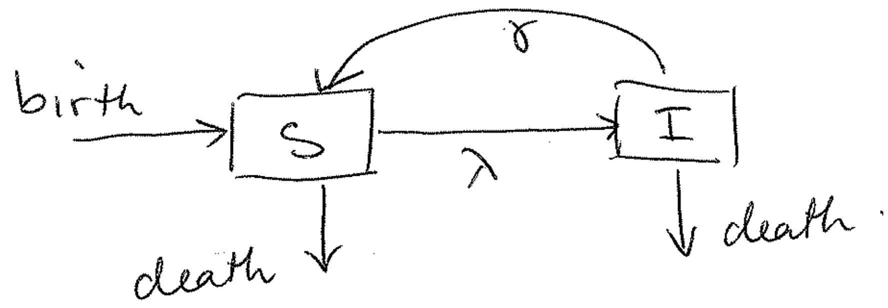
Applied Mathematics 4. - Populations and Diseases

Disease dynamics



for popn size
N constant.

so. $$\frac{\partial S}{\partial a} + \frac{\partial S}{\partial t} = -\mu S - \lambda I.$$

But $\mu \equiv 0$ except when $a = L$.

and $\frac{\partial S}{\partial t} = 0$ because we are at steady state

so $$\frac{dS}{da} = -\lambda S$$

Similarly $$\frac{dI}{da} = \lambda S - \gamma I$$

Boundary conditions: $I(0) = 0$, $S(0) = N/L$.

If $s = S/N$, $i = I/N$. then $s + i = 1$. and

$$\frac{1}{N}\frac{dI}{da} = \lambda \frac{S}{N} - \gamma \frac{I}{N}.$$

so $$\frac{di}{da} = \lambda s - \gamma i$$

$$\frac{di}{da} = \lambda(1-i) - \gamma i$$

$$= \lambda - (\lambda + \gamma) i$$

so $\frac{d}{da}\left(e^{(\gamma+\lambda)a}\, i\right) = \lambda e^{(\lambda+\gamma)a}$

$$i e^{(\gamma+\lambda)a} = \frac{\lambda}{\lambda+\gamma} e^{a(\lambda+\gamma)} + C.$$

$$i = \frac{\lambda}{\lambda+\gamma} + Ce^{-a(\lambda+\gamma)}.$$

when $a=0$ $i=0$ so $C = -\frac{\lambda}{\gamma+\lambda}$.

Hence $i(a) = \frac{\lambda}{\lambda+\gamma}\left(1 - e^{-a(\lambda+\gamma)}\right)$.

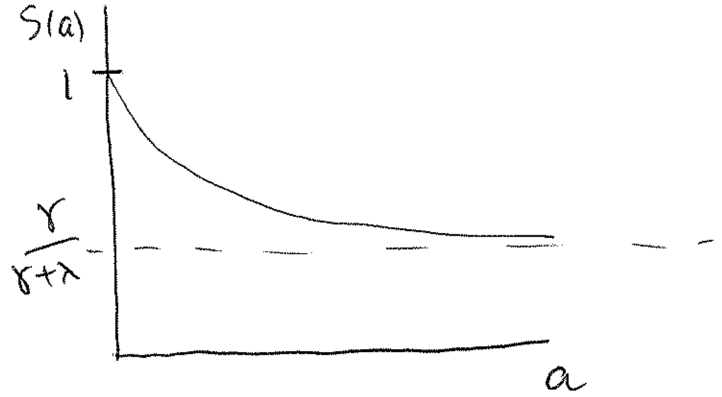
$$s(a) = 1 - i(a)$$

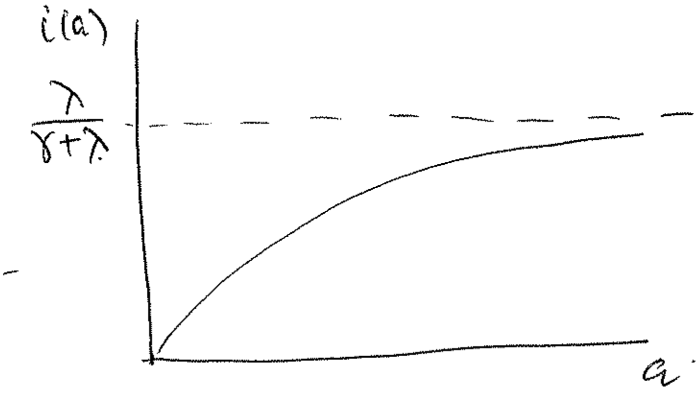
$$= 1 - \frac{\lambda}{\lambda+\gamma} + \frac{\lambda}{\lambda+\gamma} e^{-a(\lambda+\gamma)}.$$

so $s(a) = \frac{\gamma}{\lambda+\gamma} + \frac{\lambda}{\gamma+\lambda} e^{-a(\lambda+\gamma)}$.

