Algebraic Topology Notes

Part II: Fundamental Group

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Basic definitions

Definition. A space X is *locally path-connected* if every point $x \in X$ has a neighbourhood basis of path-connected open sets.

The set of path components of X is denoted $\pi_0(X)$.

Exercise 1. Let X be connected and locally path-connected. Show that X is path-connected.

Exercise 2. Let $W = \{(0, y) \mid |y| \leq 1\} \cup \{(x, sin(1/x) \mid 0 < x \leq \frac{2}{\pi}\}$ be the topologist's sine curve in $\mathbb{R}^2$. Show that W is connected, but not path-connected. If we adjoin an arc from $(0, 1)$ to $(\frac{2}{\pi}, 1)$ the resulting space is path-connected, but not locally path-connected.

Definition. A *loop at* x_0 in X is a continuous map $\sigma : [0, 1] \to X$ such that $\sigma(0) = \sigma(1) = x_0$, i.e., a path from x_0 to x_0. Two loops at x_0 are homotopic *rel* x_0 if they are homotopic through loops at x_0.

We may identify the set of homotopy classes of loops at x in X with the homotopy set $[[0, 1], \{0, 1\}; X, x_0]$. The function $e(t) = (\cos(2\pi t), \sin(2\pi t))$ maps $[0, 1]$ onto S^1, and composition with this function induces a bijection between maps from $(S^1, 1)$ to (X, x_0) and loops at x_0, and (hence) between $[S^1, 1; X, x_0]$ and $[[0, 1], \{0, 1\}; X, x_0]$. We shall use freely both versions.

Definition. Let α and β be paths in X such that $\alpha(1) = \beta(0)$. The *concatenation* of α and β is the path $\alpha.\beta : [0, 1] \to X$ given by $\alpha.\beta(t) = \alpha(2t)$ if $t \leq \frac{1}{2}$ and $= \beta(2t - 1)$ if $t \geq \frac{1}{2}$. Let $\bar{\alpha}(t) = \alpha(1 - t)$, for all $0 \leq t \leq 1$.

(Note. A minority of authors use the opposite convention for concatenation, as the fundamental groupoid defined below then has better properties).

Definition-Proposition-Exercise! If α and β are loops at x then so is $\alpha.\beta$. If $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\alpha.\beta \sim \alpha'.\beta'$, so concatenation induces an associative "multiplication" on the set of homotopy classes of loops at $*$ in X. Moreover if ϵ_x is the constant loop at x then $\alpha.\epsilon_x \sim \alpha \sim \epsilon_x\alpha$ and $\alpha.\bar{\alpha} \sim \epsilon_x \sim \bar{\alpha}.\alpha$. The *fundamental group of* X *based at* x_0 is the set of homotopy classes of loops at x_0 in X with this multiplication, and is denoted $\pi_1(X, x_0)$.

Let $f : X \to Y$ be a continuous map. If X and Y have basepoints x_0 and $y_0 = f(x_0)$, respectively, then composition with f induces a function $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. This function is a homomorphism of groups. Moreover $id_{X*} = id_{\pi_1(X,x_0)}$ and $(fg)_* = f_* g_*$, and so the fundamental group is functorial.

The subscript 1 refers to the family of higher homotopy groups ($\pi_q(X) = [S^q; X]$), which we shall not consider in this course. We shall however consider the role of the basepoint later. Note that loops at x_0 must have image in the path-component of X containing x_0, so we may usually assume X path-connected.

Definition. A space X with basepoint x_0 is *simply-connected* (or 1-*connected*) if it is path-connected and any loop at x_0 is homotopic (*rel* basepoint) to the constant loop at x_0.

Exercise 3. Show that the following are equivalent

(i) X is simply-connected;

(ii) any two paths $\alpha, \beta : [0, 1] \to X$ with the same endpoints $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ are homotopic (relative to the endpoints);

(iii) any map $f : S^1 \to X$ extends to a map from D^2 to X.

Note in particular that whether X is simply-connected or not is inependent of the choice of basepoint.

Exercise 4. Let X be a metric space, with metric ρ and basepoint x_0, and let $Map_*(S^1, X)$ be the set of all maps $f : S^1 \to X$ such that $f(1) = x_0$. Define a metric d on $Map_*(S^1, X)$ by $d(\alpha, \beta) = \max\{\rho(\alpha(s), \beta(s)) \mid s \in S^1\}$. Check that this is a metric, and show that $\pi_0(Map_*(S^1, X)) = \pi_1(X, x_0)$. (Similarly, $\pi_0(Map(S^q, X)) = \pi_q(X)$).

Exercise 5. Suppose that $X = A \cup B$ where A, B and $A \cap B$ are nonempty, open and path-connected. Let $*$ be a point in $A \cap B$. Show that $\pi_1(X, *)$ is generated by the images of $\pi_1(A, *)$ and $\pi_1(B, *)$ (under the natural homomorphisms inducedby the inclusions of these subsets into X). In other words, show that every loop at $*$ in X is homotopic to a product of finitely many loops at $*$, each of which lies either in A or in B. (Hints: you need to use the compactness of the interval $[0,1]$ and the assumption that $A \cap B$ is path-connected).

Exercise 6. Use Exercise (5) to show that $\pi_1(S^n, *) = 1$ if $n > 1$.

Exercise 7. Let X be a space with a basepoint $*$ and such that there is a map $\mu : X \times X \to X$ with $\mu(x, *) = x = \mu(*, x)$ for all $x \in X$. Show that $\pi = \pi_1(X, *)$ is abelian. (This applies in particular if X is a topological group).

[Hint. Consider the induced homomorphism $\mu_* : \pi \times \pi \to \pi$ and work on the algebraic level).

Exercise 8. (a) Let G be a topological group. (Thus G is a group and has a topology such that multiplication $G \times G \to G$ and inverse : $G \to G$ are continuous). Show that G_e, the path component of the identity, is a normal subgroup and that $\pi_0(G) \cong G/G_e$ is a group.

(b)Suppose that H is a *normal* subgroup of G which is *discrete* as a subspace. Show that if G is *path-connected* H is central in G, i.e., that $gh = hg$ for all $g \in G$ and $h \in H$. (Hint: for each $h \in H$ consider the function sending $g \in G$ to ghg^{-1} and apply each of the italicized assumptions to show that this function is constant).

(c) Give examples to show that each of the 3 italicized assumptions is needed for part (b).

The basic result: $\pi_1(S^1, 1) \cong \mathbb{Z}$

This is the central calculation of the subject. We shall identify the 1-sphere S^1 with the unit circle in the complex plane. Note that S^1 is a topological group, and we shall take its identity element 1 as the basepoint.

Let $exp : \mathbb{C} \to \mathbb{C}^\times$ be the complex exponential, given by $exp(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. Then exp restricts to a map $exp_\mathbb{R} : \mathbb{R} \to S^1$ which is an epimorphism of groups, with kernel the integers $\mathbb{Z}$. Thus $exp_\mathbb{R}$ induces an isomorphism (of groups) $\mathbb{R}/\mathbb{Z} \cong S^1$, which is easily seen to be a homeomorphism. (In fact $exp_\mathbb{R}$ maps each open interval of length < 1 homeomorphically onto an open arc on the circle).

A *lift* of a map $f : Y \to S^1$ (*through exp*) is a map $\tilde{f} : Y \to \mathbb{R}$ such that $\exp \tilde{f} = f$.

Theorem. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Sketch-proof. We show that every loop σ at 1 in S^1 has an unique lift to a path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ starting at $\sigma(0) = 0$, that every homotopy of loops lifts to a homotopy of paths, and that $\deg(\sigma) = \tilde{\sigma}(1)$ is an integer which depends only on the homotopy class of the loop σ. It is then fairly routine to show that $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ is an isomorphism. $\square$

The inclusion of S^1 into $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is a homotopy equivalence, and the degree homomorphism may be given by contour integration:

$$\deg(\sigma) = \frac{1}{2\pi i} \int_\sigma \frac{dz}{z}.$$

Exercise 9. Show that $[X; S^1]$ is an abelian group with respect to the operation determined by multiplication of maps (i.e., $fg(x) = f(x)g(x)$ for all $x \in X$).

Exercise 10. Show that forgetting the basepoint conditions determines an isomorphism from $\pi_1(S^1, 1)$ to $[S^1; S^1]$.

Exercise 11. The fundamental theorem of algebra - proof by homotopy theory. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients. Show that if r is large enough $P_t(z) = (1-t)P(z) + tz^n$ has no zeroes on the circle $|z| = r$ for any $0 \leq t \leq 1$. Hence the maps : $z \to z^n$ and : $z \to P(rz)/|P(rz)|$ are homotopic as maps from S^1 to S^1. If P has no zeroes the latter map extends to a map from the unit disc D^2 to S^1. CONTRADICTION. Why?

Exercise 12. Let $f : D \to D$ be a continuous function, where $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the closed unit disc in the plane. Show that f has a fixed point, i.e., that $f(z) = z$ for some z in D. (Hint: suppose not. Then the line from $f(z)$ through z is well defined. Let $g(z)$ be the point of intersection of this line with $S^1 = \partial D$ (so z lies between $f(z)$ and $g(z)$). Show that $g : D \to S^1$ is continuous and $g(z) = z$ for all $z \in S^1$. Obtain a contradiction by considering the induced homomorphisms of fundamental groups.

Covering maps

Definition. A map $p : E \to X$ is a *covering map* if every point $x \in X$ has an open neighbourhood U such that $p^{-1}(U)$ is a nonempty disjoint union of open sets V for which each restriction $p|_V : V \to U$ is a homeomorphism.

Such an open subset U is said to be *evenly covered* (by p) and the subsets V of E are called the *sheets* of the covering p over U. If U is *connected* the sheets of p above U are the components of $p^{-1}(U)$, but in general there may not be a canonical partition of $p^{-1}(U)$ into sheets.

A covering map p is onto, by definition, and so $p^{-1}(x)$ is nonempty, for all $x \in X$. This is a discrete subset of E, called the *fibre* over x. If X has a basepoint $*$ we shall always choose the basepoint for E to lie in the fibre over $*$, so that p is a basepoint-preserving map.

Examples.

(1) $exp : \mathbb{R} \to S^1$, given by $exp(x) = e^{2\pi ix}$ for $x \in \mathbb{R}$.

(2) $exp : \mathbb{C} \to \mathbb{C}^\times$, given by $exp(z) = e^{2\pi iz}$ for $z \in \mathbb{C}$.

(3) $: z \mapsto z^n$ for $z \in S^1$ and $n \neq 0$.

(4) The projection of $X \times F$ onto X, where F is a discrete set.

However the restriction of *exp* to a map from $(0, 2)$ to S^1 is *not* a covering map, even though it is onto and is a local homeomorphism, for no neighbourhood of 1 in S^1 is evenly covered by $exp|_{(0,2)}$.

