1. Use a triple integral to find the volume of the solid region bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = a^2 \).

Solution.

Let \( R \) be the disc \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\} \).

Then the solid is described by:

\[ (x, y) \in R, \quad (x^2 + y^2) \leq z \leq a^2. \]

Volume \( = \int \int \int_{x^2+y^2}^a dV \)

\[ = \int \int_R (a^2 - x^2 - y^2) \, dA \]

\[ = \int_0^{2\pi} \int_0^a (a^2 - r^2) r \, dr \, d\theta \quad \text{(using polar co-ordinates)} \]

\[ = \int_0^{2\pi} \left( \frac{a^4}{2} - \frac{a^4}{4} \right) \, d\theta = \frac{\pi a^4}{2}. \]

2. Find the mass of the solid bounded by the surface \( z = 9 - x^2 - y^2 \) and the xy-plane, if the density of the solid at the point \( (x, y, z) \) is \( (1 + x^2 + y^2) \).

Solution.

Let \( R \) be the disc \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\} \).

Then the solid is described by:

\( (x, y) \in R, \quad 0 \leq z \leq (9 - x^2 - y^2). \)

Mass \( = \int \int \int_R 9 - x^2 - y^2 (1 + x^2 + y^2) \, dV \)

\[ = \int \int_R (1 + x^2 + y^2)(9 - x^2 - y^2) \, dA \]

\[ = \int_0^{2\pi} \int_0^3 (1 + r^2)(9 - r^2) r \, dr \, d\theta \quad \text{(using polar co-ordinates)} \]

\[ = \int_0^{2\pi} \int_0^3 (9r + 8r^3 - r^5) \, dr \, d\theta = 162\pi. \]
3. Evaluate $\int\!\!\!\int_S \mathbf{F} \cdot \mathbf{n} \, dS$, where $\mathbf{F} = 2x \mathbf{i} + 3xy \mathbf{j} - yz^2 \mathbf{k}$ and $S$ is the surface bounded by the planes $x = 0$, $x + y = 2$, $y = 0$, $z = 0$ and $z = 2$. ($\mathbf{n}$ is the unit outward normal to the surface $S$.)

Solution.
Let $V$ be the solid enclosed by $S$. Then $V$ is described by:

- $0 \leq x \leq 2$
- $0 \leq y \leq 2 - x$
- $0 \leq z \leq 2$

Now, $\text{div} \, \mathbf{F} = \nabla \cdot \mathbf{F} = 2 + 3x - 2yz$, and so by the divergence theorem,

$$\int\!\!\!\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_V (2 + 3x - 2yz) \, dV$$

$$= \int_0^2 \int_0^2 \int_0^{2-x} (2 + 3x - 2yz) \, dy \, dx \, dz$$

$$= \int_0^2 \left[ (2 + 3x)(2 - x) - (2 - x)^2 z \right]_0^2 \, dx \, dz$$

$$= \int_0^2 \left[ 4x + 2x^2 - x^3 + z \left( \frac{2 - x)^3}{3} \right) \right]_0^2 \, dz$$

$$= \int_0^2 \left( 8 - \frac{8z}{3} \right) \, dz = \frac{32}{3}.$$

4. Evaluate $\int\!\!\!\int_S \mathbf{F} \cdot \mathbf{n} \, dS$, where $\mathbf{F} = 4xz \mathbf{i} - y^2 \mathbf{j} + yz \mathbf{k}$ and $S$ is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. ($\mathbf{n}$ is the unit outward normal to the surface $S$.)

Solution.
Let $V$ be the solid enclosed by $S$. Then $V$ is described by

- $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

By Gauss’ divergence theorem,
\[ \iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} \, dV \]
\[ = \iiint_V (4z - 2y + y) \, dV \]
\[ = \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dx \, dy \, dz \]
\[ = \int_0^1 (4z - y) \, dy \, dz \]
\[ = \int_0^1 (4z - \frac{1}{2}) \, dz = \frac{3}{2}. \]

5. Find the flux of \( \mathbf{F} = xz^2 \mathbf{i} + (x^2y - z^3) \mathbf{j} + (2xy + y^2z) \mathbf{k} \) outwards across the entire surface of the hemispherical region bounded by \( z = (a^2 - x^2 - y^2)^{1/2} \) and \( z = 0 \).

Solution.
Let \( S \) be the entire surface of the given region, and let \( V \) be the solid enclosed by \( S \). Then
\[ \iiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} \, dV \]
\[ = \iiint_V (z^2 + x^2 + y^2) \, dV. \]

Using the spherical coordinates
\[ x = r \cos \theta \sin \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \varphi, \]
\( V \) is described by:
\[ 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi/2. \]

Now, \( x^2 + y^2 + z^2 = r^2 \), and \( dV = r^2 \sin \varphi \, dr \, d\theta \, d\varphi \). So
The flux \( = \iiint_S \mathbf{F} \cdot \mathbf{n} dS \)
\[ = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a r^4 \sin \varphi \, dr \, d\theta \, d\varphi \]
\[ = \frac{2\pi a^5}{5}. \]
6. Find the flux of $\mathbf{F} = x^2(y^2+z^2) \mathbf{i} + y^2(z^2+x^2) \mathbf{j} + z^2(x^2+y^2) \mathbf{k}$ outwards across the entire surface of the cylindrical region bounded by $x^2+y^2 = 4$, $z = -2$ and $z = 3$.

Solution.

Let $S$ be the entire surface of the given region, and let $V$ be the solid enclosed by $S$. Then

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_V 2x(y^2 + z^2) + 2y(z^2 + x^2) + 2z(x^2 + y^2) \, dV.$$

Using cylindrical co-ordinates:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = t,$$

$V$ is described by:

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad -2 \leq t \leq 3.$$

So with $dV = r \, dr \, d\theta \, dt$, we have the flux across $S$ equal to

$$2 \int_{-2}^{3} \int_{0}^{2\pi} \int_{0}^{2} (r^3 (\cos \theta \sin^2 \theta + \cos^2 \theta \sin \theta) + rt^2 (\cos \theta + \sin \theta) + r^2 t) \, r \, dr \, d\theta \, dt.$$

Integrating first with respect to $\theta$, all terms but the last give zero. Hence,

$$\text{Flux across } S = 4\pi \int_{-2}^{3} \int_{0}^{2} r^3 t \, dr \, dt$$

$$= 4\pi \int_{-2}^{3} 4t \, dt = 40\pi.$$