

Pairs of permutations

Problem 1: How many pairs (u, v) of permutations of $\{1, 2, \dots, n\}$ are there such that u and v have a total of $n + 1$ cycles and $uv = (1, 2, \dots, n)$.

For $n = 1$, there is only 1 pair: $(1) \cdot (1)$

For $n = 2$, there are 2 pairs: $(1, 2) \cdot (1)(2)$ and $(1)(2) \cdot (1, 2)$

For $n = 3$, there are 5 pairs:

$$\begin{aligned} &(1, 2, 3) \cdot (1)(2)(3) \quad (1, 2)(3) \cdot (1, 3)(2) \quad (1, 3)(2) \cdot (1)(2, 3) \\ &(1)(2, 3) \cdot (1, 2)(3) \quad (1)(2)(3) \cdot (1, 2, 3) \end{aligned}$$

For $n = 4$, there are 14 pairs:

$$\begin{aligned} &(1, 2, 3, 4) \cdot (1)(2)(3)(4) \quad (1, 2, 3)(4) \cdot (1, 4)(2)(3) \quad (1, 2, 4)(3) \cdot (1)(2)(3, 4) \\ &(1, 3, 4)(2) \cdot (1)(2, 3)(4) \quad (1, 2)(3, 4) \cdot (1, 3)(2)(4) \quad (1, 2)(3)(4) \cdot (1, 3, 4)(2) \\ &(1, 3)(2)(4) \cdot (1, 4)(2, 3) \quad (1, 4)(2, 3) \cdot (1)(2, 4)(3) \quad (1, 4)(2)(3) \cdot (1)(2, 3, 4) \\ &(1)(2, 3, 4) \cdot (1, 2)(3)(4) \quad (1)(2, 3)(4) \cdot (1, 2, 4)(3) \quad (1)(2, 4)(3) \cdot (1, 2)(3, 4) \\ &(1)(2)(3, 4) \cdot (1, 2, 3)(4) \quad (1)(2)(3)(4) \cdot (1, 2, 3, 4) \end{aligned}$$

For $n = 5$, there are 42 such pairs:

$$\begin{aligned} &(1, 2, 3, 4, 5) \cdot (1)(2)(3)(4)(5) && (1, 2, 3, 4)(5) \cdot (1, 5)(2)(3)(4) \\ &(1, 2, 3, 5)(4) \cdot (1)(2)(3)(4, 5) && (1, 2, 4, 5)(3) \cdot (1)(2)(3, 4)(5) \\ &(1, 3, 4, 5)(2) \cdot (1)(2, 3)(4)(5) && (1, 2, 3)(4, 5) \cdot (1, 4)(2)(3)(5) \\ &(1, 2, 3)(4)(5) \cdot (1, 4, 5)(2)(3) && (1, 2, 4)(3)(5) \cdot (1, 5)(2)(3, 4) \\ &(1, 2, 5)(3, 4) \cdot (1)(2)(3, 5)(4) && (1, 2, 5)(3)(4) \cdot (1)(2)(3, 4, 5) \\ &(1, 3, 4)(2)(5) \cdot (1, 5)(2, 3)(4) && (1, 3, 5)(2)(4) \cdot (1)(2, 3)(4, 5) \\ &(1, 4, 5)(2, 3) \cdot (1)(2, 4)(3)(5) && (1, 4, 5)(2)(3) \cdot (1)(2, 3, 4)(5) \\ &(1, 2)(3, 4, 5) \cdot (1, 3)(2)(4)(5) && (1, 2)(3, 4)(5) \cdot (1, 3, 5)(2)(4) \\ &(1, 2)(3, 5)(4) \cdot (1, 3)(2)(4, 5) && (1, 2)(3)(4, 5) \cdot (1, 3, 4)(2)(5) \\ &(1, 2)(3)(4)(5) \cdot (1, 3, 4, 5)(2) && (1, 3)(2)(4)(5) \cdot (1, 4, 5)(2, 3) \\ &(1, 3)(2)(4, 5) \cdot (1, 4)(2, 3)(5) && (1, 4)(2, 3)(5) \cdot (1, 5)(2, 4)(3) \\ &(1, 4)(2)(3)(4) \cdot (1, 5)(2, 3, 4) && (1, 5)(2, 3, 4) \cdot (1)(2, 5)(3)(4) \\ &(1, 5)(2, 3)(4) \cdot (1)(2, 4, 5)(3) && (1, 5)(2, 4)(3) \cdot (1)(2, 5)(3, 4) \\ &(1, 5)(2)(3, 4) \cdot (1)(2, 3, 5)(4) && (1, 5)(2)(3)(4) \cdot (1)(2, 3, 4, 5) \\ &(1)(2, 3, 4, 5) \cdot (1, 2)(3)(4)(5) && (1)(2, 3, 4)(5) \cdot (1, 2, 5)(3)(4) \\ &(1)(2, 3, 5)(4) \cdot (1, 2)(3)(4, 5) && (1)(2, 4, 5)(3) \cdot (1, 2)(3, 4)(5) \\ &(1)(2, 3)(4, 5) \cdot (1, 2, 4)(3)(5) && (1)(2, 3)(4)(5) \cdot (1, 2, 4, 5)(3) \\ &(1)(2, 4)(3)(5) \cdot (1, 2, 5)(3, 4) && (1)(2, 5)(3, 4) \cdot (1, 2)(3, 5)(4) \\ &(1)(2, 5)(3)(4) \cdot (1, 2)(3, 4, 5) && (1)(2)(3, 4, 5) \cdot (1, 2, 3)(4)(5) \\ &(1)(2)(3, 4)(5) \cdot (1, 2, 3, 5)(4) && (1)(2)(3, 5)(4) \cdot (1, 2, 3)(4, 5) \\ &(1)(2)(3)(4, 5) \cdot (1, 2, 3, 4)(5) && (1)(2)(3)(4)(5) \cdot (1, 2, 3, 4, 5) \end{aligned}$$