Exercise 13. Let $p : E \to X$ be a covering map, with X connected. Show that all the fibres $p^{-1}(x)$ (for $x \in X$) have the same cardinality.

Definition. The *group of covering transformations* of a covering map $p : E \to X$ is the group

$$Aut(p) = Aut(E/X) = \{f : E \to E \mid f \textit{ is a homeomorphism and } pf = p\}.$$

(We shall see below that if the spaces involved are connected and locally path-connected the requirement that f be a homeomorphism is redundant).

Our goal is to show that the fundamental group of a "reasonable" space may be identified with the group of covering transformations of a certain covering of the space. This gives us a strong connection between two apparently quite different ideas - homotopy classes of loops in a space and groups acting as continuous permutations of a (covering) space. (Compare the identification of $\pi_1(S^1, 1)$ with $\mathbb{Z}$, which used the description of S^1 as the quotient of $\mathbb{R}$ under the action of $\mathbb{Z}$ by translation).

A *lift* of a map $f : Y \to X$ (*through* p) is a map $\tilde{f} : Y \to E$ such that $p\tilde{f} = f$.

In the following three lemmas we assume that $p : E \to X$ is a covering map such that $p(e) = x$, where e and x are basepoints.

The Uniqueness Lemma. *Let $f : Y \to X$ be a map with connected domain Y. If $f_1, f_2 : Y \to E$ are two lifts of f and there is a point $f_1(y) = f_2(y)$ then $f_1 = f_2$.*

The Existence Lemma. *Let $\sigma : [0, 1] \to X$ be a path beginning at $\sigma(0) = x$. Then there is an unique lift $\tilde{\sigma} : [0, 1] \to E$ such that $\tilde{\sigma}(0) = e$.*

The Covering Homotopy Lemma. *Let $F : Y \times [0, 1] \to X$ be a map such that $f = F_0 : Y \to X$ has a lift $\tilde{f} : Y \to E$. Then F has an unique lift $\tilde{F} : Y \times [0, 1] \to E$ such that $\tilde{F}(y, 0) = \tilde{f}(y)$ for all $y \in Y$.*

These three results are clearly generalizations of those we used in considering the case $p = exp : \mathbb{R} \to S^1$.

Theorem. *The homomorphism $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is 1-1.*

Proof. Let σ be a loop at $e \in E$ such that $p\sigma$ is homotopic to the constant loop, via a homotopy h_t. Then $\tilde{h}_t$ is a homotopy from σ (the unique lift of $p\sigma$ to a path starting at e) to the constant loop at e (the unique lift of the constant loop at x to a path starting at e). Since : $t \mapsto \tilde{h}_t(1)$ is a path in the fibre over x it is constant, and so $\tilde{h}_t$ is a homotopy of loops. $\square$

Theorem. [The Lifting Criterion] *Let $p : E \to X$ be a covering map and $f : Y \to X$ a map, where Y is connected and locally path-connected. Suppose that e and y are points of E and Y, respectively, such that $p(e) = f(y) = x$. Then f has a lift $\tilde{f} : Y \to E$ such that $\tilde{f}(y) = e$ if and only if $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(E, e))$.*

Proof. The condition is obviously necessary. For each point $z \in Y$ choose a path ω from y to z in Y. Then $f\omega$ is a path from x to $f(z)$ in X. Let $\tilde{f}(z)$

be the endpoint of $\widetilde{f\omega}$, the lift of $f\omega$ to a path starting at e in E. We must check that this is well defined and continuous.

If ω' is another path from y to z in Y then $\omega.\bar{\omega}'$ is a loop at y. Hence $f(\omega.\bar{\omega}')$ is a loop at x which by assumption lifts to a loop at e. Since $f\omega$ is homotopic to $f(\omega.\bar{\omega}'.\omega') = f(\omega.\bar{\omega}').f\omega'$ (as paths from x to $f(z)$) their lifts have the same endpoint, and so $\widetilde{f\omega}(1) = \widetilde{f\omega'}(1)$. Thus $\tilde{f}$ is well defined.

Let V be an open neighbourhood of $\tilde{f}(z)$. We may assume that V is a sheet above an evenly covered open subset $U = p(V)$ of X. Let S be a path-connected open neighbourhood of z in $f^{-1}(U)$. Let $s \in S$ and let τ be a path from z to s in S. Then $\omega.\tau$ is a path from y to s so $\tilde{f}(s) = \widetilde{f(\omega.\tau)}(1)$ which is the endpoint of the lift h of $f\tau$ beginning at $\tilde{f}(z)$ in V. But $(p|_V)^{-1}f\tau$ is such a lift, and so $h = (p|_V)^{-1}f\tau$, by uniqueness of lifts. Therefore $\tilde{f}(s) = h(1)$ is in V. Hence $\tilde{f}(S) \subseteq V$ and so $\tilde{f}$ is continuous at z. Since z was arbirtrary this completes the proof of the Theorem. $\square$

Lemma. *Let $p : E \to X$ be a covering map, where E is simply-connected and locally path-connected, and let $e, e' \in E$ be two points such that $p(e) = p(e')$. Then there is an unique homeomorphism $\phi : E \to E$ such that $p\phi = p$ and $\phi(e) = e')$.*

Proof. By the Lifting Criterion and the Uniqueness Lemma there are unique maps $\phi, \psi : E \to E$ which lift p through itself and are such that $\phi(e) = e'$ and $\psi(e') = e$. Then $\phi\psi$ and $\psi\phi$ are lifts of p through itself which agree with id_E at e' and e, respectively. Hence $\phi\psi = \psi\phi = id_E$, by the Uniqueness Lemma. $\square$

This lemma justifies our claim above that a map $f : E \to E$ such that $pf = p$ is automatically a homeomorphism.

Theorem. *Let $p : E \to X$ be a covering map, where E is simply-connected and locally path-connected. Then $G = Aut(E/X)$ and $\pi = \pi_1(X, x)$ are isomorphic.*

Proof. Define a function $\rho : G \to \pi$ as follows. Given $g \in G$, choose a path γ_g from e to $g(e)$ in E. Then $p\gamma_g$ is a loop at $x = p(e) = p(g(e))$. Let $\rho(g) = [p\gamma_g]$. Since E is simply-connected any two such paths are homotopic, and a homotopy of such paths in E projects to a homotopy of loops in X. Therefore ρ is a well defined function.

Given $g, h \in G$ and paths γ_g, γ_h from e to $g(e)$ and $h(e)$, respectively, the concatenation $\gamma_g.g(\gamma_h)$ is a path from e to $g(h(e))$. Therefore $\rho(gh) = [p(\gamma_g.g(\gamma_h)] = [p\gamma_g]p\gamma_h] = \rho(g)\rho(h)$, and so ρ is a homomorphism. If $\rho(g) = 1$ then $p\gamma_g$ is homotopic to the constant loop at x. We can lift such a homotopy to a homotopy h_t from γ_g to the constant path at e. Since the endpoint $h_t(1)$ is a path in the fibre over x it is constant, and so $g(e)(= \gamma_g(1)) = e = id_E(e)$. Hence $g = id_E$, by the above Lemma, and so ρ is 1-1.

Suppose finally that σ is a loop at x in X. Let $\tilde{\sigma}$ be the lift of σ to a path beginning at e (i.e., $\tilde{\sigma}(0) = e$). Let $e' = \tilde{\sigma}(1)$. By the above lemma there is a covering homeomorphism $\phi \in G$ such that $\phi(e) = e'$. Since we may take $\gamma_\phi = \tilde{\sigma}$ we have $\rho(\phi) = [\sigma]$. Thus ρ maps G onto π, and so is an isomorphism. $\square$

Exercise 14. Suppose that X is connected and locally path-connected, and that $*$ is a basepoint for X. Show that the map $\theta : [X, *; S^1, 1] \to Hom(\pi_1(X, *), Z)$ determined by $\theta(f) = \pi_1(f)$ is a homomorphism of abelian groups. Using the Lifting Criterion, show that it is 1-1. (It is in fact an isomorphism for reasonable spaces X).

Exercise 15. (a) Show that the map depicted below is a covering map, but that $Aut(p)$ is trivial.

(b) Find a 2-fold covering of the domain of p such that the resulting 6-fold covering of the figure eight $S^1 \vee S^1$ has automorphism group S_3 (the symmetric group on 3 letters, i.e., the nonabelian group of order 6).

Exercise 16. Let $p : X \to Y$ be a covering map, where X is connected and locally path-connected and Y is simply-connected. Show that p is a homeomorphism.

Exercise 17. Let $L = Z \oplus Zi$ be the standard lattice in $\mathbb{C}$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

be the extended complex plane. Show that every analytic function from $\hat{\mathbb{C}}$ to $T = \mathbb{C}/L$ is constant.

(NB. There is however a rich theory of analytic functions from T to $\hat{\mathbb{C}}$!).

Exercise 18. Show that if $f : S^2 \to \mathbb{R}^2$ is a function such that $f(-x) = -f(x)$ for all $x \in S^2$ then $f(y) = 0$ for some $y \in S^2$.

[Hint. Suppose not. Let $g(x) = f(x)/||f(x)||$, for all $x \in S^2$. Show that there is a commuting diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & S^1 \\ \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle p} \\ P^2(\mathbb{R}) & \xrightarrow{h} & S^1 \end{array}$$

where $a(x) = a(-x)$, $p(z) = z^2$ and $h(a(x)) = (g(x)^2$. Apply the lifting lemma to get a function $\tilde{h} : P^2(\mathbb{R}) \to S^1$ and deduce a contradiction).

The Covering Homotopy Lemma

The argument given in [Greenberg] is incomplete, although it suffices if we assume also that the space Y is locally connected. The full details of this argument (for the general case) may be in found in [Spanier], pages 67-68.

Here follows an alternative argument, based on the special case $Y = \{y\}$ (path lifting).

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\tilde{f}} & E \\ \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle p} \\ Y \times [0,1] & \xrightarrow{F} & X \end{array}$$

1. There is an unique function $\tilde{F} : Y \times [0, 1] \to E$ such that

a) $p\tilde{F} = F$;

b) $\tilde{F}(y, 0) = \tilde{f}(y)$ for all $y \in Y$; and

c) the function $\tilde{F}_y : [0, 1] \to E$ defined by $\tilde{F}_y(t) = \tilde{F}(y, t)$ for all $0 \le t \le 1$ is continuous, for all $y \in Y$.

Note that we are ignoring the topology of Y here, and so this assertion follows immediately from the existence and uniqueness of lifts of paths.

2. The function $\tilde{F}$ defined above is continuous (i.e., as a function of two variables).

Let $y \in Y$. By the usual argument involving continuity of F and compactnes of $[0,1]$ we may assume that there is an open neighbourhood N of y in Y and a partition $0 = t_0 < t_1 \cdots < t_n = 1$ of $[0,1]$ such that $F(N \times [t_{i-1}, t_i])$ is contained in an evenly covered open subset U_i (say) of X, for each $1 \leq i \leq n$.

