Short Assignment 1

1. (a) What do the following quantities represent?

   (*i*)

$$1-\exp\left(-\int_0^a \mu(s)ds\right)$$

   (*ii*)(harder)

$$\int_{t-a}^{t} b(s-(t-a))n(s-(t-a),s)ds$$

   (b) Let $b(a,t)$ be the time-dependent fecundity function for the human population of Australia where t and a are in years. Sketch a possible graph of $b(25,t)$ for the last hundred years. Explain any special features of your graph.

2. Consider an age-structured population, governed by the McKendrick-von Foerster equation. The death rate for this population is constant; that is $\mu(a)=m$ where m is a constant. The birth rate is constant during the reproductive life of each individual:

$$b(a)=\begin{cases} B & \text{for } \beta \le a \le 2\beta \\ 0 & \text{otherwise} \end{cases}$$

   where B is constant, β is the age that an individual first reproduces and 2β is the age that it ceases to reproduce.

   Individuals of age $a=0$ are continuously introduced into a closed environment at a rate of η individuals per unit time, from $t=0$ until $t=\beta$.

   (i) Show that the cohort function $w(t)$ for a cohort introduced at $t=t_0$ where $0<t_0<\beta$ is $w(t)=\eta e^{-m(t-t_0)}$.

   (ii) Hence or otherwise find an expression for $n(a,t)$ when $t-a=t_0<\beta$. Note that this expression is valid for all $t\ge 0$ provided $t-a<\beta$.

   (iii) Show that the number of individuals born at time t_1 where $\beta<t_1<2\beta$ is given by $n(0,t_1)=B\eta(e^{-m\beta}-e^{-mt_1})/m$.

   (iv) Consider a cohort born at time t_1 where $\beta<t_1<2\beta$. Find the cohort function for this cohort. Hence, or otherwise, find an expression for $n(a,t)$ where $\beta<t-a=t_1<2\beta$. Write $n(a,t)$ in terms of a and t only. Do not use t_1 in your final answer.

   (v) Find the total number of offspring born to parents who were introduced to the population in the period $t\le\beta$. Write down the average number of offspring born to each individual introduced when $t\ge\beta$.

   (vi) Let

$$N(t)=\int_0^\infty n(a,t)da$$

   be the total number of individuals in the population at time t. Find $N(\beta)$ and $N(2\beta)$. Suggest one way that parameters can be chosen to ensure that $N(2\beta)>N(\beta)$.