Note: there a number of different ways to do the above.

As $a \to \infty$ $i \to \frac{\lambda}{\lambda+\gamma}$ $s \to \frac{\gamma}{\lambda+\gamma}$.





The susceptibles decrease exponentially as a increases and the infectives show a corresponding increase. Eventually the two popns come to an equilibrium as $a \to \infty$ where the number of new infections is balanced by the number of infectives recovering and returning to the susceptible class. The prevalence of the disease increases for all ages. There is no peak prevalence:

$$\text{Average age of infection} = \frac{\int_0^\infty \frac{a\lambda}{\gamma+\lambda}\left(\gamma+\lambda e^{-a(\lambda+\gamma)}\right)da}{\frac{\lambda}{\gamma+\lambda}\int_0^\infty \gamma+\lambda e^{-a(\lambda+\gamma)}\,da}$$

$$= \frac{\int_0^L a\left(\gamma+\lambda e^{-a(\lambda+\gamma)}\right)da}{\int_0^L \gamma+\lambda e^{-a(\lambda+\gamma)}\,da} \quad \text{as } L \text{ is the max. age.}$$

$$\int_0^L a\gamma + a\lambda e^{-a(\lambda+\gamma)}\,da$$

$$= \frac{L^2\gamma}{2} + \left[\frac{-a\lambda}{\lambda+\gamma}e^{-a(\lambda+\gamma)}\right]_0^L + \frac{\lambda}{\gamma+\lambda}\int_0^4 e^{-a(\lambda+\gamma)}\,da$$

$$= \frac{L^2\gamma}{2} - \frac{L\lambda}{\lambda+\gamma}e^{-L(\lambda+\gamma)} + \frac{\lambda}{\gamma+\lambda}\left[-\frac{1}{\gamma+\lambda}e^{-a(\lambda+\gamma)}\right]_0^L$$

$$= \frac{L^2\gamma}{2} - \frac{L\lambda}{\lambda+\gamma}e^{-L(\lambda+\gamma)} - \frac{\lambda}{(\gamma+\lambda)^2}e^{-L(a+\lambda)} + \frac{\lambda}{(\gamma+\lambda)^2}.$$

$$= \frac{L^2\gamma}{2} + \frac{\lambda}{(\gamma+\lambda)^2} - \frac{\lambda e^{-L(\lambda+\gamma)}}{\lambda+\gamma}\left(L + \frac{1}{\gamma+\lambda}\right).$$

$$\int_0^L \gamma + \lambda e^{-a(\lambda+\gamma)} \, da.$$

$$= \left[\gamma a - \frac{\lambda}{\lambda+\gamma} e^{-a(\lambda+\gamma)} \right]_0^L.$$

$$= \gamma L + \frac{\lambda}{\gamma+\lambda}\left(1 - e^{-L(\lambda+\gamma)}\right).$$

So average age of infection

$$= \frac{\frac{L^2\gamma}{2} + \frac{\lambda}{(\gamma+\lambda)^2} - \frac{\lambda e^{-L(\lambda+\gamma)}}{\lambda+\gamma}\left(L + \frac{1}{\gamma+\lambda}\right)}{\gamma L + \frac{\lambda}{\gamma+\lambda}\left(1 - e^{-L(\lambda+\gamma)}\right)}.$$

$$\sim \frac{L}{2} \quad \text{as} \quad L \to \infty.$$

As L increases the average age of infection also increases becoming approx linear in L for L large. This is because individuals keep getting reinfected. The longer they live the more infections they get. Averaged over a long lifetime when the initial effects become negligible this puts the average age of infection at half-way through life i.e $L/2$.

