

Sequences with nonnegative partial sums

Problem 1: How many sequences a_1, a_2, \dots, a_{n-1} are there such that $a_i \leq 1$ and all partial sums are nonnegative?

Problem 2: How many functions

$$f : \{1, 2, \dots, n-1\} \rightarrow \{1, 0, -1, \dots, -(n-2)\}$$

are there such that $f(1) + f(2) + \dots + f(k) \geq 0$ for all $k \leq n-1$?

For $n = 2$, there are 2 such sequences; namely 1 and 0, which correspond to the balanced strings of brackets LLRR and LRLR.

For $n = 3$, there are 5 such sequences:

$$11 \quad 10 \quad 1-1 \quad 01 \quad 00$$

For $n = 4$, there are 14 such sequences:

$$\begin{array}{ccccccc} 111 & 110 & 11-1 & 11-2 & 101 & 100 & 10-1 \\ 1-11 & 1-10 & 011 & 010 & 01-1 & 001 & 000 \end{array}$$

For $n = 5$, there are 42 such sequences:

$$\begin{array}{cccccccc} 1111 & 1110 & 111-1 & 111-2 & 111-3 & 1101 & 1100 \\ 110-1 & 110-2 & 11-11 & 11-10 & 11-1-1 & 11-21 & 11-20 \\ 1011 & 1010 & 101-1 & 101-2 & 1001 & 1000 & 100-1 \\ 10-11 & 10-10 & 1-111 & 1-110 & 1-11-1 & 1-101 & 1-100 \\ 0111 & 0110 & 011-1 & 011-2 & 0101 & 0100 & 010-1 \\ 01-11 & 01-10 & 0011 & 0010 & 001-1 & 0001 & 0000 \end{array}$$

For $n = 6$, there are 132 such sequences:

11111 11110 1111-1 1111-2 1111-3 1111-4 11101
 11100 1110-1 1110-2 1110-3 111-11 111-10 111-1-1
 111-1-2 111-21 111-20 111-2-1 111-31 111-30 11011
 11010 1101-1 1101-2 1101-3 11001 11000 1100-1
 1100-2 110-11 110-10 110-1-1 110-21 110-20 11-111
 11-110 11-11-1 11-11-2 11-101 11-100 11-10-1 11-1-11
 11-1-10 11-211 11-210 11-21-1 11-201 11-200 10111
 10110 1011-1 1011-2 1011-3 10101 10100 1010-1
 1010-2 101-11 101-10 101-1-1 101-21 101-20 10011
 10010 1001-1 1001-2 10001 10000 1000-1 100-11
 100-10 10-111 10-110 10-11-1 10-101 10-100 1-1111
 1-1110 1-111-1 1-111-2 1-1101 1-1100 1-110-1 1-11-11
 1-11-10 1-1011 1-1010 1-101-1 1-1001 1-1000 01111
 01110 0111-1 0111-2 0111-3 01101 01100 0110-1
 0110-2 011-11 011-10 011-1-1 011-21 011-20 01011
 01010 0101-1 0101-2 01001 01000 0100-1 010-11
 010-10 01-111 01-110 01-11-1 01-101 01-100 00111
 00110 0011-1 0011-2 00101 00100 0010-1 001-11
 001-10 00011 00010 0001-1 00001 00000

In fact, for any integer $n \geq 1$, the number of such sequences is the Catalan number c_n .

Connection with the first bracketing problem

Given a balanced string of brackets, we obtain the corresponding sequence, with $a_i \leq 1$ and nonnegative partial sums, as follows.

Write down the sequence $1, 2, \dots, 2n$ under the left and right brackets in the string from left to right. For $0 \leq k \leq n-1$, let b_k be the integer corresponding to the $k+1$ -th left brackets. Then define a_k by

$$a_k = b_{k-1} - b_k + 2, \quad (1 \leq k \leq n-1).$$

Given a sequence a_1, a_2, \dots, a_{n-1} of integers such that $a_i \leq 1$ and all partial sums are non-negative, we obtain a balanced string of brackets by determine the positions of the left btrackets, as follows. Suppose the j -th left bracket corresponds to the integer k , then the $(j+1)$ -th left bracket corresponds to the integer $k - a_k + 2$.

1. Construct balanced strings of brackets corresponding to the following sequences of integers:
- (i) 11–1011
- (ii) 0111–10111–2

Solution.

- (i) The first left bracket always corresponds to 1. Then the second left bracket corresponds to $1 - a_1 + 2 = 1 - 1 + 2 = 2$; the 3rd is $2 - a_2 + 2 = 3$; the 4th is $3 - a_3 + 2 = 6$; the 5th is $6 - a_4 + 2 = 8$; the 6th is $8 - a_5 + 2 = 9$ and the 7th is $9 - a_6 + 2 = 10$.

Hence the corresponding balanced strings of brackets is

LLRRLRLLLRRR

- (ii) In this case, the left brackets correspond to the integers

1, 3, 4, 5, 6, 9, 11, 12, 13, 14, 18

Hence the corresponding balanced strings of brackets is

LRLLLRRLRLLLRRRLRRR

2. For the following balanced strings of brackets, construct the corresponding sequence a_1, a_2, \dots, a_{n-1} of integers such that $a_i \leq 1$, and all partial sums are non-negative.
- (i) LLRLRLLLRLRRR
- (ii) LRLLLRRLRLLLRRR

Solution.

- (i) The left brackets correspond to

$$b_0 = 1, b_1 = 2, b_2 = 4, b_3 = 6, b_4 = 7, b_5 = 8, b_6 = 10$$

Then $a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = 1, a_6 = 0$ so that the corresponding sequence is

100110 .

- (ii) The left brackets correspond to

$$b_0 = 1, b_1 = 3, b_2 = 4, b_3 = 5, b_4 = 8, b_5 = 10, b_6 = 12, b_7 = 13$$

Then $a_1 = 0, a_2 = 1, a_3 = 1, a_4 = -1, a_5 = 0, a_6 = 0, a_7 = 1$ and so the corresponding sequence is

011–1001 .