For $n = 6$, there are 132 such pairs:

$(1, 2, 3, 4, 5, 6) \cdot (1)(2)(3)(4)(5)(6)$	$(1, 2, 3, 4, 6)(5) \cdot (1)(2)(3)(4)(5, 6)$
$(1, 2, 3, 6)(4, 5) \cdot (1)(2)(3)(4, 6)(5)$	$(1, 2, 6)(3, 4, 5) \cdot (1)(2)(3, 6)(4)(5)$
$(1, 6)(2, 3, 4, 5) \cdot (1)(2, 6)(3)(4)(5)$	$(1, 2, 3, 4, 5)(6) \cdot (1, 6)(2)(3)(4)(5)$
$(1, 2, 3, 5, 6)(4) \cdot (1)(2)(3)(4, 5)(6)$	$(1, 2, 3, 6)(4)(5) \cdot (1)(2)(3)(4, 5, 6)$
$(1, 2, 6)(3, 5)(4) \cdot (1)(2)(3, 6)(4, 5)$	$(1, 6)(2, 3, 5)(4) \cdot (1)(2, 6)(3)(4, 5)$
$(1, 2, 3, 5)(4)(6) \cdot (1, 6)(2)(3)(4, 5)$	$(1, 2, 5, 6)(3, 4) \cdot (1)(2)(3, 5)(4)(6)$
$(1, 2, 6)(3, 4)(5) \cdot (1)(2)(3, 5, 6)(4)$	$(1, 6)(2, 5)(3, 4) \cdot (1)(2, 6)(3, 5)(4)$
$(1, 2, 5)(3, 4)(6) \cdot (1, 6)(2)(3, 5)(4)$	$(1, 5, 6)(2, 3, 4) \cdot (1)(2, 5)(3)(4)(6)$
$(1, 6)(2, 3, 4)(5) \cdot (1)(2, 5, 6)(3)(4)$	$(1, 5)(2, 3, 4)(6) \cdot (1, 6)(2, 5)(3)(4)$
$(1, 2, 3, 4)(5, 6) \cdot (1, 5)(2)(3)(4)(6)$	$(1, 2, 3, 4)(5)(6) \cdot (1, 5, 6)(2)(3)(4)$
$(1, 2, 4, 5, 6)(3) \cdot (1)(2)(3, 4)(5)(6)$	$(1, 2, 4, 6)(3)(5) \cdot (1)(2)(3, 4)(5, 6)$
$(1, 2, 6)(3)(4, 5) \cdot (1)(2)(3, 4, 6)(5)$	$(1, 6)(2, 4, 5)(3) \cdot (1)(2, 6)(3, 4)(5)$
$(1, 2, 4, 5)(3)(6) \cdot (1, 6)(2)(3, 4)(5)$	$(1, 2, 5, 6)(3)(4) \cdot (1)(2)(3, 4, 5)(6)$
$(1, 2, 6)(3)(4)(5) \cdot (1)(2)(3, 4, 5, 6)$	$(1, 6)(2, 5)(3)(4) \cdot (1)(2, 6)(3, 4, 5)$
$(1, 2, 5)(3)(4)(6) \cdot (1, 6)(2)(3, 4, 5)$	$(1, 5, 6)(2, 4)(3) \cdot (1)(2, 5)(3, 4)(6)$
$(1, 6)(2, 4)(3)(5) \cdot (1)(2, 5, 6)(3, 4)$	$(1, 5)(2, 4)(3)(6) \cdot (1, 6)(2, 5)(3, 4)$
