Increasing Functions on $\{1, 2, \ldots, n\}$.

Problem 1: Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a function such that

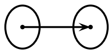
(i) f is increasing, i.e., if $i \leq j$, then $f(i) \leq f(j)$, and

(ii) for all i, $f(i) \leq i$.

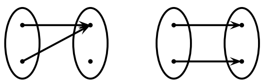
For any integer n, how many such functions are there?

Problem 2: How many sequences of the form $1 = a_1 \leq \cdots \leq a_n$ of integers are there, with $a_i \leq i$?

Problem 3: How many n-tuples $(1, a_2, \ldots, a_n)$ are there, with $a_i \leq i$, for all i, and $a_i \leq a_j$ whenever $i \leq j$?

When $n = 1$, there is only one such function which is given by the sequence (1) or the arrow diagram: 

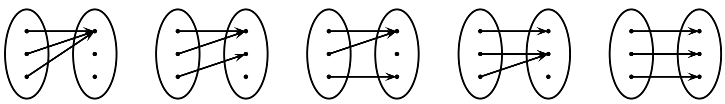
When $n = 2$, the two such functions are given by the following sequences of 2 numbers $(1, 1)$ and $(1, 2)$; or by the two arrow diagrams:



When $n = 3$, there are five such functions which are given by the sequences

$$(1, 1, 1),\ (1, 1, 2),\ (1, 1, 3),\ (1, 2, 2),\ (1, 2, 3);$$

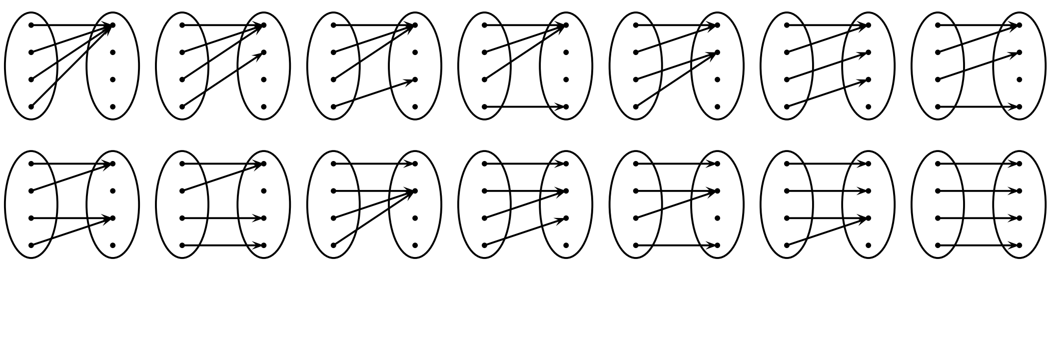
or by the arrow diagrams:



When $n = 4$, the functions are given by the following 14 sequences:

$$(1, 1, 1, 1),\ (1, 1, 1, 2),\ (1, 1, 1, 3),\ (1, 1, 1, 4),\ (1, 1, 2, 2),$$
$$(1, 1, 2, 3),\ (1, 1, 2, 4),\ (1, 1, 3, 3),\ (1, 1, 3, 4),\ (1, 2, 2, 2),$$
$$(1, 2, 2, 3),\ (1, 2, 2, 4),\ (1, 2, 3, 3),\ (1, 2, 3, 4).$$

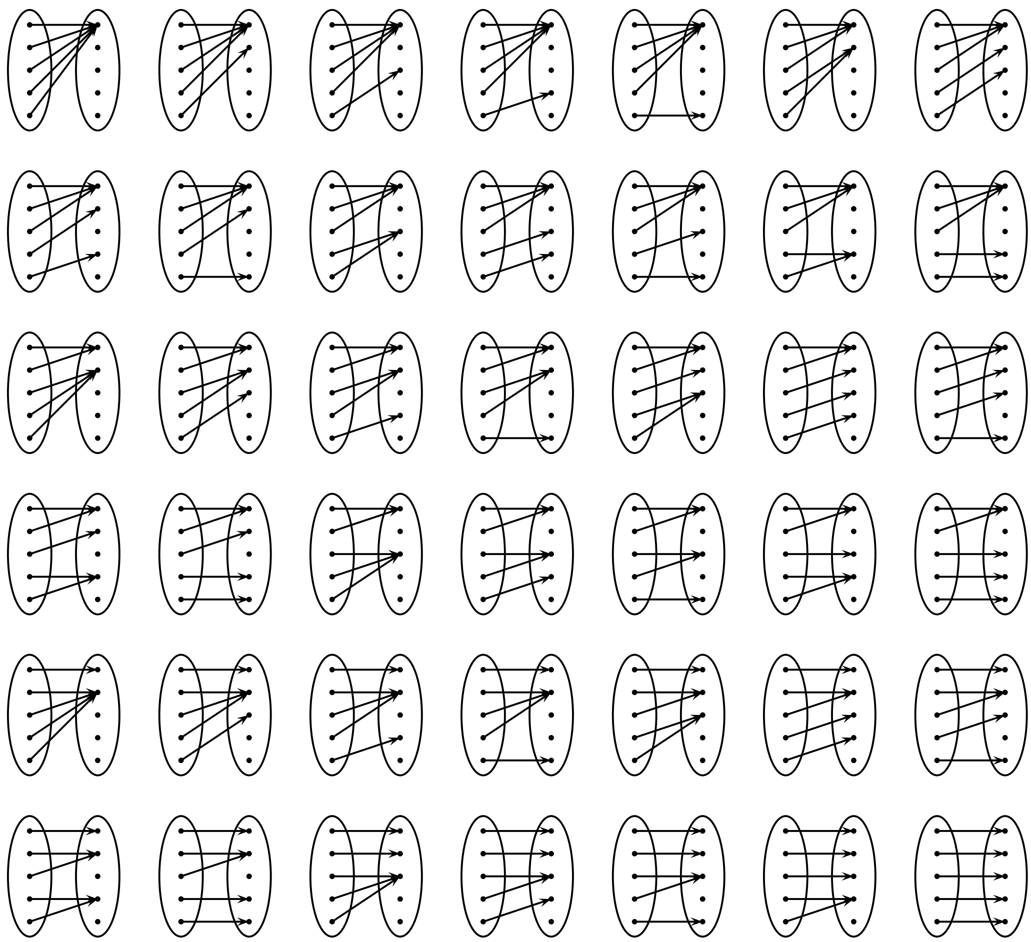
They are given by the following 14 arrow diagrams:



