

*Sequences of  $n$  Pairs of integers*

*Problem 1:* How many sequences  $a_1 a_2 \dots a_{2n}$  can be formed from the integers  $1, 1, 2, 2, \dots, n, n$  such that

- (i) the first occurrences of  $1, 2, \dots, n$  appear in increasing order, and
- (ii) there is no subsequence of the form  $jkjk$ .

*Problem 2:* How many permutations  $a_1 a_2 \dots a_{2n}$  of the multiset  $\{1^2, 2^2, \dots, n^2\}$  are there such that

- (i) the first occurrences of  $1, 2, \dots, n$  appear in increasing order, and
- (ii) there is no subsequence of the form  $\dots jk \dots jk \dots$ .

For  $n = 1$ , there is only 1 such sequence; namely 11

For  $n = 2$ , there are 2 such sequences: 1221 and 1122

For  $n = 3$  there are 5 such sequences:

123321 122331 122133 112332 112233

For  $n = 4$ , there are 14 such sequences:

12344321 12334421 12332441 12332144  
 12234431 12233441 12233144 12213443 12213344  
 11234432 11233442 11233244 11223443 11223344

For  $n = 5$ , there are 42 such sequences:

1234554321 1234455321 1234435521 1234432551 1234432155  
 1233455421 1233445521 1233442551 1233442155  
 1233245541 1233244551 1233244155 1233214554 1233214455  
 1223455431 1223445531 1223443551 1223443155  
 1223345541 1223344551 1223344155 1223314554 1223314455  
 1221345543 1221344553 1221344355 1221334554 1221334455  
 1123455432 1123445532 1123443552 1123443255  
 1123345542 1123344552 1123344255 1123324554 1122324455  
 1122345543 1122344553 1122344355 1122334554 1122334455

For  $n = 6$ , there are 132 such sequences:

123456654321  
 123455664321 123455466321 123455436621 123455432661 123455432166  
 123445665321 123445566321 123445536621 123445532661 123445532166  
 123443566521 123443556621 123443552661 123443552166 123443256651  
 123443255661 123443255166 123443215665 123443215566  
 123345665421 123345566421 123345546621 123345543661 123345543166  
 123344566521 123344556621 123344552661 123344552166 123344256651  
 123344255661 123344255166 123344215665 123344215566  
 123324566541 123324556641 123324554661 123324554166 123324456651  
 123324455661 123324455166 123324415665 123324415566 123321456654  
 123321455664 123321455466 123321445665 123321445566  
 122345665431 122345566431 122345546631 122345543661 122345543166  
 122344566531 122344556631 122344553661 122344553166 122344356651  
 122344355661 122344355166 122344315665 122344315566  
 122334566541 122334556641 122334554661 122334554166 122334456651  
 122334455661 122334455166 122334415665 122334415566 122331456654  
 122331455664 122331455466 122331445665 122331445566  
 122134566543 122134556643 122134554663 122134554366 122134456653  
 122134455663 122134455366 122134435665 122134435566 122133456654  
 122133455664 122133455466 122133445665 122133445566  
 112345665432 112345566432 112345546632 112345543662 112345543266  
 112344566532 112344556632 112344554663 112344553266 112344356652  
 112344355662 112344355266 112344325665 112344325566  
 112334566542 112334556642 112334554662 112334554266 112334456652  
 112334455662 112334455266 112334425665 112334425566 112332456654  
 112332455664 112332455466 112332445665 112332445566  
 112234566543 112234556643 112234554663 112234554366 112234456653  
 112234455663 112234455366 112234435665 112234435566 112233456654  
 112233455664 112233455466 112233445665 112233445566

*Connection with the first bracketing problem*

Given a balanced string of  $n$  left and  $n$  right brackets, we obtain the corresponding sequence of pairs as follows. There are  $2n$  positions. Label the positions of the left brackets by the increasing sequence  $1, 2, \dots, n$  and then label the matching right bracket with the same integer as the left. That is, if the  $i$ -th position of the right bracket is matching with the left bracket, labelled by the integer  $j$ , then label this  $i$ -th position by the integer  $j$ .

Given a sequence of  $n$  pairs  $1, 1, 2, 2, \dots, n, n$  with the given conditions, we obtain the balanced string of brackets as follows. We replace each of the integers in the first occurrences of  $1, 2, \dots, n$  appeared in the increasing order by the left bracket L, and then replace each of the remaining integers by the right bracket R.

1. Construct balanced strings of brackets corresponding to the following sequence of pairs.

- (i) 1233442551
- (ii) 122134455366
- (iii) 123321445665

*Solution.*

The corresponding balanced strings of brackets are:

- (i) LLLRLRRLRR
- (ii) LLRLLRLRRLR
- (iii) LLLRRRLRLLRR

2. For each of the following balanced strings of brackets, construct the corresponding sequences of pairs.

- (i) LLRRLRLLRRLR
- (ii) LLRLRLLRRLRRLR
- (iii) LLRLRRLRLLRLLRRLRRLR

*Solution.*

The corresponding sequences are:

- (i) 122133455664
- (ii) 12233145546677
- (iii) 12233441556776899(10)(10)8