From the 1998 Examinations

1. For the following balanced string of brackets

\[ \text{LLRLRLLRRLRLR,} \]

(i) construct the corresponding planar diagram or smiling face;

(ii) construct the corresponding way of pairing off 14 points on the circumference of a circle, as 7 chords where no two chords intersect. [i.e., the hand-shaking problem for 14 people!]

(iii) construct the corresponding \((7, 7)\) tableau; that is, given 2 rows of boxes with 7 boxes in each row, place the numbers 1, 2, \ldots, 14 in the boxes so that the numbers increase from left to right and so that each number in the bottom row is larger than the number in the box above it.

2. (i) In a class of 800 students, 450 are doing Integral Calculus, 250 are doing Discrete Mathematics and 143 are doing both subjects. How many are not doing any of these subjects.

(ii) How many numbers between 1 and 250 are divisible by at least one of 3, 7 or 11?

3. Consider the following Boolean function

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<tr>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(f(x, y, z))</th>
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(i) Write down the corresponding Boolean expression in its disjunctive normal form.

(ii) Using the Kanaugh map method, write down the simple Boolean expression.

(iii) Draw the switching circuit for the simple Boolean expression in (ii)
(iv) Draw the digital logic circuit, using AND and OR gates and inverters for the simple Boolean expression in (ii)

4. (i) Draw a truth table for each of the following propositions, and determine whether it is a tautology or a contradiction or neither.
(a) \((p \Rightarrow q) \Rightarrow (\sim p \lor q)\),
(b) \((p \land q) \lor \sim(p \Rightarrow q)\).

(ii) Let \(\mathbb{Z}\) be the set of all integers, \(\mathbb{N}\) the set of all natural numbers and \(\mathbb{R}\) the set of all real numbers. Determine the truth and falsity of the following propositions.
(a) \((\forall x \in \mathbb{Z})(x^2 = 9) \Rightarrow (x = 3)\)
(b) \((\forall x \in \mathbb{N})(x^2 = 9) \Rightarrow (x = 3)\)
(c) \((\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x > 0) \Rightarrow (x = y^2)\)
(d) \((\forall x \in \mathbb{Z})(\exists y \in \mathbb{R})(x > 0) \Rightarrow (x = y^2)\)

5. (i) Prove that \(13^n - 5^n\) is divisible by 8 for all integers \(n \geq 1\).
(ii) Find a closed form for the generating function of the recurrence relation
\[x_n - 5x_{n-1} + 6x_{n-2} = 0, \; n \geq 2, \; x_0 = 3, \; x_1 = 7.\]

6. Solve the following recurrence relations:
(i) \(x_n - 5x_{n-1} + 6x_{n-2} = 0, \; n \geq 2, \; x_0 = 3, \; x_1 = 7\)
(ii) \(x_n - 8x_{n-1} + 16x_{n-2} = 0, \; n \geq 2, \; x_0 = 3, \; x_1 = 20.\)

7. (i) Find a closed form for the generating function of the recurrence relation
\[x_n = 2x_{n-1} - x_{n-2} + n, \; n \geq 2, \; x_0 = 1, \; x_1 = 2\]
and then find a formula for \(x_n\).
(ii) Prove that for any integer \(n \geq 1\)
\[
\frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \cdots + \frac{n + 4}{n(n + 1)(n + 2)} = \frac{n(3n + 7)}{2(n + 1)(n + 2)}.\]