Let V_i be a sheet of E above U_i which contains $\tilde{F}(y, t_{i-1})$. Since $\tilde{f}$ is continuous there is an open neighbourhood $N_1 \subseteq N$ of y with $\tilde{f}(N_1) \subseteq V_1$. Then $(p|_{V_1})^{-1} F|_{N_1 \times [0,t_1]}$ is a map lifting $F|_{N_1 \times [0,t_1]}$. The uniqueness argument of part 1 implies that this map agrees with $\tilde{F}$ on $N_1 \times [0, t_1]$. In particular, $\tilde{F}$ is continuous on $N_1 \times [0, t_1]$.

We argue by induction. Suppose that y has an open neighbourhood $N_k \subseteq N$ such that $\tilde{F}$ is continuous on $N_k \times [0, t_k]$. Then there is an open neighbourhood $N_{k+1} \subseteq N_k$ of y with $\tilde{F}(y, t_k) \in V_{k+1}$ for all $y \in N_{k+1}$. The map $(p|_{V_{k+1}})^{-1} F|_{N_{k+1} \times [t_k, t_{k+1}]}$ lifts $F|_{N_{k+1} \times [t_k, t_{k+1}]}$. Since it agrees with $\tilde{F}$ on $N_{k+1} \times \{t_k\}$ we find as before that $\tilde{F}$ is continuous on $N_{k+1} \times [t_k, t_{k+1}]$. Hence $\tilde{F}$ is continuous on $N_{k+1} \times [0, t_{k+1}]$. After n steps we see that y has a neighbourhood N_n such that $\tilde{F}$ is continuous on $N_n \times [0,1]$. Hence $\tilde{F}$ is continuous everywhere.

Groups acting on spaces

Let S be a set and G a group. An action of G on S is a homomorphism from G to $Perm(S)$, the group of bijections from S to itself. (More precisely, this is a *left* action of G on S). Equivalently, a (left) action of G on S is a function $\mu : G \times S \to S$ such that $\mu(1_G, s) = s$ and $\mu(g, \mu(h, s)) = \mu(gh, s)$ for all $g, h \in G$ and $s \in S$. We shall write $gs = \mu(g, s)$ for simplicity. The *orbit* of G through a point $s \in S$ is the subset $Gs = \{gs \mid g \in G\}$. The set $G \backslash S$ of orbits is called the quotient of the action, and there is a natural map from S onto $G \backslash S$. (A *right* action is a function $\nu : S \times G \to S$ such that $\nu(s, 1_G) = s$ and $\nu(\nu(s, g), h) = \nu(s, gh)$ for all $g, h \in G$ and $s \in S$. Such an action corresponds to an *anti*-homomorphism from G to $Perm(S)$).

If S is a topological space we require the action to be a homomorphism from G to $Homeo(S)$. Under suitable assumptions on the action there is a natural topology on the quotient set such that the projection p of S onto $G\backslash S$ is continuous. In particular, if every point $s \in S$ has an open neighbourhood V which meets none of its translates (i.e., $gV \cap hV$ is empty if $g \neq h$) we may take the images $p(V)$ of such open sets as a basis for the topology of $G\backslash S$. As $p(V)$ is then an evenly covered open neighbourhood of $p(s)$ the map p is a covering map.

An action is *effective* if $g.s = s$ for all $s \in S$ implies $g = 1$ in G. We shall consider only effective actions.

Examples.

(1) Tori: Z^n acting on $\mathbb{R}^n$.

(2) Projective spaces: $\{\pm 1\}$ acting on S^n.

(3) Lens spaces: $Z/nZ = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$ acting on S^{2n+1}.

(4) Figure eight: $Z * Z$ acting on a tree; Z^2 acting on a rectangular lattice. (See the next section).

(5) Discrete subgroups of topological groups. (E.g., $SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$. The quotient space is homeomorphic to the complement of the trefoil knot!).

Given the simply-connected and locally path-connected covering space E and the group $G = Aut(E/X)$ of covering automorphisms, we can recover X as the quotient space $G\backslash E$. For the Lifting Criterion implies that given any two points e_1, e_2 of E with $p(e_1) = p(e_2)$ there is an unique covering automorphism g such that $g(e_1) = e_2$. Thus the orbits of the action of G on E correspond bijectively to the points of X. Similarly, if we divide out by the action of a subgroup $H \leq G$ we obtain intermediate coverings $E \to H\backslash E$ and $H\backslash E \to X$.

The infinite TV antenna

The universal covering of $W = S^1 \vee S^1$ is an infinite 4-valent tree $\widetilde{W}$. Thus it is a graph in which four edges meet at every vertex, and in which any two points are the endpoints of an unique "geodesic" path. If we give each edge the euclidean metric of the unit interval then the distance from the basepoint to any other vertex is a positive integer.

We shall construct $\widetilde{W}$ as a subset of the unit disc in $\mathbb{R}^2$.

Fix $\lambda \in (0, \frac{1}{2})$.

The vertices: Let O $= (0,0)$ be the basepoint.

The vertices at distance $n > 0$ from O (with respect to the natural graph-metric) are the points $(a,b) = (\Sigma_{k=1}^n \varepsilon_k \lambda^k, \Sigma_{k=1}^n \eta_k \lambda^k)$, where

(i) $\varepsilon_j, \eta_k \in \{-1, 0, 1\}$ for all $j, k \leq n$;

(ii) if $n > 0$ then exactly one of ε_1 and η_1 is 0;

(iii) if $\varepsilon_j \neq 0$ and $j < n$ then either $\varepsilon_{j+1} = \varepsilon_j$ and $\eta_{j+1} = 0$ or $\varepsilon_{j+1} = 0$ and $\eta_{j+1} = \pm 1$;

(iv) if $\eta_k \neq 0$ and $k < n$ then either $\eta_{k+1} = \eta_k$ and $\varepsilon_{k+1} = 0$ or $\eta_{k+1} = 0$ and $\varepsilon_{k+1} = \pm 1$.

Note that exactly one of ε_k and η_k is 0, for each $1 \leq k \leq n$, and that the vertices all lie in the open disc of radius $\Sigma_{k>0} \lambda^k = \lambda/(1-\lambda)$ (which is < 1).

The edges: Connect O to each of $(\pm\lambda, 0)$ and $(0, \pm\lambda)$. Let α and β be the edges from O to $(\lambda, 0)$ and $(0, \lambda)$, respectively.

If $n > 0$ and $\varepsilon_n \neq 0$ connect (a,b) to each of $(a + \varepsilon_n \lambda^{n+1}, b)$ and $(a, b \pm \lambda^{n+1})$. If $n > 0$ and $\varepsilon_n = 0$ connect (a,b) to each of $(a \pm \lambda^{n+1}, b)$ and $(a, b + \eta_n \lambda^{n+1})$.

Note that if (say) $\varepsilon_1 = 1$ then $b < a$, since $\lambda^2/(1-\lambda) < \lambda$. (Hence) this graph embeds in the disc of radius $\lambda/(1-\lambda)$. Note that the path-length metric on the graph is *not* the euclidean metric of the plane. (In fact $\widetilde{W}$ does not embed isometrically in $\mathbb{R}^2$).

The group: Let G be the subgroup of $Homeo(\widetilde{W})$ generated by the "horizontal" and "vertical" self-homeomorphisms X and Y defined as follows:

If $a + \lambda > 0$ let $X(a,b) = \lambda(a+1, b)$. Otherwise let $X(a,b) = \lambda^{-1}(a + \lambda, b)$.

Similarly for Y.

Then G acts freely on $\widetilde{W}$, with a single orbit of vertices $G.O$ and two orbits of edges $G\alpha$ and $G\beta$. Moreover each vertex has a small nbhd disjoint from all its translates, and the interior of each edge (i.e., the edge minus its endpoints) is disjoint from all is translates. Hence $G \backslash \widetilde{W} \cong S^1 \vee S^1$, and the projection $p\widetilde{W} \to G\backslash\widetilde{W}$ is a covering projection.

(It is more natural to define X and Y as hyperbolic transformations of the open unit disc $\mathbb{D}^2$. Let G_k be the group of homeomorphisms of $\mathbb{D}^2$ generated by $X_k(z) = \frac{kz+k-2}{(k-2)z+k}$ and $Y_k(z) = -iX_k(iz)$, and let TV be the union of the translates of the real and purely imaginary diameters of $\mathbb{D}^2$ under the action of G_k. If k is large enough then $TV \cong \widetilde{W}$ and G_k acts freely on it. It is easy to see that we must have $k > 1+\sqrt{2}$, but it is not clear how large k must be).

Intermediate coverings and subgroups of the fundamental group

Lemma. *Let $q : E \to X$ be a covering map. Suppose that V and V' are connected open subsets of E such that $q(V) = q(V')$ and $q|_V$ and $q|_{V'}$ are homeomorphisms. Then either $V = V'$ or $V \cap V' = \emptyset$.*

Proof. Suppose that $v \in V$ and $v' \in V'$ are distinct points such that $q(v) = q(v')$. Then $v \notin V'$ and $v' \notin V$. Let U be an evenly covered open neighbourhood of $q(v)$ in $q(V)$. Then v and v' are in disjoint sheets W and W' over U. We may assume $W \subseteq V$ and $W' \subseteq V'$. Then $W \cap V' = \emptyset$ (since $q|_{V'}$ is a homeomorphism). Hence $V - V'$ is open and nonempty. As $V \cap V'$ is open and V is connected it follows that V and V' are disjoint. $\square$

Let $p : E \to X$ and $q : E' \to X$ be covering maps with E and E' connected and X locally path-connected. (Then E and E' are also locally path-connected). Suppose there is a map $\tilde{p}$ such that $p = q\tilde{p}$. We shall show that $\tilde{p}$ is also a covering map.

Let U be a path-connected open subset of X which is evenly covered for both p and q. Then the sheets above U for p and q are the path components of the preimages: $p^{-1}(U) = \amalg_{\alpha\in A} W_\alpha$ and $q^{-1}(U) = \amalg_{\beta\in B} V_\beta$, where $p|_{W_\alpha}$ and $q|_{V_\beta}$ are homeomorphisms. Since each W_α is path connected and $\tilde{p}$ is continuous $\tilde{p}(W_\alpha) \subseteq V_{\rho(\alpha)}$, for some $\rho : A \to B$. Since $p|_{W_\alpha}$ and $q|_{V_{\rho(\alpha)}}$ are homeomorphisms it follows that $\tilde{p}|_{W_\alpha}$ is a homeomorphism onto $V_{\rho(\alpha)}$. It follows easily that each V_β is evenly covered for $\tilde{p}$.

