

Sequences with positive partial sums

Problem 1: How many sequences a_1, a_2, \dots, a_n of integers are there, such that

- (i) $a_i \geq -1$,
- (ii) $a_1 + a_2 + \dots + a_k \geq 0$, ($1 \leq k \leq n - 1$),
- (iii) $a_1 + a_2 + \dots + a_n = 0$?

Problem 2: How many n -tuples (a_1, a_2, \dots, a_n) of integers are there, such that

- (i) $a_i \geq -1$,
- (ii) $a_1 + a_2 + \dots + a_k \geq 0$, ($1 \leq k \leq n - 1$),
- (iii) $a_1 + a_2 + \dots + a_n = 0$?

Problem 3: How many functions

$$f : \{1, 2, \dots, n\} \rightarrow \{-1, 0, \dots, n - 1\}$$

are there such that

- (i) $f(1) + f(2) + \dots + f(k) \geq 0$ for all $k \leq n$; and
- (ii) $f(1) + f(2) + \dots + f(n) = 0$?

For $n = 1$, there is only 1 such sequence; namely 0.

For $n = 2$, there are 2 such sequences: 0, 0 and 1, -1

For $n = 3$, there are 5 such sequences:

$$0, 0, 0 \quad 0, 1, -1 \quad 1, 0, -1 \quad 1, -1, 0 \quad 2, -1, -1$$

For $n = 4$, there are 14 such sequences:

$$\begin{array}{cccccc} 0, 0, 0, 0 & 0, 0, 1, -1 & 0, 1, 0, -1 & 1, 0, 0, -1 & 0, 1, -1, 0 & \\ 0, 2, -1, -1 & 1, 1, -1, -1 & 1, 0, -1, 0 & 2, 0, -1, -1 & 1, -1, 0, 0 & \\ 1, -1, 1, -1 & 2, -1, 0, -1 & 2, -1, -1, 0 & 3, -1, -1, -1 & & \end{array}$$

For $n = 5$, there are 42 such sequences:

$$\begin{array}{ccccc} 0, 0, 0, 0, 0 & 0, 0, 0, 1, -1 & 0, 0, 1, 0, -1 & & \\ 0, 1, 0, 0, -1 & 1, 0, 0, 0, -1 & 0, 0, 1, -1, 0 & & \\ 0, 0, 2, -1, -1 & 0, 1, 1, -1, -1 & 1, 0, 1, -1, -1 & & \\ 0, 1, 0, -1, 0 & 0, 2, 0, -1, -1 & 1, 1, 0, -1, -1 & & \\ 1, 0, 0, -1, 0 & 2, 0, 0, -1, -1 & 0, 1, -1, 0, 0 & & \\ 0, 1, -1, 1, -1 & 0, 2, -1, 0, -1 & 1, 1, -1, 0, -1 & & \\ 0, 2, -1, -1, 0 & 0, 3, -1, -1, -1 & 1, 2, -1, -1, -1 & & \\ 1, 1, -1, -1, 0 & 2, 1, -1, -1, -1 & 1, 0, -1, 0, 0 & & \end{array}$$

1, 0, -1, 1, -1	2, 0, -1, 0, -1	2, 0, -1, -1, 0
3, 0, -1, -1, -1	1, -1, 0, 0, 0	1, -1, 0, 1, -1
1, -1, 1, 0, -1	2, -1, 0, 0, -1	1, -1, 1, -1, 0
1, -1, 2, -1, -1	2, -1, 1, -1, -1	2, -1, 0, -1, 0
3, -1, 0, -1, -1	2, -1, -1, 0, 0	2, -1, -1, 1, -1
3, -1, -1, 0, -1	3, -1, -1, -1, 0	4, -1, -1, -1, -1

For $n=6$, there are 132 such functions:

0, 0, 0, 0, 0, 0	0, 0, 0, 0, 1, -1	0, 0, 0, 1, 0, -1
0, 0, 1, 0, 0, -1	0, 1, 0, 0, 0, -1	1, 0, 0, 0, 0, -1
0, 0, 0, 1, -1, 0	0, 0, 0, 2, -1, -1	0, 0, 1, 1, -1, -1
0, 1, 0, 1, -1, -1	1, 0, 0, 1, -1, -1	0, 0, 1, 0, -1, 0
0, 0, 2, 0, -1, -1	0, 1, 1, 0, -1, -1	1, 0, 1, 0, -1, -1
0, 1, 0, 0, -1, 0	0, 2, 0, 0, -1, -1	1, 1, 0, 0, -1, -1
1, 0, 0, 0, -1, 0	2, 0, 0, 0, -1, -1	0, 0, 1, -1, 0, 0
0, 0, 1, -1, 1, -1	0, 0, 2, -1, 0, -1	0, 1, 1, -1, 0, -1
1, 0, 1, -1, 0, -1	0, 0, 2, -1, -1, 0	0, 0, 3, -1, -1, -1
0, 1, 2, -1, -1, -1	1, 0, 2, -1, -1, -1	0, 1, 1, -1, -1, 0
0, 2, 1, -1, -1, -1	1, 1, 1, -1, -1, -1	1, 0, 1, -1, -1, 0
2, 0, 1, -1, -1, -1	0, 1, 0, -1, 0, 0	0, 1, 0, -1, 1, -1
0, 2, 0, -1, 0, -1	1, 1, 0, -1, 0, -1	0, 2, 0, -1, -1, 0
0, 3, 0, -1, -1, -1	1, 2, 0, -1, -1, -1	1, 1, 0, -1, -1, 0
2, 1, 0, -1, -1, -1	1, 0, 0, -1, 0, 0	1, 0, 0, -1, 1, -1
2, 0, 0, -1, 0, -1	2, 0, 0, -1, -1, 0	3, 0, 0, -1, -1, -1
0, 1, -1, 0, 0, 0	0, 1, -1, 0, 1, -1	0, 1, -1, 1, 0, -1
0, 2, -1, 0, 0, -1	1, 1, -1, 0, 0, -1	0, 1, -1, 1, -1, 0
0, 1, -1, 2, -1, -1	0, 2, -1, 1, -1, -1	1, 1, -1, 1, -1, -1
0, 2, -1, 0, -1, 0	0, 3, -1, 0, -1, -1	1, 2, -1, 0, -1, -1
1, 1, -1, 0, -1, 0	2, 1, -1, 0, -1, -1	0, 2, -1, -1, 0, 0
0, 2, -1, -1, 1, -1	0, 3, -1, -1, 0, -1	1, 2, -1, -1, 0, -1
0, 3, -1, -1, -1, 0	0, 4, -1, -1, -1, -1	1, 3, -1, -1, -1, -1
1, 2, -1, -1, -1, 0	2, 2, -1, -1, -1, -1	1, 1, -1, -1, 0, 0
1, 1, -1, -1, 1, -1	2, 1, -1, -1, 0, -1	2, 1, -1, -1, -1, 0
3, 1, -1, -1, -1, -1	1, 0, -1, 0, 0, 0	1, 0, -1, 0, 1, -1
1, 0, -1, 1, 0, -1	2, 0, -1, 0, 0, -1	1, 0, -1, 1, -1, 0
1, 0, -1, 2, -1, -1	2, 0, -1, 1, -1, -1	2, 0, -1, 0, -1, 0
3, 0, -1, 0, -1, -1	2, 0, -1, -1, 0, 0	2, 0, -1, -1, 1, -1
3, 0, -1, -1, 0, -1	3, 0, -1, -1, -1, 0	4, 0, -1, -1, -1, -1

1, -1, 0, 0, 0, 0	1, -1, 0, 0, 1, -1	1, -1, 0, 1, 0, -1
1, -1, 1, 0, 0, -1	2, -1, 0, 0, 0, -1	1, -1, 0, 1, -1, 0
1, -1, 0, 2, -1, -1	1, -1, 1, 1, -1, -1	2, -1, 0, 1, -1, -1
1, -1, 1, 0, -1, 0	1, -1, 2, 0, -1, -1	2, -1, 1, 0, -1, -1
2, -1, 0, 0, -1, 0	3, -1, 0, 0, -1, -1	1, -1, 1, -1, 0, 0
1, -1, 1, -1, 1, -1	1, -1, 2, -1, 0, -1	2, -1, 1, -1, 0, -1
1, -1, 2, -1, -1, 0	1, -1, 3, -1, -1, -1	2, -1, 2, -1, -1, -1
2, -1, 1, -1, -1, 0	3, -1, 1, -1, -1, -1	2, -1, 0, -1, 0, 0
2, -1, 0, -1, 1, -1	3, -1, 0, -1, 0, -1	3, -1, 0, -1, -1, 0
4, -1, 0, -1, -1, -1	2, -1, -1, 0, 0, 0	2, -1, -1, 0, 1, -1
2, -1, -1, 1, 0, -1	3, -1, -1, 0, 0, -1	2, -1, -1, 1, -1, 0
2, -1, -1, 2, -1, -1	3, -1, -1, 1, -1, -1	3, -1, -1, 0, -1, 0
4, -1, -1, 0, -1, -1	3, -1, -1, -1, 0, 0	3, -1, -1, -1, 1, -1
4, -1, -1, -1, 0, -1	4, -1, -1, -1, -1, 0	5, -1, -1, -1, -1, -1