When $n = 5$, the functions are given by the following 42 sequences:

$$(1,1,1,1,1),\ (1,1,1,1,2),\ (1,1,1,1,3),\ (1,1,1,1,4),\ (1,1,1,1,5),\ (1,1,1,2,2),$$
$$(1,1,1,2,3),\ (1,1,1,2,4),\ (1,1,1,2,5),\ (1,1,1,3,3),\ (1,1,1,3,4),\ (1,1,1,3,5),$$
$$(1,1,1,4,4),\ (1,1,1,4,5),\ (1,1,2,2,2),\ (1,1,2,2,3),\ (1,1,2,2,4),\ (1,1,2,2,5),$$
$$(1,1,2,3,3),\ (1,1,2,3,4),\ (1,1,2,3,5),\ (1,1,2,4,4),\ (1,1,2,4,5),\ (1,1,3,3,3),$$
$$(1,1,3,3,4),\ (1,1,3,3,5),\ (1,1,3,4,4),\ (1,1,3,4,5),\ (1,2,2,2,2),\ (1,2,2,2,3),$$
$$(1,2,2,2,4),\ (1,2,2,2,5),\ (1,2,2,3,3),\ (1,2,2,3,4),\ (1,2,2,3,5),\ (1,2,2,4,4),$$
$$(1,2,2,4,5),\ (1,2,3,3,3),\ (1,2,3,3,4),\ (1,2,3,3,5),\ (1,2,3,4,4),\ (1,2,3,4,5).$$

They are given by the following 42 arrow diagrams:



In fact, for any integer $n > 0$, the number of such increasing functions is the Catalan number c_n.

Connection with the first bracketing problem

Given a balanced string of n left and n right brackets, we can construct the necessary functions by constructing a sequence of n numbers as follows: We write numbers in place of the left brackets by beginning at the left with 1 and increase the number we are writing by 1 each time you pass a right bracket.

Given a sequence of n numbers $(1, a_2, a_3, \ldots, a_n)$ with $a_i \leq i$ and $a_i \leq a_j$ for $i < j$, we can construct a balanced string of brackets as follows: For each integer 1 appeared in the sequence, we draw a left bracket; and when the number increase by k, we then draw k right brackets followed by as many left brackets as the numbers of k appeared in the sequence, and so on; and finally write down the balancing right brackets.

1. Construct the increasing functions corresponding to the following balanced strings of brackets:

(*i*) (())()(()()) (*ii*) (()())(())()()

(*iii*) (()()())()(())(()())

Solution.

The corresponding sequences are:

(*i*) $(1, 1, 3, 4, 4, 5)$ (*ii*) $(1, 1, 2, 4, 4, 6, 7)$

(*iii*) $(1, 1, 2, 3, 5, 6, 6, 8, 8, 9)$

2. Construct balanced strings of brackets corresponding to the following increasing functions:

(*i*) $(1, 1, 1, 3, 4, 4, 7, 7)$ (*ii*) $(1, 1, 2, 2, 2, 3, 4, 4, 5, 5)$

(*iii*) $(1, 1, 1, 3, 3, 4, 4, 4, 5, 5, 6)$

Solution.

The balanced strings of brackets are:

(*i*) ((())()(()))(()) (*ii*) (()((()()(()(())))))

(*iii*) ((())(()((()(()())))))