In particular, $\tilde{p}(E)$ is open in E'. Let f be a point of E' in the closure of $\tilde{p}(E)$, and let V be a neighbourhood of f which is evenly covered for $\tilde{p}$. Then $V \cap \tilde{p}(E)$ is nonempty, and so $V \subseteq \tilde{p}(E)$ (since it is evenly covered). Hence $f \in \tilde{p}(E)$ and so $\tilde{p}(E)$ is closed. Since E' is connected it follows that $\tilde{p}(E) = E'$, i.e., $\tilde{p}$ is onto. Hence $\tilde{p}$ is a covering projection.

Exercise: show that $p = qr$, p a covering, r onto does not imply that q is a covering.

Exercise 19. Let S be the circle of radius $\frac{1}{2}$ with centre $(-\frac{1}{2}, 0)$ and let $X = S \cup W = S \vee W$, where W is the topologist's sine curve. (See Exercise 2). Let $E = \mathbb{R} \vee (\mathbb{Z} \times W)$, where $(n, (0,0)) \in \mathbb{Z} \times W$ is identified with $n \in \mathbb{R}$, for all integers $n \in \mathbb{Z}$. Then $\mathbb{Z}$ acts freely and properly discontinuously on E by translation, with orbit space X. Show that the sheets above a neighbourhood of $w = (0,1) \in W$ of diameter strictly less than 1 are not determined by their intersection with the fibre over w. (Note that W fails to be locally path-connected).

It follows from the Lifting Criterion that if X has a simply-connected and locally path-connected covering $p : E \to X$ then that covering is *universal* in the sense that it factors through any other covering. For if $q : E' \to X$ is a covering map such that $q(e') = x$ for some basepoint e' then p lifts to a map $p' : E \to E'$ such that $p'(e) = e'$ and $qp' = p$. As observed above, p' is also a covering map. The argument sketched in the section on group actions implies that $E' = H \backslash E$, where $\rho(H) = q_*(\pi_1(E', e'))$.

There is a bijective correspondance between isomorphism classes of connected covering maps $q : E' \to X$ and subgroups of $\pi_1(X, x)$ which sends the covering map q to $q_*(\pi_1(E', e'))$ and the subgroup κ of $\pi_1(X, x)$ to the covering $p_\kappa : \rho^{-1}(\kappa) \backslash E \to X$.

If we attempt to carry through the construction of the isomorphism $\rho : Aut(E/X) \cong \pi_1(X, x)$ without the assumption that E be simply-connected we find the homotopy class of $[p\gamma_g]$ depends on the choice of path γ_g from e to $g(e)$ in E, and only the left coset $p_*(\pi_1(E, e))[p\gamma_g]$ in $p_*(\pi_1(E, e)) \backslash \pi_1(X, x)$ is well defined. Moreover not every coset is realised. In fact $[p\gamma_g]$ is in the normalizer $N_{\pi_1(X,x)}(p_*(\pi_1(E, e))$, since $[p\gamma_g]p_*\alpha[p\gamma_g]^{-1} = [p(\gamma_g g(\alpha)\bar{\gamma}_g)]$ is in $p_*(\pi_1(E, e))$, for any $[\alpha]$ in $\pi_1(E, e)$. (In general, if H is a subgroup of G then $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is the largest subgroup of G in which H is normal, and the coset space $N_G(H)/H = H \backslash N_G(H)$ is a group). Thus we obtain a function $\rho_E : Aut(E/X) \to N_{\pi_1(X,x)}(p_*\pi_1(E, e))/p_*(\pi_1(E, e))$.

Exercise 20. Check that ρ_E is an isomorphism.

Exercise 21. Show that every connected covering of S^1 is equivalent to either $exp : \mathbb{R} \to S^1$ or to one of the n^{th} power maps $p_n : S^1 \to S^1$ with $p_n(z) = z^n$, for some $n \geq 1$.

Existence of universal coverings

If X has a simply-connected and locally path-connected covering $p : \widetilde{X} \to X$ then X is connected (since p is onto) and locally path-connected (since p is a local homeomorphism). Moreover every point of X has an open neighbourhood U such that any loop in U is null homotopic in X. (For if U is evenly covered the inclusion of U into X factors through a map to $\widetilde{X}$).

Conversely, it can be shown that if X is connected, locally path-connected and every point of X has an open neighbourhood with the above property ("semilocally simply-connected"!) then it has a simply-connected covering (necessarily locally path-connected). We shall prove this under a somewhat more restrictive hypothesis.

Theorem. *Let X be a connected and locally path-connected space in which every point has a simply-connected open neighbourhood. Then X has a simply-connected covering.*

Proof. Choose a basepoint $* \in X$, and let P be the set of all paths $\alpha : [0,1] \to X$ starting at $\alpha(0) = *$. Define an equivalence relation $\sim$ by $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and α and β are homotopic as paths between their common endpoints. Let $\widetilde{X} = P/\sim$ and let $p : \widetilde{X} \to X$ be the function defined by $p([\alpha]) = \alpha(1)$, where $[\alpha]$ is the equivalence class of α.

We define a topology on the set $\widetilde{X}$ as follows. For each path-connected open neighbourhood U of $p([\alpha])$ let $U_\alpha = \{[\alpha\beta] \mid \beta : [0,1] \to U, \beta(0) = \alpha(1)\}$. We take the collection of all such sets U_α as a basis for the topology of $\widetilde{X}$. Since $p^{-1}(U) = \cup_{\gamma(1)=\alpha(1)} U_\gamma$ the map p is continuous, and since $p(U_\alpha) = U$ (by the path-connectedness of U) it is open.

If U is simply-connected then $p : U_\alpha \to U$ is 1-1 (check this!) and onto. Hence $p|_{U_\alpha}$ is a homeomorphism. Moreover if $\alpha(1) = \alpha'(1)$ then either $U_\alpha =$

$U_{\alpha'}$ or $U_\alpha \cap U_{\alpha'}$ is empty. Thus U is evenly covered and so p is a covering map.

Let c be the constant path at $*$. Given any other path α starting at $*$, we may define a path $\tilde{\alpha} = [\alpha_s]$ from $[c]$ to $[\alpha]$ by $\alpha_s(t) = \alpha(st)$ for all $0 \le s, t \le 1$. (Note that $\tilde{\alpha}$ lifts α, i.e., $p([\alpha_s]) = \alpha(s)$). Thus $\widetilde{X}$ is path-connected.

If τ is a loop in $\widetilde{X}$ at $[c]$ and $\alpha = p\tau$ then τ lifts α so $\tau = \tilde{\alpha}$ so $[\alpha] = \tilde{\alpha}(1) = \tau(1) = [c]$ so $\alpha \sim c$. A homotopy between these loops lifts to a homotopy from τ to the constant loop at $[c]$. Thus $\widetilde{X}$ is simply-connected. $\square$

Exercise 22. Check that the condition that U be simply-connected can be weakened to requiring only that every loop in U be homotopic to a constant loop in X.

The above argument is analogous to a classical construction by "analytic continuation" of the Riemann surface associated to a somewhere convergent power series.

Example. (The *Hawaiian earring*). Let C_n be the circle in $\mathbb{R}^2$ with radius $1/n$ and centre $(1/n, 0)$, and let $X = \cup_{n \ge 1} C_n$, with its natural topology as a subset of $\mathbb{R}^2$. Then X is not semilocally simply-connected at $(0, 0)$, as each C_n represents a nontrivial element of $\pi_1(X, (0, 0))$.

(Note that the obvious bijection from the semilocally simply-connected space $\vee^{n \ge 1} S^1$ onto X is continuous, but is *not* a homeomorphism).

Exercise 23. Show that $(S^1)^{\mathbb{N}}$ (the product of countably many circles) does not have a universal covering space.

Varying the basepoint

A path $\omega : [0, 1] \to X$ from $\omega(0) = x$ to $\omega(1) = x'$ determines an isomorphism $\omega_\# : \pi_1(X, x) \to \pi_1(X, x')$ by $\omega_\#([\alpha]) = [\bar{\omega}.\alpha.\omega]$. This isomorphism depends only on the homotopy class (*rel* endpoints) of the path. If ω is a loop at x then $\omega_\#$ is just conjugation by $[\omega]$. In particular, if X is path-connected the isomorphism class of $\pi_1(X, x)$ is independent of the choice of base-point, and if $\pi_1(X, x)$ is abelian for some (hence every) basepoint these groups are canonically isomorphic. (In the latter case we really can ignore the basepoint!).

The *fundamental groupoid* of X is the category whose set of objects is X itself (forgetting the topology) and whose morphisms are given by $Hom(x, y) =$ the set of homotopy classes (*rel* endpoints) of paths from x to y. Note however that the rule for composing morphisms *reverses* the order of concatenation. In particular, $Hom(x, x) \cong \pi_1(X, x)$, for all $x \in X$, but the isomorphism sends $[\alpha] \in Hom(x, x)$ to $[\bar{\alpha}] \in \pi_1(X, x)$, i.e., is inversion rather than the identity.

Cardinality of π_1

A metric space X is *separable* if it has a countable dense subset, or equivalently if its topology has a countable basis. (The metric hypothesis below can be relaxed).

Theorem. *Let $p : E \to X$ be a covering map, where E is connected and X is a separable, locally path-connected metric space. Then E is a separable metric space.*

Proof. The space X has a countable basis $\mathcal{B}$ consisting of evenly covered path-connected open sets, and each of these sets is separable. Let $\mathcal{V}_E$ be the family of open subsets of E which are sheets over members of $\mathcal{B}$. Thus each $V \in \mathcal{V}_E$ is a path-component of $p^{-1}(U) = \amalg V_\alpha$, for some $U \in \mathcal{B}$. For each such $V \in \mathcal{V}_E$ and $U \in \mathcal{B}$ the intersection $V \cap p^{-1}(U) = \amalg(V \cap V_\alpha)$ is a disjoint union of open subsets of V. Since V is separable at most countably many of these subsets $V \cap V_\alpha$ can be nonempty. Since $\mathcal{B}$ is countable it follows that each $V \in \mathcal{V}_E$ meets only countably many other members of this family.

Given $W \subseteq E$ let $V^0(W) = W$ and $V(W) = \cup\{V \in \mathcal{V}_E \mid V \cap W \neq \emptyset\}$. Then for each $e \in E$ the set $V^\infty(e) = \cup_{n \geq 0} V^n(\{e\})$ is open, and is separable, since it is a countable union of sets in $\mathcal{V}_E$. If $V^\infty(e) \cap V^\infty(e')$ is nonempty then $V^\infty(e) = V^\infty(e')$. Hence these sets are also closed. Since E is connected these sets are all equal to E, and so E is separable.