The basic differential equation for i remains the same with λ replaced by λ_v:

$$\frac{di}{da} = \lambda_v s - \gamma i \qquad \text{where now } s = 1 - i - r.$$

so
$$\frac{di}{da} = \lambda_v (1 - i - r) - \gamma i$$
$$= \lambda_v(1-r) - (\gamma + \lambda_v) i \quad \text{for } a > a_v.$$

Solving this

$$\frac{d}{da}\left(i e^{(\gamma+\lambda_v)a}\right) = \lambda_v(1-r) e^{(\gamma+\lambda_v)a}.$$

$$i = \frac{+\lambda_v}{(\gamma + \lambda_v)}(1-r) + Ce^{-(\gamma+\lambda_v)a}$$

Now $i(a)$ must be cts at $a = a_v$. (Vaccination doesn't change the number of infectives.) So $i(a_v^-) = i(a_v^+)$. where $i(a_v^-)$ is calculated using the previous expression for $i(a)$ but with $\lambda = \lambda_v$ (as vaccination affects the whole population's contact rate not just those of the age-class post vaccination).

So
$$\frac{\lambda_v}{\lambda_v + \gamma}\left(1 - e^{-a_v(\lambda_v + \gamma)}\right) = \frac{\lambda_v}{\gamma + \lambda_v}(1-r) + Ce^{-(\gamma+\lambda_v)a_v}$$

or $$C e^{-(\gamma+\lambda_v)a_v} = \frac{\lambda_v}{\lambda_v+\gamma}\left(r - e^{a_v(\gamma+\lambda_v)}\right).$$

$$C = \frac{\lambda_v}{\lambda_v+\gamma}\left(r e^{(\gamma+\lambda_v)a_v} - 1\right).$$

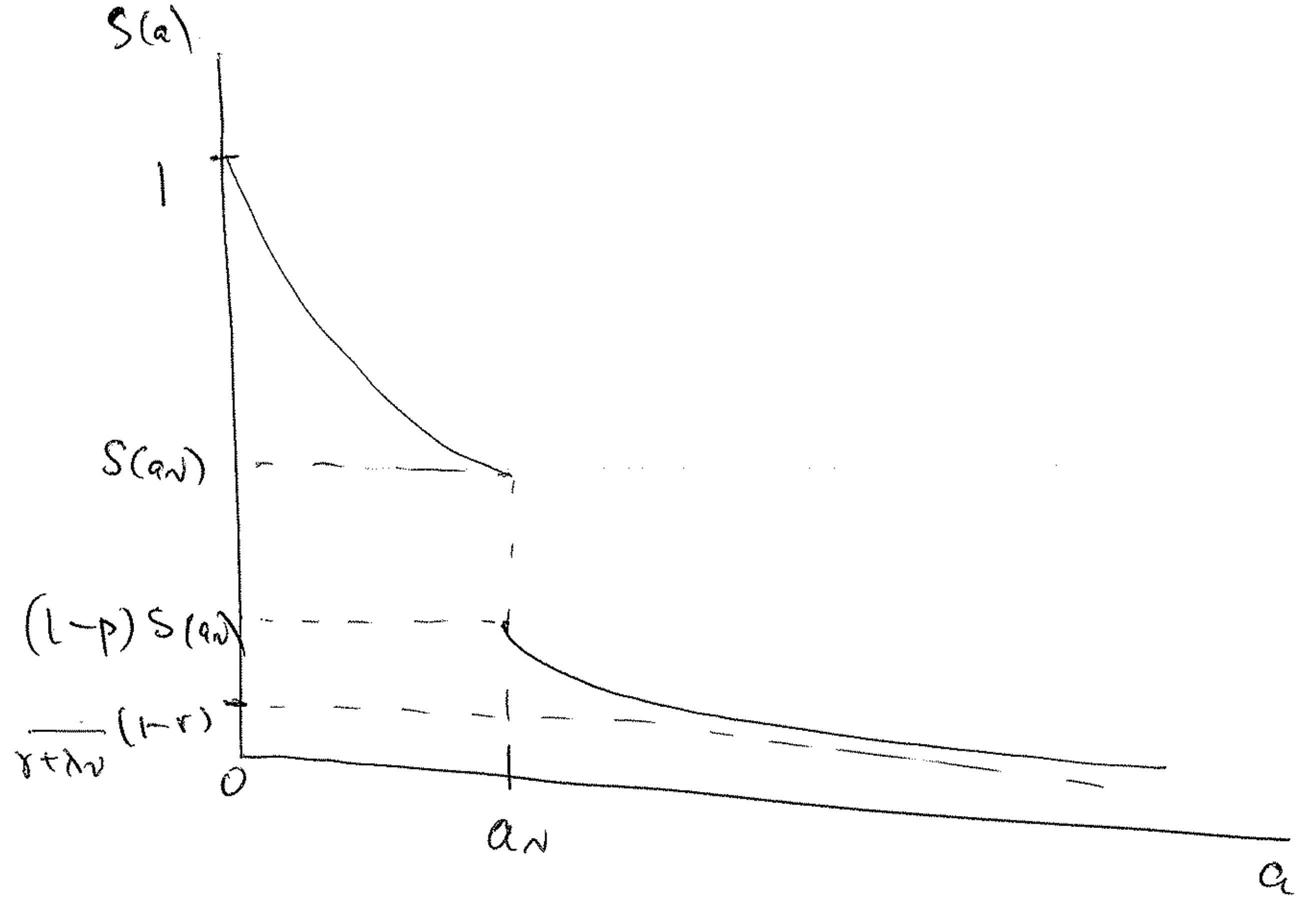
so $$i(a) = \begin{cases} \frac{\lambda_v}{\gamma+\lambda_v}\left(1-r + \left(r e^{(\gamma+\lambda_v)a_v} - 1\right) e^{-(\lambda_v+\gamma)a}\right) & \text{for } a \geq a_v. \\ \frac{\lambda_v}{\gamma+\lambda_v}\left(1 - e^{-a(\lambda_v+\gamma)}\right) & \text{for } a < a_v \end{cases}$$

