Short Assignment 1

1. (a) What do the following quantities represent?
   (i) \[ 1 - \exp \left( - \int_0^a \mu(s) \, ds \right) \]
   (ii) (harder) \[ \int_{t-a}^t b(s - (t - a)) n(s - (t - a), s) \, ds \]

(b) Let \( b(a, t) \) be the time-dependent fecundity function for the human population of Australia where \( t \) and \( a \) are in years. Sketch a possible graph of \( b(25, t) \) for the last hundred years. Explain any special features of your graph.

2. Consider an age-structured population, governed by the McKendrick-von Foerster equation. The death rate for this population is constant; that is \( \mu(a) = m \) where \( m \) is a constant. The birth rate is constant during the reproductive life of each individual:
   \[
   b(a) = \begin{cases} 
   B & \text{for } \beta \leq a \leq 2\beta \\
   0 & \text{otherwise}
   \end{cases}
   \]
   where \( B \) is constant, \( \beta \) is the age that an individual first reproduces and \( 2\beta \) is the age that it ceases to reproduce.

   Individuals of age \( a = 0 \) are continuously introduced into a closed environment at a rate of \( \eta \) individuals per unit time, from \( t = 0 \) until \( t = \beta \).

   (i) Show that the cohort function \( w(t) \) for a cohort introduced at \( t = t_0 \) where \( 0 < t_0 < \beta \) is \( w(t) = \eta e^{-m(t-t_0)} \).

   (ii) Hence or otherwise find an expression for \( n(a, t) \) when \( t - a = t_0 < \beta \). Note that this expression is valid for all \( t \geq 0 \) provided \( t - a < \beta \).

   (iii) Show that the number of individuals born at time \( t_1 \) where \( \beta < t_1 < 2\beta \) is given by \( n(0, t_1) = B\eta(e^{-m\beta} - e^{-mt_1})/m \).

   (iv) Consider a cohort born at time \( t_1 \) where \( \beta < t_1 < 2\beta \). Find the cohort function for this cohort. Hence, or otherwise, find an expression for \( n(a, t) \) where \( \beta < t - a = t_1 < 2\beta \). Write \( n(a, t) \) in terms of \( a \) and \( t \) only. Do not use \( t_1 \) in your final answer.

   (v) Find the total number of offspring born to parents who were introduced to the population in the period \( t \leq \beta \). Write down the average number of offspring born to each individual introduced when \( t \geq \beta \).

   (vi) Let \( N(t) = \int_0^\infty n(a, t) \, da \) be the total number of individuals in the population at time \( t \). Find \( N(\beta) \) and \( N(2\beta) \). Suggest one way that parameters can be chosen to ensure that \( N(2\beta) > N(\beta) \).