Let $n(x, y) = \min\{n \geq 0 \mid y \in V^n(\{x\})\}$, and let $d_E : E \times E \to [0, \infty)$ be the function defined by $d_E(x, y) = d(p(x), p(y) + n(x, y)$. Then d_E is a metric on E which determines the given topology. $\square$

Corollary. *$\pi_1(X, x)$ is countable.*

Proof. The fibre of the universal covering is a discrete subset of a separable metric space and so must be countable. □

The theorem and its corollary apply in particular if X is compact. Since a connected locally separable metric space is separable, it also applies to any metrizable manifold. We shall have more to say about fundamental groups of compact manifolds later.

Can one extend the idea of this theorem to bound the cardinality of $\pi_1(X, *)$ when X is connected and locally separable, but not separable?

Free groups

Theorem. *Let X be a set. There is a group $F(X)$ and a function $j : X \to F(X)$ such that for every group H and function $f : X \to H$ there is an unique homomorphism $h : F(X) \to H$ such that $f = hj$.*

The pair $(F(X), j)$ is unique up to an unique isomorphism.

Sketch-proof. Let W be the set of words of finite length of the form $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, where $x_i \in X$, $\epsilon_i = \pm 1$ and n is a non-negative integer. (If $n = 0$ write 1 for the "empty" word). Define a composition from $W \times W$ to W by juxtaposition. Say two words v and w are equivalent, $v \sim w$, if they can each be reduced to the same word u by contracting subwords of the form $x^\epsilon x^{-\epsilon}$ (e.g., $x_1 x_2^{-1} x_3^{-1} x_3 x_2 \sim x_1 x_2^{-1} x_2 \sim x_1 \sim x_4^{-1} x_4 x_1$, etc.). Then the composition respects equivalence classes, and the set $W/\sim$ with the induced composition is a group, which we shall call $F(X)$. There is an obvious 1-1 function $j : X \to F(X)$ and clearly (i) holds.

Given $f : X \to H$ we may extend it to a function from W to H by sending $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ to $f(x_1)^{\epsilon_1} \dots f(x_n)^{\epsilon_n}$ (where the product is formed in H). It is not hard to check that equivalent words have the same image, and that we get a homomorphism $h : F(X) \to H$ such that $hj = f$. It is clear that h is uniquely determined by f. □

Addendum. *The function j is 1-1 and the group $F(X)$ is generated by $j(X)$.* □

In the language of categories, writing $|H|$ for the *set* of elements of H, we have

$$Hom_{((Grp))}(F(X), H) = Hom_{((Set))}(X, |H|)$$

- the free group functor F from $((Set))$ to $((Grp))$ is left adjoint to the gorgetful functor $|-|$ from $((Grp))$ to $((Set))$. The group $F(X)$ is called the *free group* with *basis* X.

Each element of $F(X)$ has an unique *normal form*, obtained by contracting all possible subwords $x^{\epsilon}x^{-\epsilon}$. The length of an element w is the number $\ell(w)$ of letters $x^{\pm 1}$ in the normal form. (Note also that the notions of normal form and length are defined in terms of the basis X, and may not be preserved under isomorphisms).

Exercise 24. Show that $F(X) \cong F(Y)$ if and only if X and Y have the same cardinality. (Note that if X is finite then $\#Hom_{((Grp))}(F(X), Z/2Z) = \#Hom_{((Set))}(X, |Z/2Z|) = 2^{\#X}$, while if X is infinite $\#F(X) = \#X$).

Notation: let $F(r)$ be the free group with basis of cardinality r. It is easy to see that $F(0) = F(\emptyset) = 1$ and that $F(1) = F(\{x\}) \cong Z$.

Combinatorial group theory

The groups that arise in topology are often infinite, and so it is not practical to give their multiplication tables. Instead we use presentations, giving a set of generators (usually finite) and a list of relations between these generators that are sufficient to characterize the group. Combinatorial group theory is the study of groups in terms of such presentations. It has close connections with topology on the one hand and with the theory of formal languages on the other.

Every group is a quotient of a free group. Consider the identity function from $|H|$ to H. By property (*ii*) of free groups there is a homomorphism from $F(|H|)$ to H extending this function, and this homomorphism is clearly onto. In general, this construction is grossly inefficient. We say that H is *finitely generated* if there is a finite set X and an epimorphism $f : F(X) \to H$. The set X may be viewed as a set of *generators* for H, as every element of H is the image of a word in the $x_i^{\pm 1}$. If f is an isomorphism, we call X a *basis* for H. In general, $\mathrm{Ker}(f)$ represents the *relations* between the generators X.

Let $F(X)$ be the free group with basis X, and let R be a subset of $F(X)$. Let $\langle\langle R\rangle\rangle$ denote the smallest normal subgroup of $F(X)$ containing R. Then the quotient $F(X)/\langle\langle R\rangle\rangle$ is a group, and is said to be presented with generators X and *relators* R. Moreover any group has such a presentation, for if $f : F(X) \to H$ is an epimorphism from a free group onto H then $\mathrm{Ker}(f)$ is a normal subgroup of $F(X)$, and there are many possible choices for the subset R. (For instance, R could be all of $\mathrm{Ker}(f)$!). We usually write $H = \langle X \mid R\rangle$, although strictly speaking we should also specify the epimorphism f. Any word in X which represents an element of $\langle\langle R\rangle\rangle$ is said to be a *consequence* of R.

Each relator r corresponds to an equation $f(r) = 1$ in H. It is often convenient to write the relators as *relations* between words, so that the relation $r = s$ corresponds to the relator rs^{-1}. We shall do this without further comment below.

Examples.

(1) $Z = \langle x \mid \emptyset\rangle$.

(2) $F(x, y) = \langle x, y \mid \emptyset\rangle = \langle x, y, z \mid z\rangle$.

(3) $F(r) = \langle x_1, \ldots , x_r \mid \emptyset\rangle$.

(4) Z^2 is the quotient of $F(x, y)$ via $h(x) = (1, 0)$ and $h(y) = (0, 1)$. As $\mathrm{Ker}(h)$ is the smallest normal subgroup containig $xyx^{-1}y^{-1}$ we have $Z^2 = \langle x, y \mid xyx^{-1}y^{-1}\rangle$. We may also write $Z^2 = \langle x, y \mid xy = yx\rangle$.

(5) $S_3 = \langle a, b \mid a^2, aba = b^2\rangle$

We say that the group H is *finitely presentable* if there is a presentation $\langle X \mid R\rangle$ in which both X and R are finite sets. (The *normal* subgroup $\langle\langle R\rangle\rangle$ is only finitely generated if the quotient $F(X)/R$ is finite).

Two presentations define isomorphic groups if and only if they can be related by a chain of "elementary Tietze transformations" of the following kinds:

(a) introduce a new generator x and a new relation $x = w$ where w is a word in the old generators;

(b) adjoin a relation that is a consequence of the other relations;

and their inverses. (A relation $u = v$ is a *consequence* of R if $uv^{-1} \in \langle\langle R\rangle\rangle$).

Example. Define words in $F = F(a, b)$ by $r = a^2$, $s = abab^{-2}$, $t = aba^{-1}b^{-2}$, $u = b^3$ and $v = aba^{-1}b$. Then $S_3 = \langle a, b \mid r, s\rangle$ (via $f(a) = (1, 2)(3)$ and $f(b) = (1, 2, 3)$). Now $t = sb^2r^{-1}b^{-2}$ and $s = tb^2rb^{-2}$, so $\langle\langle r, s\rangle\rangle = \langle\langle r, t\rangle\rangle$ in F. Hence $S_3 = \langle a, b \mid r, t\rangle$. Now $u = (b^2t^{-1}b^{-2})t^{-1}(at^{-1}a^{-1})r(brb^{-1})$ is in $\langle\langle r, t\rangle\rangle$, and so $S_3 = \langle a, b \mid r, t, u\rangle$. Now $v = tu$ and $t = vu^{-1}$, so $\langle\langle r, t\rangle\rangle = \langle\langle r, t, u\rangle\rangle = \langle\langle r, u, v\rangle\rangle$. Hence $S_3 = \langle a, b \mid r, u, v\rangle = \langle a, b \mid a^2, b^3, aba^{-1}b\rangle$. (Note: we cannot delete the relator b^3 from the latter presentation, for the group with presentation $\langle a, b \mid a^2, aba^{-1}b\rangle$ is isomorphic to the group of homeomorphisms of the real line $\mathbb{R}$ generated by $A(x) = -x$ and $B(x) = x + 1$, for all $x \in \mathbb{R}$, and so is infinite).

If some object or notion defined in terms of presentations for a group has the same value for Tietze-equivalent presentations it is an invariant for the group.

For instance, let P be a presentation with finitely many generators for a group G and let $w_P(n)$ be the number of elements of G represented by words of length at most n in the given generators and their inverses. It can be shown that the asymptotic rate of growth of w_P depends only on G, and not on P. If G is a free group, or is solvable but not virtually nilpotent then any such function has exponential growth; if G is abelian or (more generally) virtually nilpotent the growth rate is polynomial.

However there are logical difficulties in using presentations to study groups. For instance, it can be shown that it is impossible to construct a (universal) algorithm which will decide (in every case) whether a given presentation represents the trivial group. One must rely on one's ingenuity, experience and luck. (See exercises 27 and 28 below).

Exercise 25. Find presentations for the groups $Z \oplus (Z/3Z)$ and Z^3.

Exercise 26. Verify that the group with presentation $\langle x, y \mid x^2 = y^2 = (xy)^2\rangle$ is isomorphic to the "quaternion group" $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. (Hint: show that $x^2 = (xy)^2$ implies that $x^{-1}yx = y^{-1}$, hence $x^{-1}y^2x = y^{-2}$, hence

$x^2 = x^{-2}$. Thus the cyclic group generaed by the image of x^2 has order (at most) 2. Adding the extra relation $x^2 = 1$ gives a presentation for the quotient group, which thus has order 4. Thus our group has order dividing 8. Show that there is a homomorphism from this group onto Q_8, i.e., that Q_8 is generated by a pair of elements satisfying these relations).

Exercise 27. Show that $\langle x, y \mid xy^2x^{-1} = y^3, yx^2y^{-1} = x^3 \rangle$ is a presentation of the trivial group.

Exercise 28. Show that $\langle x, y \mid xy^m x^{-1} = y^{m+1}, yx^n y^{-1} = x^{n+1} \rangle$ is a presentation of the trivial group.

Exercise 29. Let P, Q be presentations for groups G, H, respectively. Use these to give a presentation for the direct product $G \times H$.

Exercise 30. Show that the group with presentation
$\langle x, y \mid xyx^{-1} = y^3, yxy^{-1} = x^3 \rangle$ has order 16.

Coproducts and pushouts

By a similar procedure to that used in constructing free groups we may construct the *coproduct* of a family $\{G_x \mid x \in X\}$ of groups indexed by a set X. (Use words in the elements of the disjoint union of sets $|G_x| - \{1\}$, contracting subwords formed from consecutive elements from the same group in the obvious way). Then the coproduct $*_{x \in X} G_x$ is generated by the images of the G_x, and any family$\{f_x : G_x \to H \mid x \in X\}$ of homomorphisms gives rise to an unique homomorphism from $*_{x \in X} G_x$ to H.

