Planar Diagrams or Smiling faces

Consider a row of $2n$ dots. For each pair of dots, we join them by a unique arc from below. We call such a diagram a planer diagram. How many ways can we join the dots with nonintersecting arcs? That is, how many such planer diagrams with nonintersecting curves?

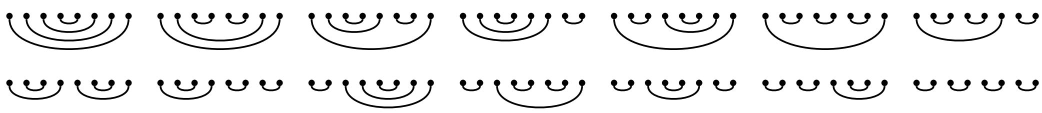
Find the number of ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting 2 of the points and lying below the points.

For $n = 1$, there is only 1 such planar diagram:

For $n = 2$, there are two such planar diagrams:  and

For $n = 3$, there are five such planar diagrams:

For $n = 4$, there are 14 such planars diagrams.



For $n = 5$, there are 42 ways:

In fact, for $2n$ dots, the number of such planar diagrams is the Catalan number c_n.

Connection with the first bracketing problem

Given a balanced string of brackets, we obtain the corresponding planar diagram as follows. Find the rightmost (remaining) $L-$bracket and the first remaining $R-$bracket to its right. We replace this pair of brackets by two dots which we join by an arc. This procedure is repeated until no brackets remain.

Given a planar diagram with nonintersecting arcs. To obtain the balanced string of brackets, we replace the left end of a loop with an $L-$ bracket and the right end with an $R-$bracket.

1. Construct balanced strings of brackets corresponding to the following planar diagrams.

(*i*)  (*ii*)  (*iii*) 

Solution.

The corresponding balanced strings of brackets are:

(*i*) ((()())()) (*ii*) (())(()())() (*iii*) ((()))()(()).

2. For each of the following balanced strings of brackets, construct the corresponding planar diagrams.

(*i*) (())()(()())

(*ii*) (()())(())()()

(*iii*) (()()())()(())(()())

Solution.

The corresponding planar diagrams are:

(*i*)  (*ii*)  (*iii*)