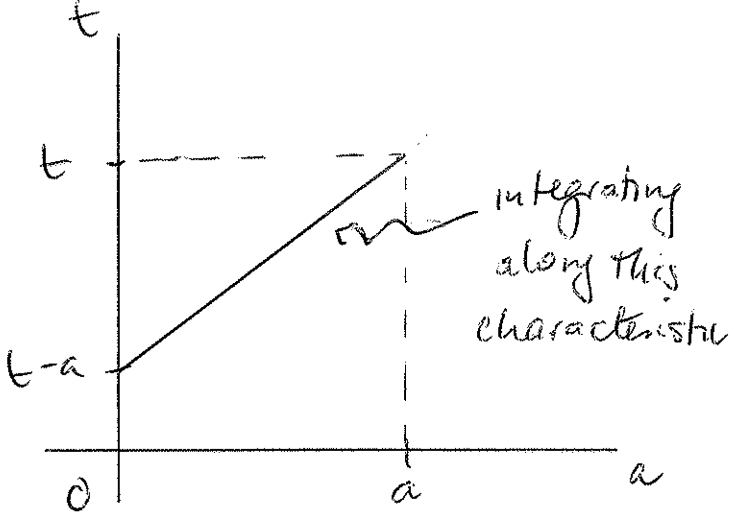
Assignment 1. - Populations and Disease.

Applied Maths 4.

1 (a). (i). $1 - \exp\left(-\int_0^a \mu(s)\,ds\right)$ is the proportion of the cohort that dies before age a.

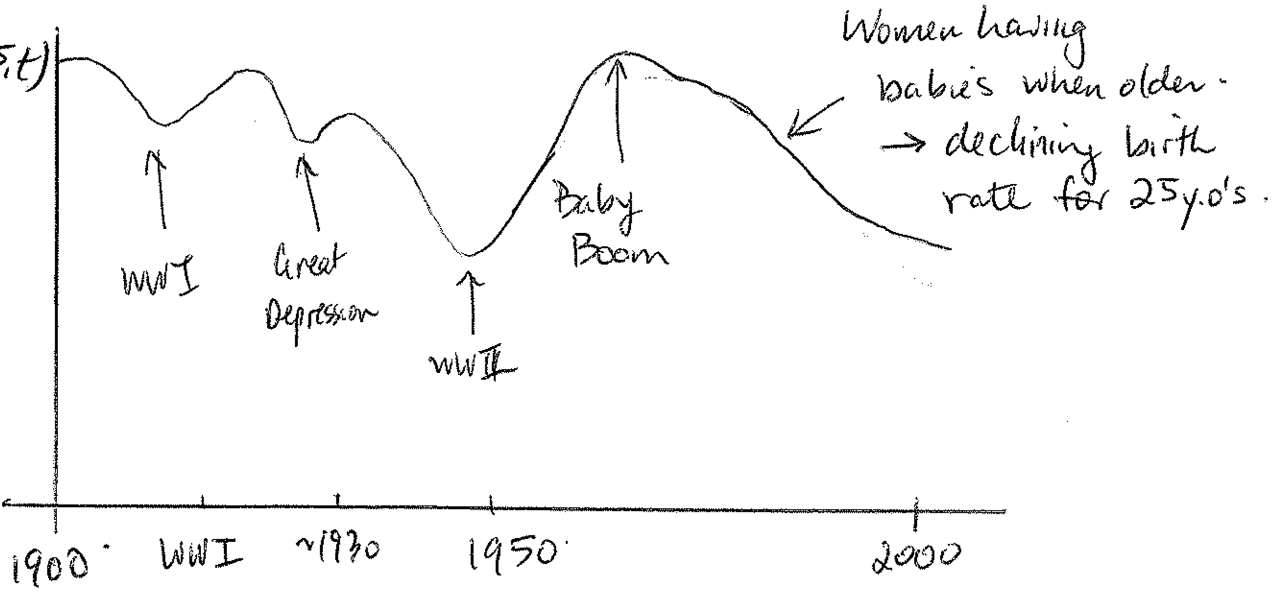
(ii)