Exercise 31. Show that $F(X) \cong *_{x \in X} Z_x$, where $Z_x = Z$ for all $x \in X$.

There is a further extension of this construction which is useful in describing how the fundamental group of a union of two spaces with a common subspace is determined by the fundamental groups of the constituent spaces. Suppose given a group H and two homomorphisms $\alpha_1 : H \to G_1$ and $\alpha_2 : H \to G_2$. There is an essentially unique "pushout" of this data: A group P with homomorphisms $\omega_1 : G_1 \to P$ and $\omega_2 : G_2 \to P$ with $\omega_1\alpha_1 = \omega_2\alpha_2$ and such that given any other group Q and homomorphisms $\psi_1 : G_1 \to Q$ and $\psi_2 : G_2 \to Q$ with $\psi_1\alpha_1 = \psi_2\alpha_2$ there is an unique homomorphism $\Psi : P \to Q$ such that $\Psi\omega_i = \psi_i$ for $i = 1, 2$.

The square with vertices H, G_1, G_2, P and edges α_1, α_2, ω_1, ω_2 is called a "pushout square"; loosely speaking P is called the pushout of G_1 and G_2 over H. (Strictly speaking, the maps of G_1 and G_2 to P are part of the pushout data).

The pushout of a diagram may be constructed from the coproduct as follows. Let $P = G_1 * G_2/\langle\langle \alpha_1(h)\alpha_2(h)^{-1} \mid h \in H\rangle\rangle$. (Thus P is the biggest quotent of $G_1 * G_2$ in which both images of H become identified). Let $\omega_i : G_i \to G_1 * G_2 \to P$ be the obvious maps, for $i = 1, 2$.

If α_1 and α_2 are each injective we write $P = G_1 *_H G_2$ and call this pushout the *generalized free product of* G_1 *and* G_2 *with amalgamation over* H. The coproduct of G_1 and G_2 is just the pushout of these groups over the trivial group: $G_1 * G_2 = G_1 *_{\{1\}} G_2$.

Exercise 32. Show that $(Z/4Z) *_{Z/2Z} (Z/4Z)$ has a composition series with quotients $Z/2Z$, Z and $Z/2Z$.

Exercise 33. Describe the pushout of two homomorphisms $\alpha_i : H \to G_i$ (for $i = 1, 2$) in terms of amalgamated products of certain quotient groups.

The notions of product, coproduct and pushout square are all essentially categorical, as is the notion of a "free" object - an adjoint to the forgetful functor from $((Grp))$ or $((Mod_R))$ (etc.) to $((Set))$. See under the headings "limit" and "adjoint functors" in any reference on Categories.

The Van Kampen Theorem

The Van Kampen Theorem is one of those theorems where it is very important to understand the result, but not at all necessary to know the techniques of proof to use efficiently. (I've survived thus for years). The statement is essentially that "the fundamental group of a pushout of spaces is the pushout of their fundamental groups". More precisely, suppose that X and Y are path-connected spaces with union Z, and that their intersection $X \cap Y$ is also path-connected. Choose a point $* \in X \cap Y$ to be the basepoint for all these spaces. Then $\pi_1(Z, *)$ is the pushout of the homomorphisms determined by the inclusions of $X \cap Y$ into X and Y. Thus it is generated by the images of $\pi_2(X, *)$ and $\pi_1(Y, *)$, and the only relations between the generator of $\pi_1(Z, *)$

are those which follow from relations in $\pi_2(X,*)$ or $\pi_1(Y,*)$ and from setting the images of $\pi_1(X\cap Y,*)$ in $\pi_2(X,*)$ and $\pi_1(Y,*)$ to be equal. The statement is more complicated if one or more of the spaces involved is not connected. (One should use groupoids rather than groups. There is not yet a good analogue for the higher homotopy groups, which is one reason why they are hard to compute, but the Mayer-Vietoris Theorem gives a satisfactory analogue for homology).

Examples.

(1) If Y is simply-connected then $\pi_1(X\cup Y,*)$ is the largest quotient of $\pi_1(X,*)$ in which the image of $\pi_1(X\cap Y,*)$ is trivial.

(2) Let $S^1\vee S^1$ be the figure eight and let $*$ be the common point of the two circles. Then $\pi_1(S^1\vee S^1,*)$ is free on two generators (one for each circle).

If X is path connected then $X\cup_f e^1\simeq X\vee S^1$, and hence $\pi_1(X\cup_f e^1)\cong \pi_1(X)*Z$.

Exercise 34. Define isometries t,u of the euclidean plane $\mathbb{R}^2$ by $t(x,y)=(x,y+1)$ and $u(x,y)=(x+1,-y)$ for all $(x,y)\in\mathbb{R}^2$. Verify that $tut=u$ and that t and u^2 commute. Let G be the group of homeomorphisms of R^2 generated by t and u. Show that the orbit of G through (x,y) is the set $\{(x+m,(-1)^m y+n)\mid m,n\in\mathbb{Z}\}$. Show that the projection of $\mathbb{R}^2$ onto the orbit space $G\backslash\mathbb{R}^2$ is a covering map. This orbit space is the Klein bottle Kb. Show that $\langle t,u\mid tut=u\rangle$ is a presentation for $\pi_1(Kb)\cong G$.

Exercise 35. (a) The Möbius band Mb may be constructed by identifying opposite sides of a rectangle with a half-twist. Explicitly, let

$$Mb=\{(x,y)\in\mathbb{R}^2\mid 0\le x\le 1,|y|\le 1\}/\sim .$$

where $(0,y)\sim(1,-y)$ for all $-1\le y\le 1$. The image of the line segment $\{(x,0)\mid 0\le x\le 1\}$ in Mb is a circle C, the centreline of Mb. Show that the inclusion of the centreline C is a homotopy equivalence. Show that the inclusion of the boundary ∂Mb sends a generator of $\pi_1(\partial Mb)\cong Z$ to twice a generator of $\pi_1(Mb)\cong Z$.

(b) The Klein bottle Kb may also be constructed by gluing two Möbius bands together along their boundaries. Use Van Kampen's Theorem to obtain a presentation of $\pi_1(Kb)$. (Hint: let M, M' be the two Möbius bands and ∂M, $\partial M'$ their boundary circles. Apply the observations of part (a)).

(c) Show that the presentation obtained in this way is equivalent to that of question 34.

We may reduce the study of the case with $X \cap Y$ disconnected to the connected case as follows. Let A and X be path-connected spaces, and suppose that $h_0, h_1 : A \to X$ are embeddings with disjoint images. Let $Y = (X \amalg A \times [0,1])/\sim$, where $(a,0) \sim h_0(a)$ and $(a,1) \sim h_1(a)$, for all $a \in A$. (In other words, glue $A \times [0,1]$ to X along its ends). Fix a basepoint $a \in A$ and take $x = h_0(a)$ as the basepoint for each of X and Y. Since X is path-connected there is a path $\omega : [0,1] \to X$ from $h_1(a)$ to $h_0(a)$.

[insert figure]

We shall assume that ω is an embedding, with image I. (This is OK for reasonable spaces, and can always be achieved after replacing X by a homotopy equivalent space, if necessary). Let $\alpha(t) = (a,t)$ for $0 \leq t \leq 1$ and let J be the image of α in Y. Let $\hat{X} = X \cup J \subseteq Y$. Note that $I \cup J \cong S^1$. Then $Y = \hat{X} \cup (A \times [0,1])$ is a union of connected sets with connected intersection $\hat{X} \cap (A \times [0,1]) = h_0(A) \cup (\{a\} \times (0,1)) \cup h_1(A)$. Moreover $\hat{X}$ is also such a union: $\hat{X} = X \cup (I \cup J)$ where $X \cap (I \cup J) = I$. Thus we can use the above version of the Van Kampen Theorem to compute $\pi_1(Y,x)$.

For simplicity of notation let $C = \pi_1(X,x)$ and $H = \pi_1(A,a)$. Let t be the loop $\alpha.\omega$ in $\hat{X} \subseteq Y$. Then $\pi_1(\hat{X},x) \cong G * Z$, where the second factor is generated by the image of t and $\pi_1(\hat{X} \cap (A \times [0,1]),x) \cong H * \bar{\alpha}_\# H$. The geometry determines two homomorphisms from H to G: $\theta_0 = h_{0*}$ and $\theta_1 = \omega_\# h_{1*}$. (We are being careful about basepoints!). In $\pi_1(Y,*)$ these are related by $t^{-1}h_{0*}(h)t = \omega_{\#}h_{1*}(h)$, for all $h \in H$. Hence

$$\pi_1(Y,*) = \langle G, t \mid t^{-1}\theta_0(h)t = \theta_1(h),\ \forall h \in H\rangle.$$

If θ_0 and θ_1 are monomorphisms this construction is called the *HNN extension* with *base* G, *associated subgroups* $H_0 = \theta_0(H)$ and $H_1 = \theta_1(H)$, and *defining isomorphism* $\phi = \theta_1\theta_0^{-1}$.
Notation: $HNN(G; \phi : H_0 \cong H_1)$, $G *_H \phi$ or just $G*_\phi$.

Exercise 36. Show how we may compute the fundamental group of a union $X \cup Y$ where $X \cap Y$ has finitely many components. (Hint: constructing the union amounts to identifying certain connected subsets of X with subsets of Y. Identify one pair at a time).

Exercise 37. Let $G = Z*_2$ be the HNN extension with base Z and associated subgroups Z and $2Z$. Show that $G/G' \cong Z$ and that $G' \cong \mathbb{Z}[\frac{1}{2}]$, the additive group of dyadic rationals $\{m/2^k \in \mathbb{Q} \mid m \in \mathbb{Z}, k \geq 0\}$.

The following proof of the "connected" case of the Van Kampen Theorem is taken from *Elements de Topologie Algébrique*, by C.Godbillon. The proof applies only to spaces for which covering space theory works, but they are the most important. (I've no idea whether the Van Kampen Theorem is true in much greater generality). The one basic fact that Godbillon uses is that if $\pi_1(X,*) = H$ then for every set S with a left H-action there is an essentially unique covering $p : E \to X$ such that $p^{-1}(*) \cong S$ as H-sets.

Let X and Y be connected open sets of $Z = X \cup Y$ with connected intersection $X \cap Y$, and let $j_1 : \pi_1(X \cap Y,*) \to \pi_1(X,*)$, $j_2 : \pi_1(X \cap Y,*) \to \pi_1(Y,*)$ $k_1 : \pi_1(X,*) \to \pi_1(Z,*)$ and $k_2 : \pi_1(Y,*) \to \pi_1(Z,*)$ be the homomorphisms induced by the inclusions. Suppose given a group G and homomorphisms $h_1 : \pi_1(X,*) \to G$ and $h_2 : \pi_1(Y,*) \to G$ such that $h_1j_1 = h_2j_2$. We shall show that there is an unique homomorphism $h : \pi_1(Z,*) \to G$ such that $hk_1 = h_1$ and $hk_2 = h_2$.

