Consider a model for an SIS epidemic which has continuous age structure. We make the following assumptions

- the total population is stable with total size N;
- every individual that is born survives until age L then dies;
- all individuals are born susceptible;
- the age structure of the disease has reached steady state;
- the force of infection λ is constant;
- the removal rate from the infective to susceptible class is given by γI.

Let $s(a) = S(a)/N$ be the fraction of susceptibles in the population and $i(a) = I(a)/N$ be the fraction of infectives. Show that

$$\frac{di}{da} = \lambda(1-i) - \gamma i.$$

Solve this equation to get expressions for $i(a)$ and $s(a)$. What happens as $a \to \infty$? Sketch $s(a)$ and $i(a)$ as a function of a.

Comment on the likely progress of the disease through an age cohort. Will there be an age where the prevalence of the disease peaks?

The average age of infection is conventionally given by

$$\frac{\int_0^\infty a\lambda s(a)da}{\int_0^\infty \lambda s(a)da}.$$

Find the average age of infection for this model in terms of L. What happens to the age of infection as L increases? Why does this occur?

A vaccination program is implemented where a proportion p of the susceptible population is a vaccinated at age a_v. This effectively introduces a removed fraction, for $a > a_v$ so that $r = ps(a_v)$, such that $r + s + i = 1$. The force of infection also decreases after vaccination (fewer contacts will produce infections) and becomes $\lambda_v < \lambda$.

Show that, for $a > a_v$,

$$\frac{di}{dt} = \lambda_v(1-r) - (\gamma + \lambda_v)i.$$

Hence or otherwise show that the number of infectives when $a > a_v$ is given by

$$i(a) = \frac{\lambda_v}{\gamma + \lambda_v}\left((1-r) + \left(re^{(\lambda_v+\gamma)a_v} - 1\right)e^{-(\lambda_v+\gamma)a}\right).$$

Sketch $i(a)$ and $s(a)$ for $a \in [0, L]$ when vaccination occurs.

Find i and s as $a \to \infty$. Hence show that vaccination will always reduce the number of infectives as $t \to \infty$ and will increase the number of susceptibles as $t \to \infty$ if $1 - r > (\gamma + \lambda_v)/(\gamma + \lambda)$.

What effect will vaccination have on the average age of infection? (You do not have to calculate the new average age of infection, but you should explain your answer.)