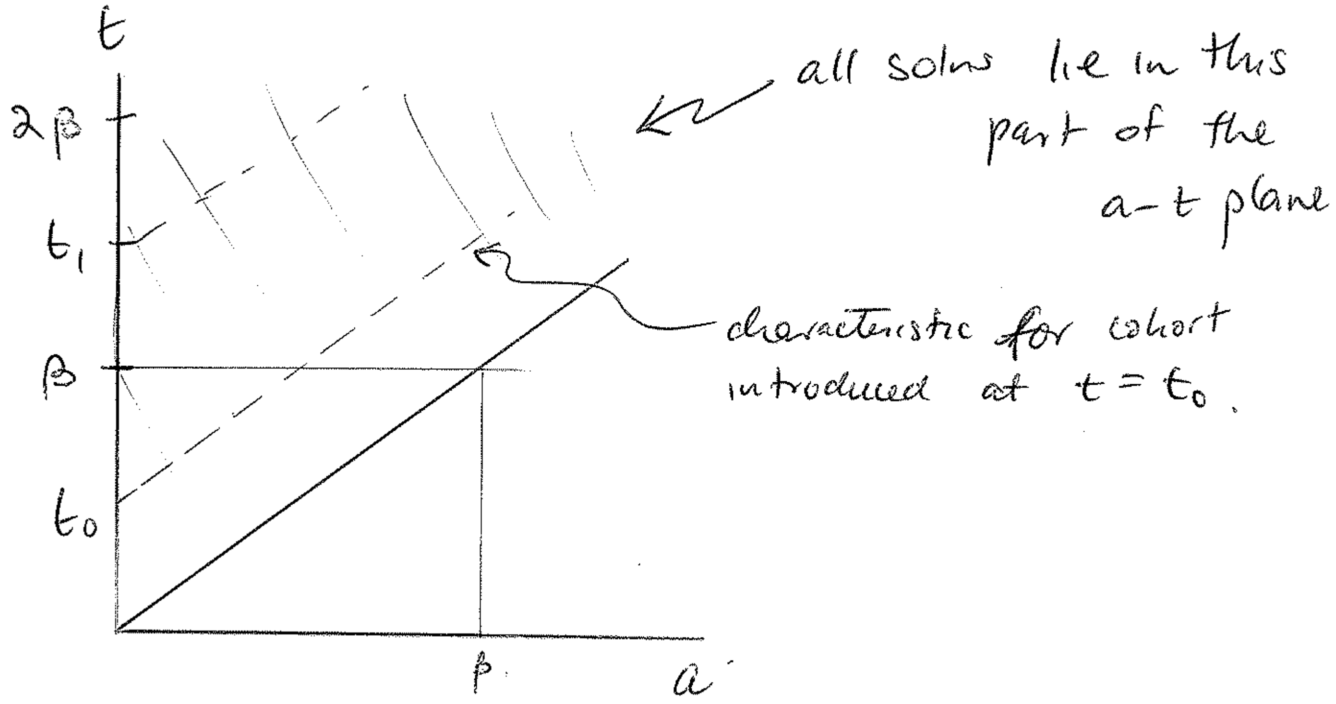


$$\int_{t-a}^{t} b(s-(t-a))\,n(s-(t-a),s)\,ds.$$

is the number of offspring born to a cohort who themselves were born at $t-a$ by the time that cohort has reached age a.

(b). $b(25,t)$





(i) Let $t = a + t_0$ so $a = t - t_0$

$w_{t_0}(t) = n(t - t_0, t)$ and

$$\frac{dw_{t_0}}{dt} = -\mu(t - t_0)\, w_{t_0}(t) = -m\, w_{t_0}(t)$$

So $w_{t_0}(t) = Ae^{-mt}$.

When $t = t_0$ $w(t_0) = Ae^{-mt_0} = \eta$, the rate of introduction.