$$S(a) = 1 - r - i(a) \qquad \text{for } a \geq a_v.$$

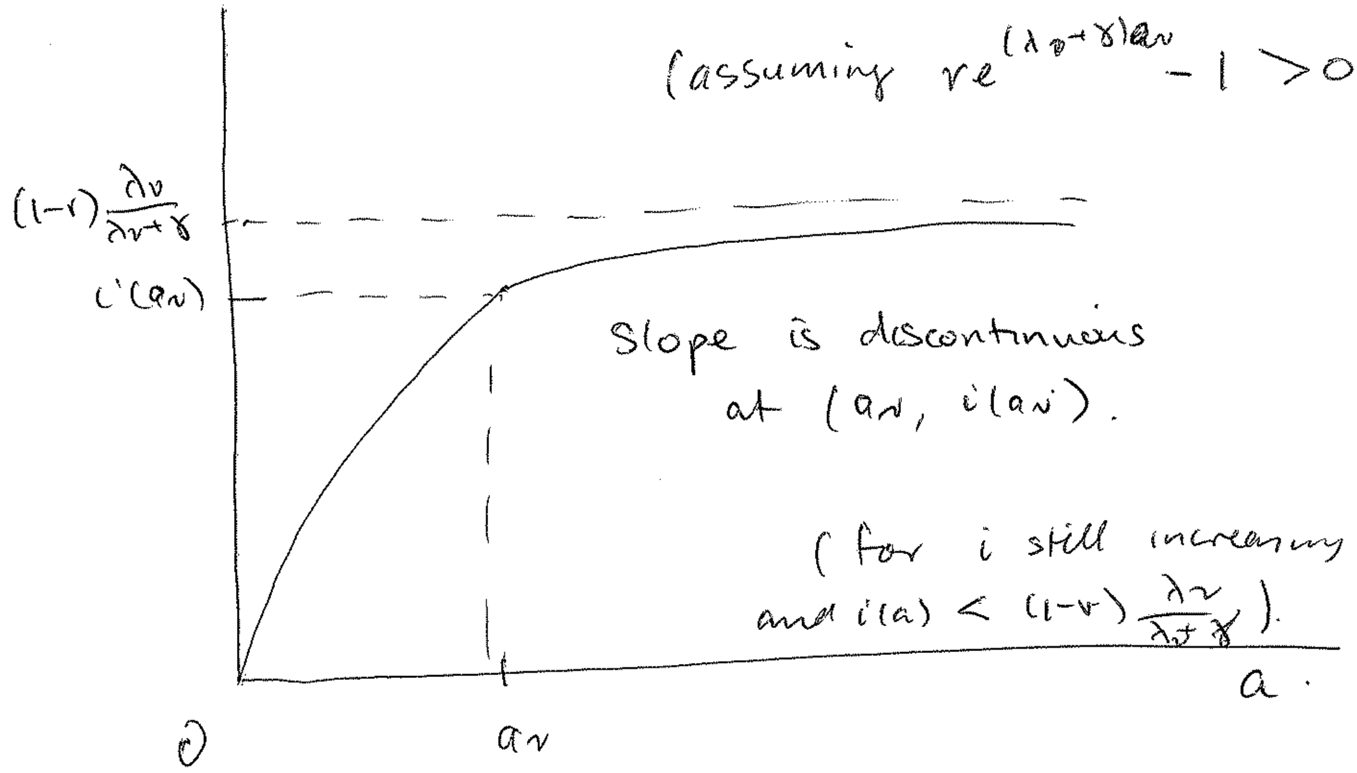
so $$S(a) = \begin{cases} \frac{\gamma}{\lambda_v+\gamma}(1-r) - \frac{\lambda_v}{\gamma+\lambda_v}\left(r e^{(\gamma+\lambda_v)a_v} - 1\right) e^{-(\lambda_v+\gamma)a} \\ \frac{\gamma}{\lambda_v+\gamma} + \frac{\lambda_v}{\gamma+\lambda_v} e^{-a(\lambda_v+\gamma)} & \text{for } a < a_v \end{cases}$$

