Sequences with nonnegative partial sums

Problem 1: How many sequences $a_1, a_2, \ldots, a_{n-1}$ are there such that $a_i \leq 1$ and all partial sums are nonnegative?

Problem 2: How many functions

$$f : \{1, 2, \ldots, n - 1\} \rightarrow \{1, 0, -1, \ldots, -(n - 2)\}$$

are there such that $f(1) + f(2) + \cdots f(k) \geq 0$ for all $k \leq n - 1$?

For $n = 2$, there are 2 such sequences; namely 1 and 0, which correspond to the balanced strings of breackets LLRR and LRLR.

For $n = 3$, there are 5 such sequences:

11 10 1-1 01 00

For $n = 4$, there are 14 such sequences:

111 110 11-1 11-2 101 100 10-1
1-11 1-10 011 010 01-1 001 000

For $n = 5$, there are 42 such sequences:

1111 1110 111-1 111-2 111-3 1101 1100
110-1 110-2 11-11 11-10 11-1-1 11-21 11-20
1011 1010 101-1 101-2 1001 1000 100-1
10-11 10-10 1-111 1-110 1-11-1 1-101 1-100
0111 0110 011-1 011-2 0101 0100 010-1
01-11 01-10 0011 0010 001-1 0001 0000
For $n = 6$, there are 132 such sequences:

\[
\]

In fact, for any integer $n \geq 1$, the number of such sequences is the Catalan number $c_n$.

**Connection with the first bracketing problem**

Given a balanced string of brackets, we obtain the corresponding sequence, with $a_i \leq 1$ and nonnegative partial sums, as follows.

Write down the sequence $1, 2, \ldots, 2n$ under the left and right brackets in the string from left to right. For $0 \leq k \leq n-1$, let $b_k$ be the integer corresponding to the $k + 1$-th left brackets. Then define $a_k$ by

\[
a_k = b_{k-1} - b_k + 2, \quad (1 \leq k \leq n-1).
\]

Given a sequence $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are non-negative, we obtain a balanced string of brackets by determine the positions of the left brackets, as follows. Suppose the $j$-th left bracket corresponds to the integer $k$, then the $(j + 1)$-th left bracket corresponds to the integer $k - a_k + 2$. 
1. Construct balanced strings of brackets corresponding to the following sequences of integers:

(i) $11-1011$

(ii) $0111-10111-2$

Solution.

(i) The first left bracket always corresponds to 1. Then the second left bracket corresponds to $1-a_1 + 2 = 1-1+2 = 2$; the 3rd is $2-a_2 + 2 = 3$; the 4th is $3-a_3 + 2 = 6$; the 5th is $6-a_4 + 2 = 8$; the 6th is $8-a_5 + 2 = 9$ and the 7th is $9-a_6 + 2 = 10$.

Hence the corresponding balanced strings of brackets is

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  LLLRRLRLLRRRR
```

(ii) In this case, the left brackets correspond to the integers

$1, 3, 4, 5, 6, 9, 11, 12, 13, 14, 18$

Hence the corresponding balanced strings of brackets is

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  LRLLLRRLRLLRRRRLRRRR
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2. For the following balanced strings of brackets, construct the corresponding sequence $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$, and all partial sums are non-negative.

(i) $LLRLRLRRLRRRR$

(ii) $LRLLLRRLRLLRRRR$

Solution.

(i) The left brackets correspond to

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  b_0 = 1, b_1 = 2, b_2 = 4, b_3 = 6, b_4 = 7, b_5 = 8, b_6 = 10
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Then $a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = 1, a_6 = 0$ so that the corresponding sequence is

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  100110
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(ii) The left brackets correspond to

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  b_0 = 1, b_1 = 3, b_2 = 4, b_3 = 5, b_4 = 8, b_5 = 10, b_6 = 12, b_7 = 13
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Then $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = -1, a_5 = 0, a_6 = 0, a_7 = 1$ and so the corresponding sequence is

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  011-1001
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