So $A = \eta e^{mt_0}$ and

$$w_{t_0}(t) = \eta e^{-m(t-t_0)}.$$

(ii) Hence $n(a,t) = \eta e^{-m(t-t_0)}$ where $t_0 = t - a$

$$= \eta e^{-m(t-(t-a))}$$

$$= \eta e^{-ma}$$ when $a > t - \beta$

(iii). For $\beta < t_1 < 2\beta$

$$n(0,t_1) = \int_0^\infty b(a)\, n(a,t)\, da$$

$$= \int_\beta^{t_1} B\eta e^{-ma}\, da$$ since the oldest individuals are age t_1 and the youngest reproductive individuals are age β.

$$= \frac{B\eta}{m}\left[-e^{-ma}\right]_\beta^{t_1}$$

$$= \frac{B\eta}{m}\left(e^{-m\beta} - e^{-mt_1}\right)$$

(iv) For a cohort function born at $t = t_1$

$$\frac{dw_{t_1}}{dt} = -m w_{t_1} \quad \text{so} \quad w_{t_1}(t) = C e^{-mt}$$

When $t = t_1$

$$w_{t_1}(t_1) = Ce^{-mt_1} = \frac{B\eta}{m}\left(e^{-m\beta} - e^{-mt_1}\right).$$

so $$C = \frac{B\eta}{m}\left(e^{-m(\beta - t_1)} - 1\right).$$

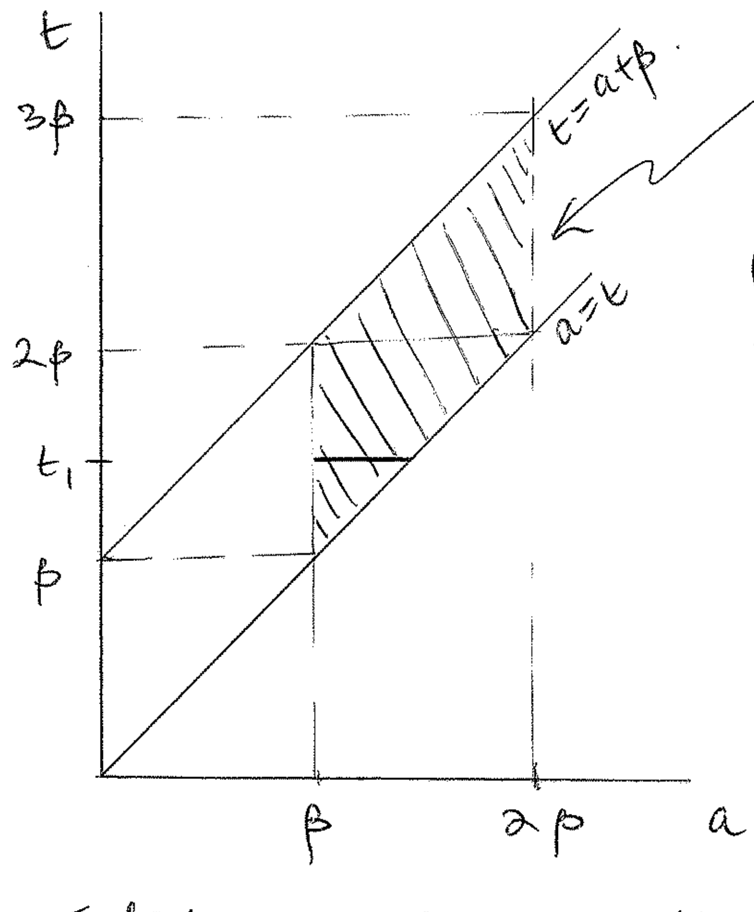
so. $$w_{t_1}(t) = \frac{B\eta}{m}\left(e^{m(t_1-\beta)} - 1\right) e^{-mt}.$$

Since $a = t - t_1$, so $t_1 = t - a$.

$$n(a,t) = \frac{B\eta}{m}\left(e^{m(t-a-\beta)} - 1\right) e^{-mt}$$

$$= \frac{B\eta}{m}\left(e^{-m(a+\beta)} - e^{-mt}\right) \text{ for } \beta < t-a < 2\beta.$$

(v) There are a number of different ways this can be done – here is one of them.



We want to integrate $b(a)n(a,t)$ over this area.