$(1, 2, 4)(3)(5, 6) \cdot (1, 5)(2)(3, 4)(6)$	$(1, 2, 4)(3)(5)(6) \cdot (1, 5, 6)(2)(3, 4)$
$(1, 4, 5, 6)(2, 3) \cdot (1)(2, 4)(3)(5)(6)$	$(1, 4, 6)(2, 3)(5) \cdot (1)(2, 4)(3)(5, 6)$
$(1, 6)(2, 3)(4, 5) \cdot (1)(2, 4, 6)(3)(5)$	$(1, 4, 5)(2, 3)(6) \cdot (1, 6)(2, 4)(3)(5)$
$(1, 5, 6)(2, 3)(4) \cdot (1)(2, 4, 5)(3)(6)$	$(1, 6)(2, 3)(4)(5) \cdot (1)(2, 4, 5, 6)(3)$
$(1, 5)(2, 3)(4)(6) \cdot (1, 6)(2, 4, 5)(3)$	$(1, 4)(2, 3)(5, 6) \cdot (1, 5)(2, 4)(3)(6)$
$(1, 4)(2, 3)(5)(6) \cdot (1, 5, 6)(2, 4)(3)$	$(1, 2, 3)(4, 5, 6) \cdot (1, 4)(2)(3)(5)(6)$
$(1, 2, 3)(4, 6)(5) \cdot (1, 4)(2)(3)(5, 6)$	$(1, 2, 3)(4, 5)(6) \cdot (1, 4, 6)(2)(3)(5)$
$(1, 2, 3)(4)(5, 6) \cdot (1, 4, 5)(2)(3)(6)$	$(1, 2, 3)(4)(5)(6) \cdot (1, 4, 5, 6)(2)(3)$
$(1, 3, 4, 5, 6)(2) \cdot (1)(2, 3)(4)(5)(6)$	$(1, 3, 4, 6)(2)(5) \cdot (1)(2, 3)(4)(5, 6)$
$(1, 3, 6)(2)(4, 5) \cdot (1)(2, 3)(4, 6)(5)$	$(1, 6)(2)(3, 4, 5) \cdot (1)(2, 3, 6)(4)(5)$
$(1, 3, 4, 5)(2)(6) \cdot (1, 6)(2, 3)(4)(5)$	$(1, 3, 5, 6)(2)(4) \cdot (1)(2, 3)(4, 5)(6)$
$(1, 3, 6)(2)(4)(5) \cdot (1)(2, 3)(4, 5, 6)$	$(1, 6)(2)(3, 5)(4) \cdot (1)(2, 3, 6)(4, 5)$
$(1, 3, 5)(2)(4)(6) \cdot (1, 6)(2, 3)(4, 5)$	$(1, 5, 6)(2)(3, 4) \cdot (1)(2, 3, 5)(4)(6)$
$(1, 6)(2)(3, 4)(5) \cdot (1)(2, 3, 5, 6)(4)$	$(1, 5)(2)(3, 4)(6) \cdot (1, 6)(2, 3, 5)(4)$
$(1, 3, 4)(2)(5, 6) \cdot (1, 5)(2, 3)(4)(6)$	$(1, 3, 4)(2)(5)(6) \cdot (1, 5, 6)(2, 3)(4)$
$(1, 4, 5, 6)(2)(3) \cdot (1)(2, 3, 4)(5)(6)$	$(1, 4, 6)(2)(3)(5) \cdot (1)(2, 3, 4)(5, 6)$
$(1, 6)(2)(3)(4, 5) \cdot (1)(2, 3, 4, 6)(5)$	$(1, 4, 5)(2)(3)(6) \cdot (1, 6)(2, 3, 4)(5)$

