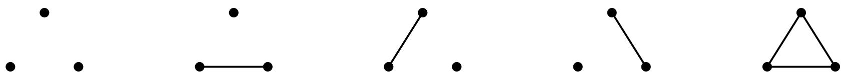
Partitions

Problem: How many partitions $\{B_1, B_2, \ldots, B_n\}$ of $\{1, 2, \ldots, n\}$ are there such that if the numbers $1, 2, \ldots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, B_2, \ldots, B_n$ are pairwise disjoint?

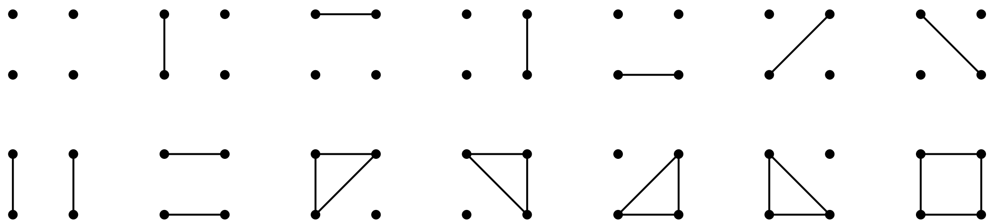
For $n = 1$, there is only 1 ways; namely

For $n = 2$, there are 2 ways; namely and

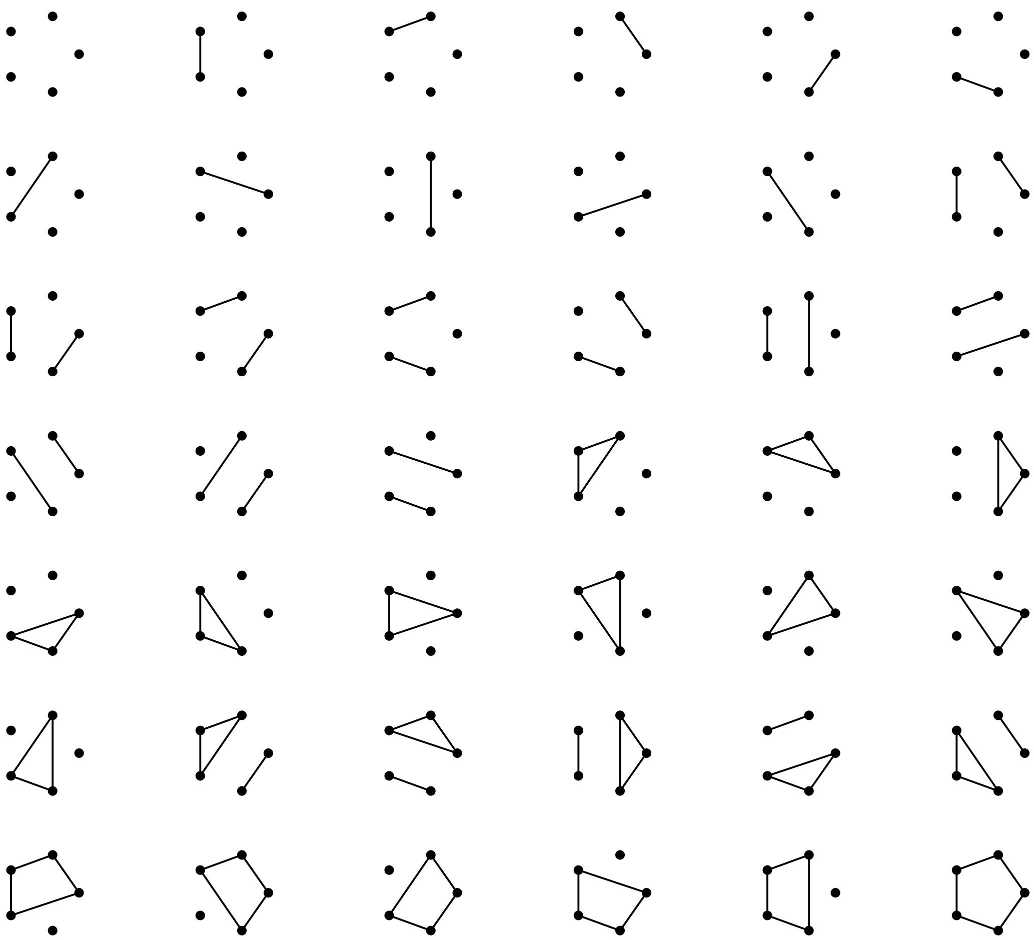
For $n = 3$, there are 5 ways:



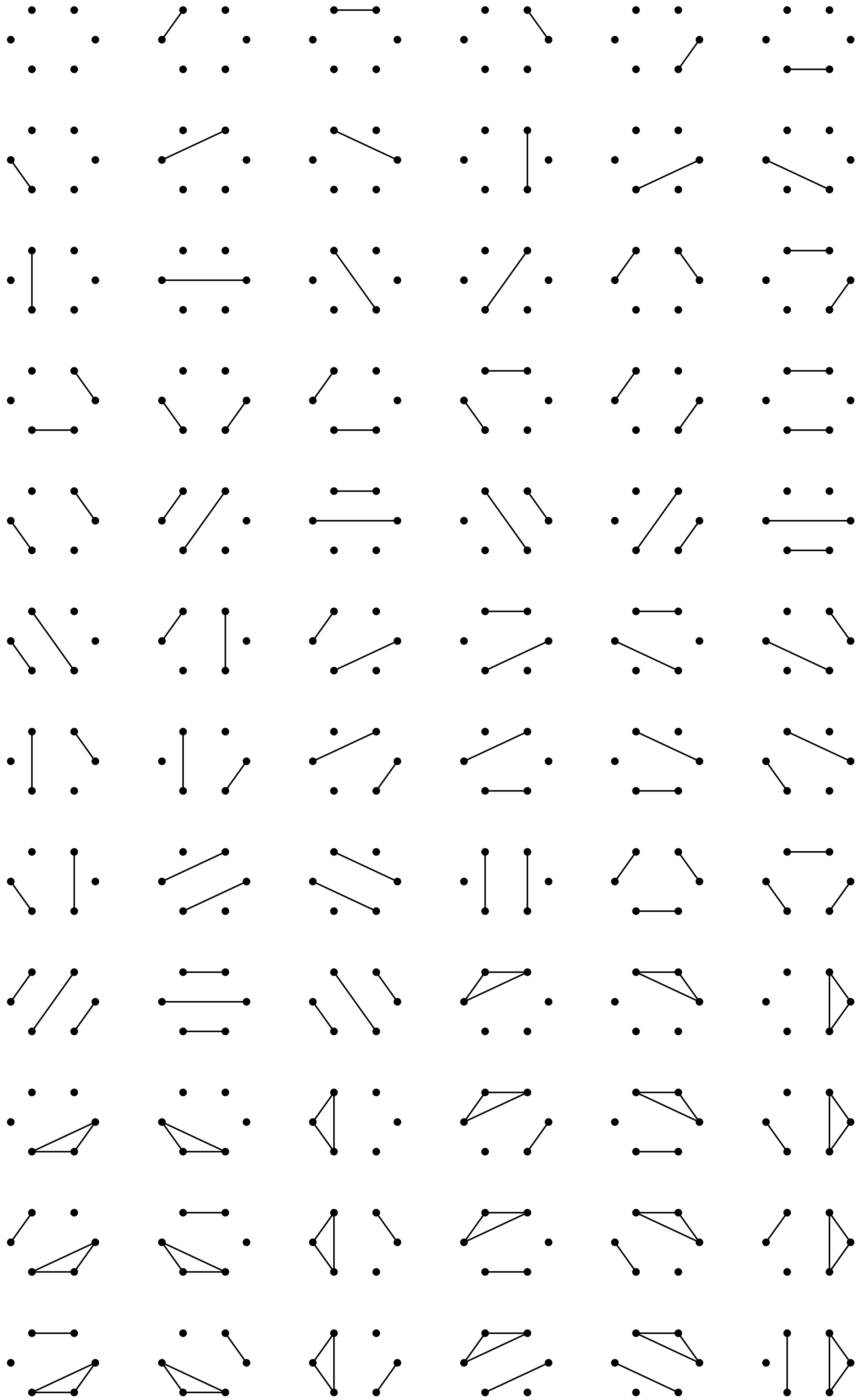
For $n = 4$, there are 14 ways:

