\,.$$

(*v*) Replace $|a_{\ell_1}a_{\ell_1+1}$ by $a_{\ell_1}(a_{\ell_1}+a_{\ell_1+1})a_{\ell_1+1}$ and repeat k_2 times, in repalcing $|a_{\ell_1}b$ by $|a_{\ell_1}(a_{\ell_1}+b)b$ to get

$$1a_1a_2\cdots a_{k_1}a_{k_1+1}\cdots a_{k_1+k_2}1\,.$$

(*vi*) Draw k_3 vertical lines after $\ell_1+\ell_2$ numbers; i.e., draw k_3 vertical lines before the number $a_{\ell_1+\ell_2-1}$ to get

$$1a_1a_2\cdots a_{\ell_1}\cdots a_{\ell_1+\ell_2-1}||\cdots|a_{\ell_1+\ell_2}\cdots a_{k_1+k_2}1\,.$$

(*vii*) Repeat the above procedures for k_3 and so on until we finish up to S_m.

(*viii*) Then we will obtain the necessary sequence $1a_1a_2\cdots a_n1$.

Given a sequence $1a_1a_2\cdots a_n1$ with $a_i|(a_{i-1}+a_{i+1}$, $a_0 = a_{n+1} = 1$, we obtain the corresponding balanced string of brackets as follows:

(*i*) Reading from the left, replace $\max a_i$ with $a_i|(a_{i-1}+a_{i+1}$ by

$$La_{i-1}Ra_{i+1}\,.$$

(*ii*) Repeat the above procedure for the next $\max a_i$, ignoring L and R; and so on, until only 1,1 left.

(*iii*) Replace 1 by R.

(*iv*) Ignoring the last two R, we then obtain the necessary balanced string of n left and n right brackets.

1. Find the corresponding sequence for each of the following balanced string of brackets:

(i) LLRLRLLRRRLR

(ii) LLLRLRLLRRLLRRRR

Solution.

(i) In this case,

$$k_1 = 2,\ k_2 = 1,\ k_3 = 2,\ k_4 = 1,\ \ell_1 = 1,\ \ell_2 = 1,\ \ell_3 = 3,\ \ell_4 = 1\,.$$

We obtain the sequence as follows:

$$\begin{aligned} 11 \quad &\Longrightarrow \quad ||11 \\ &\Longrightarrow \quad |121 \\ &\Longrightarrow \quad 1321 \\ &\Longrightarrow \quad 1|321 \\ &\Longrightarrow \quad 13521 \\ &\Longrightarrow \quad 13||521 \\ &\Longrightarrow \quad 13|5721 \\ &\Longrightarrow \quad 135(12)721 \\ &\Longrightarrow \quad 135(12)7|21 \\ &\Longrightarrow \quad 135(12)7231 \end{aligned}$$

(ii) In this case,

$$k_1 = 3,\ k_2 = 1,\ k_3 = 2,\ k_4 = 2,\ \ell_1 = 1,\ \ell_2 = 1,\ \ell_3 = 2,\ \ell_4 = 4\,.$$

We obtain the sequence as follows:

$$\begin{aligned} 11 \quad &\Longrightarrow \quad |||11 \\ &\Longrightarrow \quad ||121 \\ &\Longrightarrow \quad |1321 \\ &\Longrightarrow \quad 14321 \\ &\Longrightarrow \quad 1|4321 \\ &\Longrightarrow \quad 147321 \\ &\Longrightarrow \quad 14||7321 \\ &\Longrightarrow \quad 14|7(10)321 \\ &\Longrightarrow \quad 147(17)(10)321 \\ &\Longrightarrow \quad 147(17)||(10)321 \\ &\Longrightarrow \quad 147(17)|(10)(13)321 \\ &\Longrightarrow \quad 147(17)(10)(23)(13)321 \end{aligned}$$

2. For each of the sequences, write down the corresponding balanced string of brackets:

(i) 135(12)7231

(ii) 147(17)(10)(23)(13)321

Solution.

(i) We obtain the balanced string of brackets as follows:

$$\begin{aligned}
135(12)7231 &\Longrightarrow 13L5R7231 \quad (12|(5+7)\\
&\Longrightarrow 13LL5RR231 \quad (7|5+2)\\
&\Longrightarrow 1L3LLRRR231 \quad (5|3+2)\\
&\Longrightarrow L1LRLLRRR231 \quad (3|1+2)\\
&\Longrightarrow L1LRLLRRRL2R1 \quad (3|2+1)\\
&\Longrightarrow LL1LRLLRRRLRR1 \quad (2|1+1)\\
&\Longrightarrow LLRLRLLRRRLRRR \quad (1 \to R)\\
&\Longrightarrow LLRLRLLRRRLR \qquad \text{(drop the last 2 R)}
\end{aligned}$$

(ii) We obtain the balanced string of brackets as follows:

$$\begin{aligned}
147(17)(10)(23)(13)321 &\Longrightarrow 147(17)L(10)R(13)321\\
&\Longrightarrow 14L7RL(10)R(13)321\\
&\Longrightarrow 14L7RLL(10)RR321\\
&\Longrightarrow 14LL7RLLRRR321\\
&\Longrightarrow 1L4LLRRLLRRR321\\
&\Longrightarrow L1LRLLRRLLRRR321\\
&\Longrightarrow LL1LRLLRRLLRRRR21\\
&\Longrightarrow LLL1LRLLRRLLRRRRR1\\
&\Longrightarrow LLLRLRLLRRLLRRRRRR\\
&\Longrightarrow LLLRLRLLRRLLRRRR
\end{aligned}$$