$(1, 5, 6)(2)(3)(4) \cdot (1)(2, 3, 4, 5)(6)$	$(1, 6)(2)(3)(4)(5) \cdot (1)(2, 3, 4, 5, 6)$
$(1, 5)(2)(3)(4)(6) \cdot (1, 6)(2, 3, 4, 5)$	$(1, 4)(2)(3)(5, 6) \cdot (1, 5)(2, 3, 4)(6)$
$(1, 4)(2)(3)(5)(6) \cdot (1, 5, 6)(2, 3, 4)$	$(1, 3)(2)(4, 5, 6) \cdot (1, 4)(2, 3)(5)(6)$
$(1, 3)(2)(4, 6)(5) \cdot (1, 4)(2, 3)(5, 6)$	$(1, 3)(2)(4, 5)(6) \cdot (1, 4, 6)(2, 3)(5)$
$(1, 3)(2)(4)(5, 6) \cdot (1, 4, 5)(2, 3)(6)$	$(1, 3)(2)(4)(5)(6) \cdot (1, 4, 5, 6)(2, 3)$
$(1, 2)(3, 4, 5, 6) \cdot (1, 3)(2)(4)(5)(6)$	$(1, 2)(3, 4, 6)(5) \cdot (1, 3)(2)(4)(5, 6)$
$(1, 2)(3, 6)(4, 5) \cdot (1, 3)(2)(4, 6)(5)$	$(1, 2)(3, 4, 5)(6) \cdot (1, 3, 6)(2)(4)(5)$
$(1, 2)(3, 5, 6)(4) \cdot (1, 3)(2)(4, 5)(6)$	$(1, 2)(3, 6)(4)(5) \cdot (1, 3)(2)(4, 5, 6)$
$(1, 2)(3, 5)(4)(6) \cdot (1, 3, 6)(2)(4, 5)$	$(1, 2)(3, 4)(5, 6) \cdot (1, 3, 5)(2)(4)(6)$
$(1, 2)(3, 4)(5)(6) \cdot (1, 3, 5, 6)(2)(4)$	$(1, 2)(3)(4, 5, 6) \cdot (1, 3, 4)(2)(5)(6)$
$(1, 2)(3)(4, 6)(5) \cdot (1, 3, 4)(2)(5, 6)$	$(1, 2)(3)(4, 5)(6) \cdot (1, 3, 4, 6)(2)(5)$
$(1, 2)(3)(4)(5, 6) \cdot (1, 3, 4, 5)(2)(6)$	$(1, 2)(3)(4)(5)(6) \cdot (1, 3, 4, 5, 6)(2)$
$(1)(2, 3, 4, 5, 6) \cdot (1, 2)(3)(4)(5)(6)$	$(1)(2, 3, 4, 6)(5) \cdot (1, 2)(3)(4)(5, 6)$
$(1)(2, 3, 6)(4, 5) \cdot (1, 2)(3)(4, 6)(5)$	$(1)(2, 6)(3, 4, 5) \cdot (1, 2)(3, 6)(4)(5)$
$(1)(2, 3, 4, 5)(6) \cdot (1, 2, 6)(3)(4)(5)$	$(1)(2, 3, 5, 6)(4) \cdot (1, 2)(3)(4, 5)(6)$
$(1)(2, 3, 6)(4)(5) \cdot (1, 2)(3)(4, 5, 6)$	$(1)(2, 6)(3, 5)(4) \cdot (1, 2)(3, 6)(4, 5)$
$(1)(2, 3, 5)(4)(6) \cdot (1, 2, 6)(3)(4, 5)$	$(1)(2, 5, 6)(3, 4) \cdot (1, 2)(3, 5)(4)(6)$