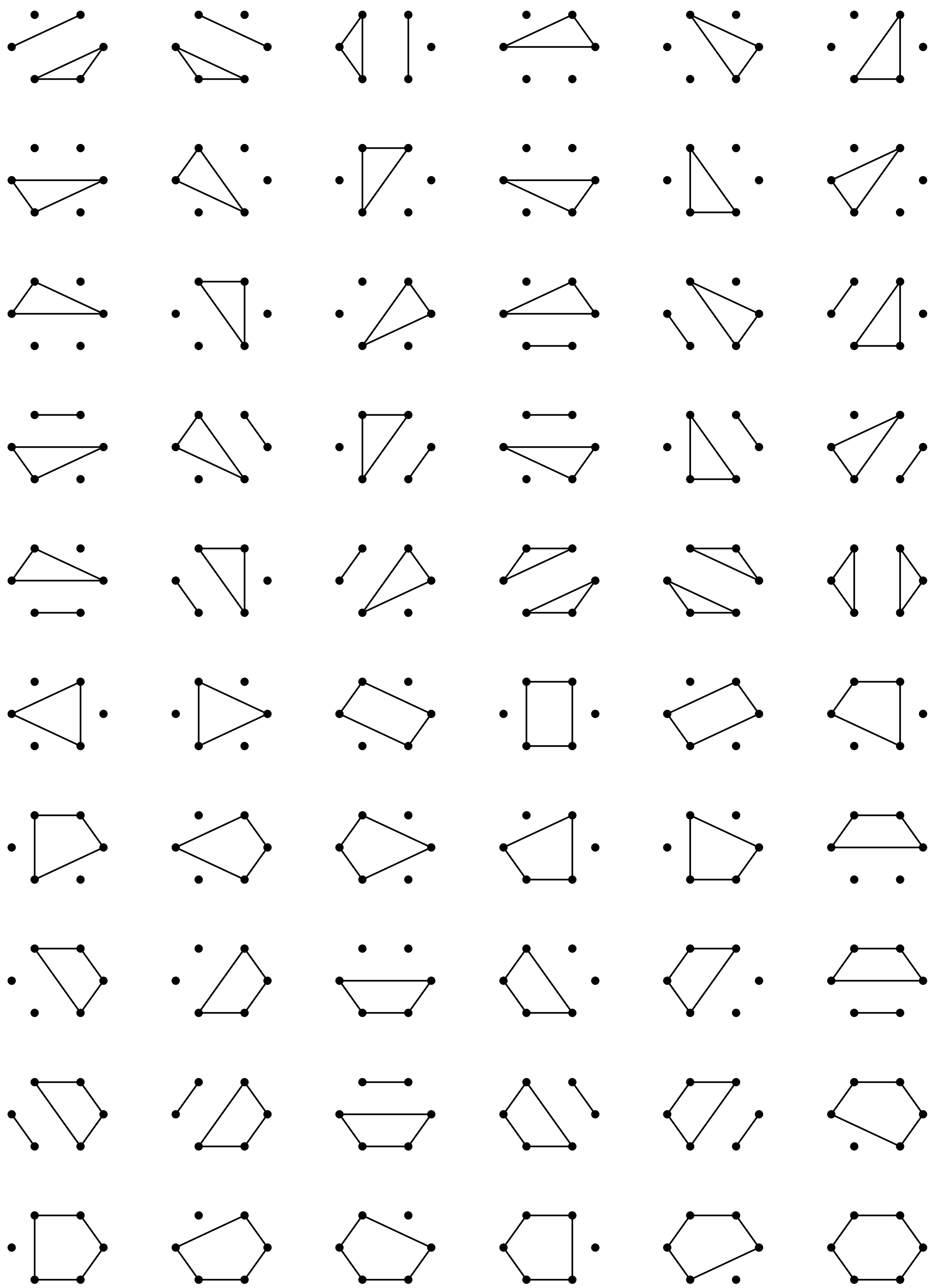


For $n = 5$ there are 42 ways:



For $n = 6$, there are 132 such partitions:





In fact, for any n, the number of such partitions is the Catalan number c_n.

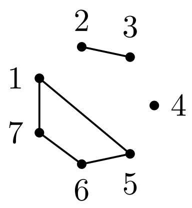
Connection with the first bracket problem

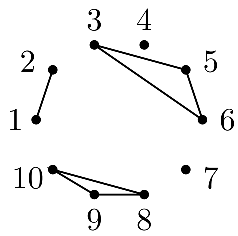
Given a balanced string of n left and n right brackets, we obtain the corresponding partition as follows. First we choose a starting position, choose clockwise direction to draw n dots and name these dots from 1 to n. Also name the left brackets in the balanced string from 1 to n. Read from the left of the balanced string: if there is a block of k right brackets R, then join the integers corresponding to the matching L by a $k-$gon [1-gon is a point, 2-gon is a line].

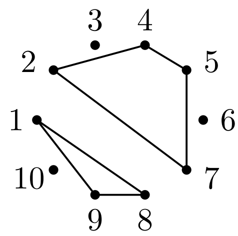
Given such a partition, we first choose a starting position for the partition and choose clockwise direction to construct the corresponding balanced string of bracket, by reversing the above procedures.

Remark: This is closely related to the noncrossing partitions problem and the noncrossing Murasaki diagrams problem.

1. Construct balanced strings of brackets corresponding to the following partition:

(i) 

(ii) 

(iii) 

Solution.

The corresponding balanced strings of brackets are:

(i) LLLRRLRLLLRRRR

(ii) LLRRLLRLLRRRLRLLLRRR

(iii) LLLRLLLRLRRRRLLLRRRR

2. For the following balanced strings of brackets, construct the corresponding partitions:

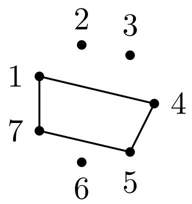
(i) LLRLRLLLRLRRRR

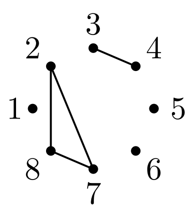
(ii) LRLLLRRLRLRLLRRR

(iii) LLLRRRLLLLLLLRRRRLLRRRRR

Solution.

The corresponding partitions are:

(i) 

(ii) 

(iii)