As $a \to \infty$ $\quad i \to \frac{\lambda_v}{\gamma+\lambda_v}(1-r).$

$$s \to \frac{\gamma}{\lambda_v+\gamma}(1-r).$$



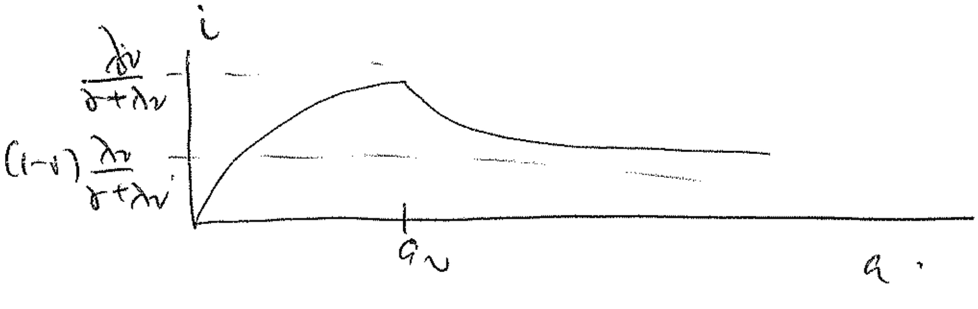
(assuming $re^{(\lambda v+\gamma)a_v} - 1 > 0$).



Slope is discontinuous at $(a_v, i(a_v))$.

(for i still increasing and $i(a) < (1-r)\frac{\lambda v}{\lambda v+\gamma}$).

If $i \sim \frac{\lambda v}{\lambda v+\gamma}$ at a_v.



In the unvaccinated population as $a \to \infty$

$$i \to \frac{\lambda}{\gamma+\lambda} = 1 - \frac{\gamma}{\gamma+\lambda}$$

$$< 1 - \frac{\gamma}{\gamma+\lambda_v} \quad \text{since } \lambda_v < \lambda$$

$$= \frac{\lambda_v}{\gamma+\lambda_v}$$

$$< \frac{\lambda_v}{\gamma+\lambda_v}(1-r).$$

So vaccination always reduces infective numbers as $a \to \infty$.

As $a \to \infty$ $\quad s \to \dfrac{\gamma}{\lambda+\gamma}$ in the unvaccinated case.

If this is smaller than in the vaccinated case

$$\frac{\gamma}{\lambda+\gamma} < (1-r)\frac{\gamma}{\gamma+\lambda_v}.$$

$$\text{so } (1-r) > \frac{\gamma+\lambda_v}{\gamma+\lambda}$$

<u>Note</u> for this to occur we need $\lambda_v < \lambda$.
It is not likely to occur when vaccinating a small proportion of the population dramatically reduces contact rates and hence force of infection.

Vaccination will lower the average age of infection as a proportion of the population will not become infected again after age $a = a_v$.