Murasaki diagrams

Problem 1: Given \( n \) vertical lines. A diagram obtained by joining some of the \( n \) vertical lines with horizontal lines, is called a Murasaki diagram. How many noncrossing Murasaki diagrams with \( n \) vertical lines are there?

For \( n = 1 \), there is only 1 such diagram, just a vertical line:

For \( n = 2 \), there are 2 such diagrams:

For \( n = 3 \), there are 5 such diagrams:

For \( n = 4 \), there are 14 such diagrams:

For \( n = 5 \), there are 42 such diagrams:
For $n = 6$, there are 132 such diagrams:
In fact, for any $n$, the number of noncrossing Murasaki diagrams is the Catalan number $c_n$.

Connection with the first bracket problem

Given a balanced strings of $n$ left and $n$ right brackets, we obtain the corresponding Murasaki diagram as follows: Draw $n$ vertical lines and label them from 1 to $n$ from the left. Label the left brackets from 1 to $n$ in the given balanced string of brackets. Then each vertical line corresponds to a left bracket. For each group of right brackets together, we obtain the positions of the matching left brackets and then we join the corresponding vertical lines by a horizontal line from the top of the lines.

Given a Murasaki diagram, we obtain a balanced string of left and right brackets as follows: Label the $n$ lines from 1 to $n$ from the left. To each line, we associate a left bracket. For each group of the lines joined by a horizontal line, we associate the corresponding left brackets by the matching right brackets group together.

Remark: The noncrossing Murasaki diagrams correspond exactly to the noncrossing partitions. This is also closely related to the other partitions problem.
1. For each of the following balanced strings of brackets, construct the corresponding Murasaki diagram:

   (i) LLRLRLLLRLRR
   (ii) LLRLRLRLRLRLLR
   (iii) LLRLRLRLRLRLRLLRR

Solution.

The left brackets are grouped together, respectively, as follows:

(i) 12–3–46–5
(ii) 13–2–45–6–7
(iii) 14–2–3–5–67–8(10)–9

and so the corresponding Murasaki diagrams are

\[ \begin{array}{c}
\quad \\
(i) \\
\quad \\
(ii) \\
(iii) \\
\end{array} \]

2. For each of the following Murasaki diagrams, construct the corresponding balanced string of brackets:

\[ \begin{array}{c}
\quad \\
(i) \\
\quad \\
(ii) \\
(iii) \\
\end{array} \]

Solution.

The lines are grouped together as follows:

(i) 15–24–3–689–7
(ii) 15–234–69(10)–78
(iii) 13–2–456(10)(11)–789

Hence the corresponding balanced strings of brackets are:

(i) LLLRLRLLRLLRLLRR
(ii) LLLRRRLRLLLRRRLLRR
(iii) LLRLRLLLLLRRRLLLRRR