(Previously in part (iii) of the question we integrated $b(a)n(a,t)$ only along the line $t = t_1$ in the shaded area)

Total number of offspring born to parents introduced when $t \le \beta$

$$= \int_{\beta}^{2\beta}\left(\int_{a}^{a+\beta} b(a)\, n(a,t)\, dt\right) da$$

$$= \int_{\beta}^{2\beta} \int_{a}^{a+\beta} B\eta e^{-ma}\, dt\, da$$

$$= \int_{\beta}^{2\beta} \left[t B\eta e^{-ma} \right]_{a}^{a+\beta} da$$

$$= \int_{\beta}^{2\beta} \beta B\eta e^{-ma}\, da$$

$$= \left[-\frac{1}{m} \beta B\eta e^{-ma} \right]_{\beta}^{2\beta}$$

$$= \frac{\beta B\eta}{m} \left[-e^{-2\beta m} + e^{-\beta m} \right]$$

$$= \frac{\beta e^{-\beta m} B\eta}{m} \left(1 - e^{-\beta m} \right).$$

Number of offspring born to individuals introduced when $t \leq \beta$ is $\frac{\beta e^{-\beta m} B\eta}{m} (1 - e^{-\beta m})$.

Here is another approach: In part we found $n(0, t_1) = \int_{\beta}^{t_1} b(a) n(a, t_1)\, da$ for $\beta < t_1 < 2\beta$. To find the number of offspring born before $t = 2\beta$ we integrate this w.r.to t_1:

$$\int_\beta^{2\beta} \left(\int_\beta^{t_1} b(a)\, n(a,t)\, da \right) dt$$

$$= \int_\beta^{2\beta} \frac{B\eta}{m} \left(e^{-m\beta} - e^{-mt_1} \right) dt_1 \qquad \text{(from part iii)}$$

$$= \frac{B\eta}{m} \left(t_1 e^{-m\beta} - \frac{1}{m} e^{-mt_1} \right) \Big|_\beta^{2\beta}$$

$$= \frac{B\eta}{m} \left((2\beta - \beta) e^{-m\beta} - \frac{1}{m} \left(e^{-2\beta m} - e^{-\beta m} \right) \right)$$

$$= \frac{B\eta}{m} e^{-m\beta} \left(\beta - \frac{1}{m} \left(e^{-\beta m} - 1 \right) \right).$$

We also have to integrate the total number born in the upper triangle. For fixed $t = t_2$ where $2\beta < t_2 < 3\beta$ we have:

number of offspring born at $t = t_2$ to parents introduced before $t = \beta$

$$= \int_{t_2 - \beta}^{2\beta} b(a)\, n(a, t_2)\, da$$

$$= \int_{t_2 - \beta}^{2\beta} B\eta e^{-ma}\, da$$

$$= \frac{B\eta}{m} \left(e^{-(t_2 - \beta)m} - e^{-2\beta m} \right).$$

ver t_2 to get the total number in
n triangle

$$/ e^{-(t_2-\beta)m} - e^{-2\beta m}) dt_2$$

$$\frac{1}{m} e^{-(t_2-\beta)m} - t_2 e^{-2\beta m}) \Big|_{2\beta}^{3\beta}$$

$$(e^{-2\beta m} - e^{-\beta m}) - \beta e^{-2\beta m})$$

$$^{n} \left(\frac{1}{m} (e^{-\beta m} - 1) - \beta e^{-\beta m} \right).$$

esults from the two triangles to get
umber of offspring born to parents
before $t = \beta$:

$$- \frac{1}{m} (e^{-\beta m} - 1)) + \frac{B\gamma}{m} e^{-\beta m} \left(\frac{1}{m} (e^{-\beta m} - 1) - \beta e^{-\beta m} \right)$$

$e^{-\beta m})$ which agrees with the

The average number of offspring born to any individual is:

$$\frac{\text{total number of offspring born per age cohort}}{\text{total number in cohort at birth}}$$

$$= \frac{\int_{\beta}^{2\beta} B\eta e^{-ma}\, da}{\eta}$$

$$= \frac{\frac{B\eta}{m}\left(e^{-\beta m} - e^{-2\beta m}\right)}{\eta}$$

$$= \frac{B}{m}\left(e^{-\beta m} - e^{-2\beta m}\right).$$

(vii) At $t = \beta$ the max age is β so

$$N(\beta) = \int_0^{\beta} \eta e^{-ma}\, da$$

$$= \frac{\eta}{m}\left(1 - e^{-m\beta}\right).$$

contd....

