

Another type of noncrossing partitions

Problem 1: How many noncrossing partitions $\{B_1, B_2, \dots, B_k\}$ of $\{1, 2, \dots, n\}$ are there, such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$?

For $n = 1$, there is 1 such partition: 1

For $n = 2$, there are 2 such partitions: 12 1-2

For $n = 3$, there are 5 such partitions:

123 13-2 12-3 1-23 1-2-3

For $n = 4$, there are 14 such partitions:

1234 124-3 14-23 123-4 134-2 14-2-3 13-2-4
12-34 12-3-4 1-234 1-24-3 1-23-4 1-2-34 1-2-3-4

For $n = 5$, there are 42 such partitions:

12345 1235-4 125-34 15-234 1234-5 1245-3
125-3-4 15-24-3 124-3-5 145-23 15-23-4 14-23-5
123-45 123-4-5 1345-2 135-2-4 15-2-34 134-2-5
145-2-3 15-2-3-4 14-2-3-5 13-2-45 13-2-4-5 12-345
12-35-4 12-34-5 12-3-45 12-3-4-5 1-2345 1-235-4
1-25-34 1-234-5 1-245-3 1-25-3-4 1-24-3-5 1-23-45
1-23-4-5 1-2-345 1-2-35-4 1-2-34-5 1-2-3-45 1-2-3-4-5

For $n = 6$, there are 132 such partitions:

123456	12346-5	1236-45	126-345	16-2345	12345-6
12356-4	1236-4-5	126-35-4	16-235-4	1235-4-6	1256-34
126-34-5	16-25-34	125-34-6	156-234	16-234-5	15-234-6
1234-56	1234-5-6	12456-3	1246-3-5	126-3-45	16-245-3
1245-3-6	1256-3-4	126-3-4-5	16-25-3-4	125-3-4-6	156-24-3
16-24-3-5	15-24-3-6	124-3-56	124-3-5-6	1456-23	146-23-5
16-23-45	145-23-6	156-23-4	16-23-4-5	15-23-4-6	14-23-56
14-23-5-6	123-456	123-46-5	123-45-6	123-4-56	123-4-5-6
13456-2	1346-2-5	136-2-45	16-2-345	1345-2-6	1356-2-4
136-2-4-5	16-2-35-4	135-2-4-6	156-2-34	16-2-34-5	15-2-34-6
134-2-56	134-2-5-6	1456-2-3	146-2-3-5	16-2-3-45	145-2-3-6
156-2-3-4	16-2-3-4-5	15-2-3-4-6	14-2-3-56	14-2-3-5-6	13-2-456
13-2-46-5	13-2-45-6	13-2-4-56	13-2-4-5-6	12-3456	12-346-5
12-36-45	12-345-6	12-356-4	12-36-4-5	12-35-4-6	12-34-56
12-34-5-6	12-3-456	12-3-46-5	12-3-45-6	12-3-4-56	12-3-4-5-6
1-23456	1-2346-5	1-236-45	1-26-345	1-2345-6	1-2356-4
1-236-4-5	1-26-35-4	1-235-4-6	1-256-34	1-26-34-5	1-25-34-6
1-234-56	1-234-5-6	1-2456-3	1-246-3-5	1-26-3-45	1-245-3-6
1-256-3-4	1-26-3-4-5	1-25-3-4-6	1-24-3-56	1-24-3-5-6	1-23-456
1-23-46-5	1-23-45-6	1-23-4-56	1-23-4-5-6	1-2-3456	1-2-346-5
1-2-36-45	1-2-345-6	1-2-356-4	1-2-36-4-5	1-2-35-4-6	1-2-34-56
1-2-34-5-6	1-2-3-456	1-2-3-46-5	1-2-3-45-6	1-2-3-4-56	1-2-3-4-5-6

Connection with the first bracket problem

Given a balanced strings of n left and n right brackets, we obtain the noncrossing partition as follows: Label the left brackets from 1 to n in the given balanced string of brackets. For each group of right brackets together, we obtain the positions of the matching left brackets and group them together as part of the partition.

Given a noncrossing partition, we obtain a balanced string of left and right brackets as follows: For each part of the partitions, we associate the corresponding left brackets by the matching right brackets group together.

Remark: The noncrossing partitions correspond exactly to the noncrossing Murasaki diagrams. This is also closely related to the other partitions problem.

1. For each of the following balanced strings of brackets, construct the corresponding noncrossing partitions:
 - (i) LLRRLRLLRRLR
 - (ii) LLRLRRLRRLRRLR
 - (iii) LLRLRRLRRLRRLRRLRRLR

Solution.

The left brackets are grouped together, respectively, to obtain the corresponding noncrossing partitions as follows:

- (i) 12–3–46–5
- (ii) 13–2–45–6–7
- (iii) 14–2–3–5–67–8(10)–9

2. For each of the following noncrossing partitions, construct the corresponding balanced strings of left and right brackets:

- (i) 15–24–3–689–7
- (ii) 15–234–69(10)–78
- (iii) 13–2–456(10)(11)–789

Solution.

The corresponding balanced strings of brackets are:

- (i) LLLRLRRLRRLRRLRRLRRLR
- (ii) LLLLRRLRRLRRLRRLRRLR
- (iii) LLRLRRLRRLRRLRRLRRLRRLRRLRRLR

3. For each of the following noncrossing partitions, construct the corresponding balanced strings of left and right brackets:

- (i) 1567–23–4
- (ii) 12–356–4–7–89(10)
- (iii) 189(10)–2457–3–6

Solution.

The corresponding balanced strings of brackets are:

- (i) LLLRRLRLLLRRRR
- (ii) LLRRLRLLRRRLRLLLRRR
- (iii) LLLRLLLRLRRRLLLRRRR

4. For the following balanced strings of brackets, construct the corresponding partitions:

- (i) LLRLRLLLRLRRRR
- (ii) LRLLLRRLRLLRRR
- (iii) LLLRRLLLLLLLRRRLLRRRRR

Solution.

The corresponding noncrossing partitions are:

- (i) 1457–2–3–6
- (ii) 1–278–34–5–6
- (iii) 123–456(11)(12)–789(10)