Connection with the rooted plane tree problem

Given a rooted plane tree with $n + 1$ vertices and n edges, we obtain a sequence of integers a_1, a_2, \dots, a_n such that $a_i \geq -i$, all partial sums are nonnegative and $a_1 + a_2 + \dots + a_n = 0$, as follows:

Start from the bottom vertex and when a vertex is encountered for the first time, record one less than its number of successors, except that the last vertex is ignored, reading the vertex from left to right.

Given a sequence of integers a_1, a_2, \dots, a_n such that $a_i \geq -i$, all partial sums are nonnegative and $a_1 + a_2 + \dots + a_n = 0$, we obtain a rooted plane tree as follows:

Counting the vertex from bottom up and from left to right, the number of successors is $a_i + 1$, for each vertex i .

Connection with the first bracketing problem

Given a balanced string of brackets, we obtain first the corresponding rooted plane tree and then use the above procedure to obtain the corresponding sequence of integers with the given properties.

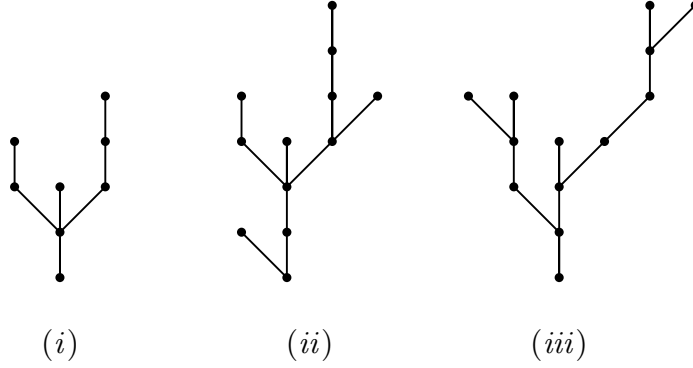
Given a sequence a_1, a_2, \dots, a_n of integers such that $a_i \geq -1$, all partial sums are non-negative and $a_1 + a_2 + \dots + a_n = 0$, we use the above procedure to obtain first the corresponding rooted plane tree and then obtain the corresponding balanced string of brackets.

1. Construct balanced strings of brackets corresponding to the following sequences of integers:

- (i) 0, 2, 0, -1, -1, 0, 0
(ii) 1, -1, 0, 2, 0, -1, -1, 1, 0, 0, -1
(iii) 0, 1, 0, 1, -1, -1, 1, -1, 0, 0, 1, -1

Solution.

The corresponding rooted plane trees are:



Hence the corresponding balanced strings of brackets are

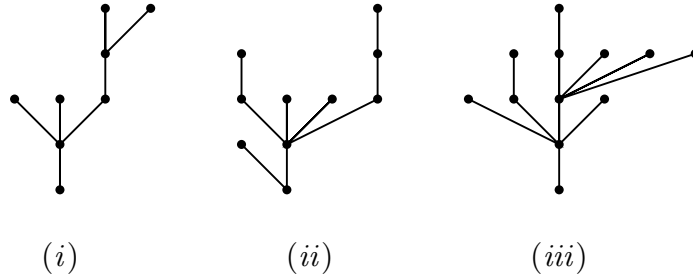
- (i) LLLRRLRLLLRRRR
(ii) LRLLLRRLRLLLRRRLLRRRR
(iii) LLLRRLRRLLRLLLRLRRRRRR

2. For the following balanced strings of brackets, construct the corresponding sequence a_1, a_2, \dots, a_n of integers such that $a_i \geq -1$, all partial sums are non-negative and $a_1 + a_2 + \dots + a_n = 0$.

- (i) LLRLRLLLRLRRRR
(ii) LRLLLRRLRLLLRRRR
(iii) LLRLRLLLRRLLRRLRRLRR

Solution.

The corresponding rooted plane trees are:



The corresponding sequences are

- (i) 0, 2, -1, -1, 0, 1, -1 (ii) 1, -1, 3, 0, -1, -1, -1, 0, 0
(iii) 0, 3, -1, 0, -1, 3, 0, -1, -1, -1