$(1)(2, 6)(3, 4)(5) \cdot (1, 2)(3, 5, 6)(4)$	$(1)(2, 5)(3, 4)(6) \cdot (1, 2, 6)(3, 5)(4)$
$(1)(2, 3, 4)(5, 6) \cdot (1, 2, 5)(3)(4)(6)$	$(1)(2, 3, 4)(5)(6) \cdot (1, 2, 5, 6)(3)(4)$
$(1)(2, 4, 5, 6)(3) \cdot (1, 2)(3, 4)(5)(6)$	$(1)(2, 4, 6)(3)(5) \cdot (1, 2)(3, 4)(5, 6)$
$(1)(2, 6)(3)(4, 5) \cdot (1, 2)(3, 4, 6)(5)$	$(1)(2, 4, 5)(3)(6) \cdot (1, 2, 6)(3, 4)(5)$
$(1)(2, 5, 6)(3)(4) \cdot (1, 2)(3, 4, 5)(6)$	$(1)(2, 6)(3)(4)(5) \cdot (1, 2)(3, 4, 5, 6)$
$(1)(2, 5)(3)(4)(6) \cdot (1, 2, 6)(3, 4, 5)$	$(1)(2, 4)(3)(5, 6) \cdot (1, 2, 5)(3, 4)(6)$
$(1)(2, 4)(3)(5)(6) \cdot (1, 2, 5, 6)(3, 4)$	$(1)(2, 3)(4, 5, 6) \cdot (1, 2, 4)(3)(5)(6)$
$(1)(2, 3)(4, 6)(5) \cdot (1, 2, 4)(3)(5, 6)$	$(1)(2, 3)(4, 5)(6) \cdot (1, 2, 4, 6)(3)(5)$
$(1)(2, 3)(4)(5, 6) \cdot (1, 2, 4, 5)(3)(6)$	$(1)(2, 3)(4)(5)(6) \cdot (1, 2, 4, 5, 6)(3)$
$(1)(2)(3, 4, 5, 6) \cdot (1, 2, 3)(4)(5)(6)$	$(1)(2)(3, 4, 6)(5) \cdot (1, 2, 3)(4)(5, 6)$
$(1)(2)(3, 6)(4, 5) \cdot (1, 2, 3)(4, 6)(5)$	$(1)(2)(3, 4, 5)(6) \cdot (1, 2, 3, 6)(4)(5)$
$(1)(2)(3, 5, 6)(4) \cdot (1, 2, 3)(4, 5)(6)$	$(1)(2)(3, 6)(4)(5) \cdot (1, 2, 3)(4, 5, 6)$
$(1)(2)(3, 5)(4)(6) \cdot (1, 2, 3, 6)(4, 5)$	$(1)(2)(3, 4)(5, 6) \cdot (1, 2, 3, 5)(4)(6)$
$(1)(2)(3, 4)(5)(6) \cdot (1, 2, 3, 5, 6)(4)$	$(1)(2)(3)(4, 5, 6) \cdot (1, 2, 3, 4)(5)(6)$
$(1)(2)(3)(4, 6)(5) \cdot (1, 2, 3, 4)(5, 6)$	$(1)(2)(3)(4, 5)(6) \cdot (1, 2, 3, 4, 6)(5)$
$(1)(2)(3)(4)(5, 6) \cdot (1, 2, 3, 4, 5)(6)$	$(1)(2)(3)(4)(5)(6) \cdot (1, 2, 3, 4, 5, 6)$

In fact, for any integer $n \geq 1$, the number of such pairs is the Catalan number c_n .

Note that this is just the noncrossing partitions problem.