$$N(2\beta) = \int_0^\beta n(a, 2\beta)\,da \quad + \quad \int_\beta^{2\beta} n(a, 2\beta)\,da$$

Individuals born after $t = \beta$ (see part iv) — Individuals born before $t = \beta$ (see part ii).

$$= \int_0^\beta \frac{B\eta}{m}\left(e^{-(a+\beta)m} - e^{-2m\beta}\right) da + \int_\beta^{2\beta} \eta e^{-ma}\,da$$

$$= \frac{B\eta}{m}\left(-\frac{1}{m} e^{-(a+\beta)m} - a e^{-2m\beta}\right]_0^\beta + \frac{\eta}{m}\left(-e^{-ma}\right]_\beta^{2\beta}$$

$$= \frac{B\eta}{m}\left[-\frac{1}{m} e^{-2\beta m} - \beta e^{-2m\beta} + \frac{1}{m} e^{-\beta m}\right] + \frac{\eta}{m}\left(e^{-m\beta} - e^{-2m\beta}\right)$$

$$= \frac{\eta}{m} e^{-\beta m}\left(\frac{B}{m}\left(1 - e^{-m\beta}(1+\beta m)\right) + \left(1 - e^{-m\beta}\right)\right)$$

$$= \frac{\eta B}{m^2} e^{-\beta m}\left(1 - e^{-m\beta}(1+\beta m)\right) + e^{-\beta m}\left(N(\beta)\right).$$

$$= -\frac{\eta B}{m}\beta e^{-2\beta m} + \frac{B}{m} e^{-\beta m} N(\beta) + e^{-\beta m} N(\beta).$$

$$= -\frac{\eta B}{m}\beta e^{-2\beta m} + \left(1 + \frac{B}{m}\right) e^{-\beta m} N(\beta).$$

For $N(2\beta) > N(\beta)$ we would take B large, m small and η & β small. This last requirement will reduce the effect of the $-\frac{\eta B}{m}\beta e^{-2\beta m}$ term.

If we want to get more specific, we can look at the equality $N(\beta) = N(2\beta)$. and find a more precise algebraic condition so.

$$\frac{\eta}{m}\left(1-e^{-m\beta}\right) = \frac{\eta}{m} e^{-m\beta}\left(\frac{B}{m}\left(1-e^{-m\beta}(1+\beta m)\right)+\left(1-e^{-m\beta}\right)\right)$$

$$\left(1-e^{-m\beta}\right) = \left(\frac{B}{m}+1\right)e^{-m\beta} - e^{-2m\beta}\left(\frac{B}{m}(1+\beta m)+1\right).$$

So if $N(\beta) < N(2\beta)$.

$$1 - e^{-m\beta}\left(\frac{B}{m}+2\right) + e^{-2\beta m}\left(\frac{B}{m}(1+\beta m)+1\right) < 0.$$

or $$e^{2m\beta} - e^{m\beta}\left(\frac{B}{m}+2\right) + \left(\frac{B}{m}(1+\beta m)+1\right) < 0.$$

This is a quadratic in $e^{m\beta}$ which will have real roots if $$\left(\frac{B}{m}+2\right)^2 - 4\left(\frac{B}{m}(1+\beta m)+1\right) > 0$$

or $$\left(\frac{B}{m}-2\right)^2 - 4(B\beta+1) > 0. \quad (1)$$

We also require the largest root to be positive as $e^{\beta m}$ is always positive, but since $\frac{B}{m}+2>0$ and $\frac{B}{m}(1+\beta m)+1 > 0$ this is always satisfied and so we require condition (1) for real roots and

$$(B/m+2) - \sqrt{\Delta} < 2e^{m\beta} < \left(\frac{B}{m}+2\right)+\sqrt{\Delta}$$

for $N(2\beta) > N(\beta)$. (The solution on this page was devised by David Kelly, Philip Howes and Sam Butler AM4 2008.)