The groups $\pi_1(X,*)$ and $\pi_1(Y,*)$ act on G by left translations ($g \mapsto h_1(x)g$, etc.). Therefore there are regular coverings $p_1 : E_1 \to X$ and $p_2 : E_2 \to Y$ and isomorphisms $r : G \to p_1^{-1}(*)$ and $s : G \to p_2^{-1}(*)$ of sets with group actions.

Let $D_1 = p_1^{-1}(X \cap Y)$ and $D_2 = p_2^{-1}(X \cap Y)$. Since $h_1 j_1 = h_2 j_2$ the induced coverings $p|_{D_1} : D_1 \to X \cap Y$ and $p|_{D_2} : D_2 \to X \cap Y$ are isomorphic, and thee is an isomorphism $t : D_2 \to D_1$ such that $ts = r$. Let $E = E_1 \cup E_2$. Then p_1 and p_2 together determine a covering map $p : E \to Z$. Let $q : E_1 \amalg E_2 \to E$ be the natural map. Then $q|_{E_1}$ and $q|_{E_2}$ determine isomorphisms of E_1 and E_2 onto the induced covers $p : p^{-1}(X) \to X$ and $p : p^{-1}(Y) \to Y$, respectively.

The covering $p : E \to Z$ determines an action θ of $\pi_1(Z, *)$ on the fibre $p^{-1}(*)$. Taking into account the above isomorphisms we have $\theta(k_1(x))qr(g) = qr(h_1(x)g)$ and $\theta(k_2(x))qs(g) = qs(h_2(x)g)$. Let $u = qs = qts = qr$. the function $\hat{h}$ defined by $\hat{h}(\gamma) = u^{-1}\theta(\gamma)u$ determines an action of $\pi_1(Z, *)$ on G such that $\hat{h}(k_1(x))(g) = h_1(x)g$ and $\hat{h}(k_2(y))(g) = h_2(y)(g)$. Since $\pi_1(Z, *)$ is generated by the images of k_1 and k_2 the permutation $\hat{h}(\gamma)$ is translation by an element $h(\gamma)$ of G and the map sending γ to $h(\gamma)$ is a homomorphism of $\pi_1(Z, *)$ to G such that $h_1 = hk_1$ and $h_2 = hk_2$.

Since $\pi_1(Z, *)$ is generated by the images of k_1 and k_2 the homomorphism h is unique. Hence $\pi_1(Z, *)$ is the pushout.

P.J.Higgins has used groupoids to give a uniform treatment of the general case [Math. Proc. Cambridge Phil. Soc. 60 (1964), 7-20].

Realizing groups by 2-complexes and 4-manifolds

It follows from the Van Kampen Theorem that the fundamental group of a finite complex is finitely presentable. As it can be shown that every compact manifold is homotopy equivalent to a finite complex, compact manifolds also have finitely presentable fundamental groups.

Let $P = \langle X \mid R \rangle$ be a finite presentation for a group G. This presentation determines a finite 2-dimensional cell complex $C(P)$ with one 0-cell, a 1-cell for each generator and a 2-cell for each relator. Adjoining the 1-cells to the 0-cell (which we take as the basepoint) gives a wedge $\vee^{|X|}S^1$, with fundamental group $F(|X|)$. Each relator is a word in $F(|X|)$, and so corresponds to a homotopy class of loops in $\vee^{|X|}S^1$. We attach a 2-cell along a representative loop, for each relator. It then follows from Van Kampen's Theorem that $\pi_1(C(P)) \cong G$. Conversely, every connected 2-dimensional complex with a single 0-cell arises in this way.

Exercise 37. A (finitely generated) group is free if and only if it is the fundamental group of a (finite) connected graph, i.e.a connected 1-dimensional cell complex. Use covering space theory to conclude that subgroups of free groups are free. (This is quite delicate to prove algebraically!)

Definition. The *deficiency* of a finite presentation $P = \langle X \mid R\rangle$ is $def(P) = |X| - |R|$. The deficiency of a finitely presentable group is G is $\max\{def(P) \mid P\ presents\ G\}$.

The 2-complex $C(P)$ has Euler characteristic $\chi(C(P)) = 1 - |X| + |R| = 1 - def(P)$. Every connected 2-dimensional complex is homotopy equivalent to one with a single 0-cell.

Example. Suppose G has presentation $\langle x, y \mid r\rangle$. Let $V = S^1 \vee S^1$. Then $\pi_1(V) \cong F(\{x, y\}) = F(2)$. The word r determines a homotopy class of maps $r : S^1 = \partial D^2 \to V$. Let $C = V \cup_r e^2 = (S^1 \vee S^1 \cup D^2)/\sim$, where we identify points of ∂D^2 with their images under r.

Let $A = int\frac{2}{3}D^2$ and $B = C - \frac{1}{2}D^2$. Then $A \cap B = S^1 \times (\frac{1}{2}, \frac{2}{3}) \simeq S^1$ and $V \simeq B$, so Van Kampen's Theorem gives $\pi_1(C) \cong F(\{x, y\}/\langle\langle r\rangle\rangle \cong G$. The general case is similar.

We must work slightly harder to realize (finitely presentable) groups as fundamental groups of manifolds.

The circle S^1 is the only compact connected 1-manifold without boundary, and $\pi_1(S^1, 1) \cong \mathbb{Z}$ is the only 1-dimensional Poincaré duality group.

There is a complete list of closed surfaces (compact connected 2-manifolds without boundary), and their fundamental groups may be considered well understood. Each may be constructed by identifying the sides of a polygon in pairs. This leads to the presentations:

(i) orientable surfaces: $\langle a_i, b_i, 1 \leq i \leq g \mid \Pi[a_i, b_i] = 1\rangle$. The case $g = 1$ corresponds to the torus.

(ii) nonorientable surfaces: $\langle v_j, 1 \leq j \leq c \mid \Pi v_j^2 = 1\rangle$. The cases $c = 1$ and $c = 2$ correspond to the projective plane and Klein bottle, respectively.

Excepting only $\pi_1(P^2(\mathbb{R})) = Z/2Z$, they are all torsion free, and the torsion free surface groups have an intrinsic algebraic characterization as the 2-dimensional Poincaré duality groups. The fundamental group determines the surface up to homeomorphism.

Less is known about 3-manifolds and their groups. The indecomposable factors of a 3-manifold group are either Z, $Z \oplus (Z/2Z)$, finite (and conjecturally one of a known infinite list of subgroups of $SO(4)$) or a 3-dimensional Poincaré duality group. (It is not known whether the latter are always 3-manifold groups). It is conceivable that 3-manifold groups are sufficiently special that geometric methods may provide algorithmic solutions to the standard decision problems of combinatorial group theory.

Example. If a, b, c, d be integers such that $ad - bc = \pm 1$ the map $f(w, z) = (w^a z^b, w^c, z^d)$ determines a self homeomorphism of the torus $S^1 \times S^1$. Then $\pi_1((S^1 \times D^2) \cup_f (S^1 \times D^2))$ is cyclic.

$\pi_1(S^1 \times S^1 \times S^1) \cong Z^3$.

$\pi_1(S^1 \times P^2(\mathbb{R})) \cong Z \oplus (Z/2Z)$.

No other abelian group is the fundamental group of a closed 3-manifold.

On the other hand, every finitely presentable group is the fundamental group of some closed (orientable) 4-manifold. There are two natural approaches, each essentially "thickening" the above construction so that we may assume the attaching maps are embeddings.

(a) With a little more care, we may in fact assume that $C(P)$ is a 2-dimensional polyhedron. Now every n-dimensional polyhedron can be embedded in R^{2n+1}. (This is a "general position" argument. The expected dimension of the intersection of linear subspaces $P, Q \subseteq R^n$ is $dim(P)+dim(Q)-n$). Thus if $C(P)$ is a 2-dimensional polyhedron it may be embedded as a union of polygon subsets of R^5. For ϵ sufficiently small the set $N_\epsilon = \{x \in R^5 \mid |x - C(P)| \leq \epsilon\}$ of all points within distance ϵ of $C(P)$ is a compact 5-manifold, and the inclusion $C(P) \subset N_\epsilon$ is a homotopy equivalence. (Moreover any two such "regular neighbourhoods" are homeomorphic). Then ∂N_ϵ is a closed orientable 4-manifold, and it can be shown that $\pi_1(\partial N_\epsilon) \cong \pi_1(N_\epsilon) \cong G$.

(b) We may mimic the construction of a cell complex by "attaching 1- and 2-handles" to the 4-disc, to obtain a compact, bounded 4-manifold homotopy equivalent to $C(P)$. We attach the 1-handles $D^1 \times D^3$ by embedding $|X|$ disjoint copies of $S^0 \times D^3$ in ∂D^4, to get the boundary connected sum $\natural^{|X|}(S^1 \times D^3) \simeq \vee^{|X|} S^1$. We then attach the 2-handles $D^2 \times D^2$ by embedding $|R|$ disjoint copies of $S^1 \times D^2$ in $\partial \natural^{|X|}(S^1 \times D^3) = \sharp^{|X|}(S^1 \times S^2)$. Note that $\pi_1(\sharp^{|X|}(S^1 \times S^2) \cong F(|X|)$. Then $M = D^4 \cup^{|X|} (D^1 \times D^3) \cup^{|R|} (D^2 \times D^2) \simeq C(P)$, and $\pi_1(\partial M)$ maps onto $\pi_1(M)$. The double $D(M) = M \cup_{\partial M} M$ is then a closed 4-manifold and we again have $\pi_1(M) \cong G$.

The Hurewicz Theorem and homology in low degrees

Let X be path-connected, and let $*$ be a basepoint for X. For each $q \geq 0$ let $*_q : \Delta_q \to X$ be the constant map with value $*$, considered as a singular q-simplex in X.

The map $d : [0,1] \to \Delta_1$ given by $d(u) = (1-u, u)$ is a homeomorphism such that $d(0) = (1,0)$ and $d(1) = (0,1)$. The inverse homeomorphism is given by projection on the y-axis: $d^{-1}(x,y) = y$. (In Part I we used the symbol p for $d^{-1} : \Delta \to [0,1]$). A path $\alpha : [0,1] \to X$ determines a singular 1-simplex $\hat{\alpha} = \alpha d^{-1}$, with boundary $\partial\hat{\alpha} = \alpha(1) - \alpha(0)$. Clearly every singular 1-simplex arises in this way. Note also that $\hat{\alpha}$ is a 1-cycle if and only if α is a loop (i.e., $\alpha(1) = \alpha(0)$).

Degree 0. Let $\epsilon : C_0(X) \to R$ be the homomorphism defined by $\epsilon(x) = 1$ for all $x \in X$. Then $\mathrm{Ker}(\epsilon) = B_0(X)$, and so ϵ induces an isomorphism $H_0(X;R) \cong R$. (If σ is a singular 1-simplex then $\epsilon(\partial\sigma) = 0$. Conversely, suppose that $\epsilon(\Sigma r_x x) = 0$. Since X is path-connected we may choose a path γ_x from $*$ to x for each point $x \in X$. Since $0 = \epsilon(\Sigma r_x x) = \Sigma r_x$ we have $\Sigma r_x x = \Sigma r_x (x - *) = \partial(\Sigma r_x \hat{\gamma}_x)$. In particular, $H_0(X;R)$ is generated by the image of any point of X (considered as a singular 0-cycle in X).

If X is *not* path-connected then $H_0(X;R)$ is the free R-module with basis $\pi_0(X)$. Note also that there is a natural bijection $\pi_0(X) \cong [S^0, 1; X, *]$.

Degree 1. Let h be the function given by $h(\alpha) = [\hat{\alpha}]$ (the homology class of the cycle $\hat{\alpha}$) for all loops α at $*$ in X. The Hurewicz Theorem in degree 1 asserts that h induces a homomorphism from $\pi_1(X,*)$ to $H_1(X;\mathbb{Z})$, which in turn induces an isomorphism $\pi_1(X,*)^{ab} \cong H_1(X;\mathbb{Z})$, on abelianization.

Let α be a loop at $*$ in X. Define a singular 2-simplex $\sigma : \Delta_2 \to X$ by $\sigma(x,y,z) = \alpha(x+z)$. Then $\sigma F_2^0 = \hat{\alpha}$, $\sigma F_2^1 = *_1$ and $\sigma F_2^2 = \hat{\bar{\alpha}}$, so $\partial\sigma = \hat{\alpha} - *_1 + \hat{\bar{\alpha}}$. Since $\partial *_2 = *_1 - *_1 + *_1 = *_1$, it follows that $\hat{\alpha} + \hat{\bar{\alpha}} = \partial(\sigma + *_2)$ and hence that $h(\bar{\alpha}) = -h(\alpha)$.

Suppose now that $A : [0,1]^2 \to X$ is a homotopy of loops from α to α'. Define singular 2-simplices by $\sigma_I(x,y,z) = A(y,z)$ and $\sigma_{II}(x,y,z) = A(1-z, 1-y)$. Then $\partial(\sigma_I - \sigma_{II} + 2*_2) = \hat{\alpha} + \widehat{\bar{\alpha}'}$, so $h(\alpha) = -h(\bar{\alpha}') = h(\alpha')$. Thus h gives rise to a function hwz from $\pi_1(X,*)$ to $H_1(X;\mathbb{Z})$.

Let β be another loop at $*$ in X. Define a singular 2-simplex $\tau : \Delta_2 \to X$ by $\tau(x,y,z) = \alpha(1-x+z)$ if $x \geq z$ and $\tau(x,y,z) = \beta(z-x)$ if $x \leq z$. Then $\tau F_2^0 = \hat{\beta}$, $\tau F_2^1 = \widehat{\alpha.\beta}$ and $\tau F_2^2 = \hat{\alpha}$, so $\partial\tau = \hat{\alpha} - \widehat{\alpha.\beta} + \hat{\beta}$. In particular, if α and β are loops at $*$ then $hwz(\alpha.\beta) = hwz(\alpha) + hwz(\beta)$. Thus hwz is a homomorphism, and hence $\pi_1(X,*)' \leq \mathrm{Ker}(hwz)$, since homology groups are abelian. We shall show that hwz is onto and that $\mathrm{Ker}(hwz) = \pi_1(X,*)'$ (i.e., is no larger).

Choose a path γ_x from $*$ to x for each $x \in X$. Then for any path α in X the path $\gamma_{\alpha(0)}.\alpha.\bar{\gamma}_{\alpha(1)}$ is a loop at $*$, and $hwz(\gamma_{\alpha(0)}.\alpha.\bar{\gamma}_{\alpha(1)})$ is the homology class of $\hat{\gamma}_{\alpha(0)} + \hat{\alpha} + \hat{\bar{\gamma}}_{\alpha(1)}$ (which is a 1-cycle). Let $\xi = \Sigma r_\alpha \hat{\alpha}$ be a singular 1-cycle in X. (Here the sum is over a finite set of paths α in X and the coefficients r_α are integers). Then $0 = \partial\xi = \Sigma r_\alpha \alpha(1) - \Sigma r_\alpha \alpha(0)$, i.e., the endpoints of these paths match in pairs, with the same multiplicities. Therefore $\Sigma r_\alpha(\hat{\gamma}_{\alpha(0)} - \hat{\gamma}_{\alpha(1)}) = 0$ and so $\xi = \Sigma r_\alpha(\hat{\gamma}_{\alpha(0)} + \hat{\alpha} - \hat{\gamma}_{\alpha(1)})$, which has the same homology class as $\Sigma r_\alpha(\hat{\gamma}_{\alpha(0)} + \hat{\alpha} + \hat{\bar{\gamma}}_{\alpha(1)})$. It follows that hwz is onto.

It remains to prove that if α is a loop such that $\hat{\alpha} = \partial\zeta$ for some singular 2-chain ζ in X then α is in $\pi_1(X,*)'$, the commutator subgroup. Let $\zeta = \Sigma_{i \in I} \varepsilon(i)\sigma_i$ where the σ_i are singular 2-simplices and $\varepsilon(i) = \pm 1$, and write $\partial\sigma_i = \sigma_{i0} - \sigma_{i1} + \sigma_{i2}$. Let $I_+ = \{i \in I \mid \varepsilon(i) = 1\}$ and $I_- = \{i \in I \mid \varepsilon(i) = -1\}$. Let Δ_1 have the standard orientation (from $(1,0)$ to $(0,1)$, and orient (the boundary of) Δ_2 so that the face maps F_0 and $F_2 : \Delta_1 \to \Delta_2$ are orientation preserving. (Then F_1 reverses the orientation). Note that this orientation corresponds to the sequence (cycle) of faces $(2,0,1) = (0,1,2)$. Reversing the orientation of the boundary of Δ_2 gives the cycle $(2,1,0) = (1,0,2)$. Let

$S = I \times \Delta_2$ and give each component of $I_+ \times \Delta_2$ the standard orientation and each component of $I_- \times \Delta_2$ the opposite orientation. Then S is a finite family of oriented affine 2-simplices. In the equation $\hat{\alpha} = \Sigma_{i \in I} \varepsilon(i)(\sigma_{i0} - \sigma_{i1} + \sigma_{i2})$ the faces σ_{in} must match in pairs (with opposite signs), except for one equal to $\hat{\alpha}$.

CLAIM: if $\sigma_{im} = \sigma_{jn}$ and cancel in this sum (i.e., have opposite signs) then exactly one of the two corresponding face maps $F_m : \Delta_1 \to \{i\} \times \Delta_2$ and $F_n : \Delta_1 \to \{j\} \times \Delta_2$ is orientation preserving. (There are two cases to check).

Identifying these two 2-simplices along this pair of edges gives a quadrilateral Q such that

(i) $\sigma_i \cup \sigma_j$ defines a continuous function from Q to X;

(ii) the chosen orientations determine a consistent orientation for (the boundary of) Q;

(iii) the CLAIM remains valid for other pairs of faces yet to be identified in this way.

After finitely many such pairwise gluings we obtain a finite set of polygons $P = P_0, \dots P_k$ with

(a) oriented boundaries $\partial P, \dots \partial P_k$;

(b) the edges making up the boundary of each polygon correspond to the as yet unused faces of $\{\sigma_i \mid i \in I\}$;

(c) these faces are matched in pairs (except for one equal to $\hat{\alpha}$, which corresponds to an edge of ∂P);

(d) exactly one edge of each pair is oriented consistently with ∂P.

We discard the components $P_1, \dots P_k$. (These correspond to disjoint summands of ζ which are 2-cycles, and thus have algebraic boundary 0). With a little more work we can arrange that all the edges represent loops at $*$. Thus ∂P is a product of loops at $*$, and clearly represents the identity elements of $\pi_1(X, *)$ (since it extends to a map from P to X). We thus get an equation $1 = \alpha\Pi$ where Π is the product of all the edges except α. Clearly Π is in $\pi_1(X, *)'$, since each factor occurs twice with exponents 1 and -1, and so α is also in $\pi_1(X, *)'$. (Alternatively, continue gluing pairs of free edges until only $\hat{\alpha}$ is left. We obtain an orientable surface R with one boundary component, and a map $r : R \to X$ agrreing with $\hat{\alpha}$ on ∂R. A separate argument shows that ∂R represents a product of commutators in $\pi_1(R)$. Hence $\hat{\alpha} \in r_*(\pi_1(R,*)') \leq \pi_1(X,*)')$.

The Hurewicz homomorphism in degree 1 is basepoint-independent in the sense that if ω is a path from x to x' and α is a loop at x then $hwz(\widehat{\omega_\# \alpha}) = hwz(\hat{\alpha})$.

Degree 2 or more. Each homology class in $H_2(X;\mathbb{Z})$ may be represented by a "singular surface" in X (i.e., a map $f : F \to X$, where F is a closed orientable surface). To see this, note that a singular 2-cycle is a formal sum of maps of triangles into X whose algebraic boundary is 0. The boundary condition implies that the edges of the triangles may be matched in pairs, on which the restrictions of the defining maps agree. We may glue the triangles together to get a closed surface, and the maps together give a map of this surface into X. It can be shown that two such singular surfaces f and f' represent the same homology class if and only if they together extend across some compact orientable 3-manifold W with boundary the disjoint union of F and $-F'$, i.e., there is a map $F : W \to X$ which restricts to f and f' on the boundaries. (The sign refers to an orientation condition). There is a related "geometric" interpretation of the higher homology groups, involving maps of polyhedral manifolds (with "controlled" singularities of lower dimension). In tha analytic context, a Riemannian metric on a smooth manifold determines canonical "harmonic" forms representing real cohomology classes (Hodge theory).

In every dimension $q \geq 1$ there is a function from $\pi_q(X) = [S^q; X]$ to $H_q(X;\mathbb{Z})$, sending $[f] \in \pi_q(X)$ to $H_q(f;\mathbb{Z})([S^q])$, where $[S^q]$ is a fixed generator for $H_q(S^q;\mathbb{Z}) \cong \mathbb{Z}$. The Hurewicz Theorem in degree $q > 1$ asserts that this is a homomorphism, and if X is path-connected and all maps from S^k to X are homotopic to constant maps, for all $k < q$, then $H_k(X;\mathbb{Z}) = 0$ for $1 \leq k < q$ and the Hurewicz homomorphism in degree q is an isomorphism.