Representations of cyclotomic algebras

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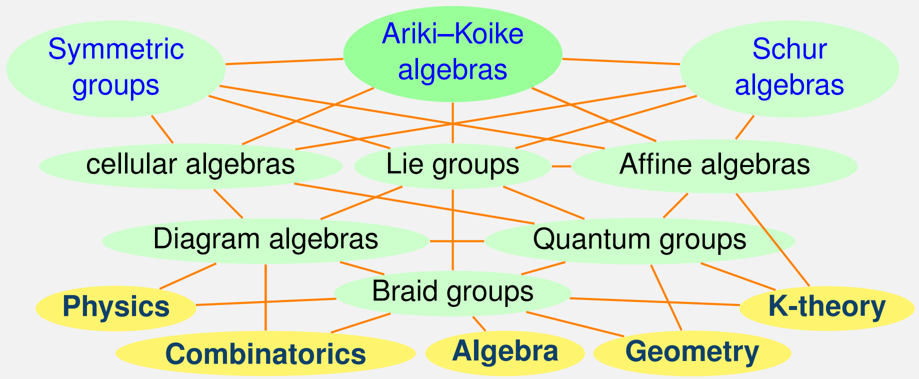
Combinatorial representation theory

Most of my work is concerned with the representation theory of Hecke algebras associated with complex reflection groups.

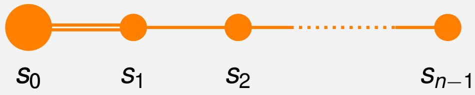


Motivating question

Compute the decomposition matrices of the symmetric groups for fields of positive characteristic.

The Ariki–Koike algebras

The complex reflection group $W_{r,n}$ of type $G(r,1,n)$ is the group with Coxeter diagram



$$\Longrightarrow \quad W_{r,n} = \langle s_0, s_1, \dots, s_{n-1} \mid s_0^r = s_i^2 = 1 \text{ + braid relations}\rangle$$

Concretely, $W_{r,n} = \left\langle \begin{pmatrix} \sqrt[r]{1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \right\rangle$

The Ariki–Koike algebra is the associative R–algebra

$$\mathscr{H}_{r,n} = \left\langle T_0, T_1, \dots, T_{n-1} \;\middle|\; \begin{array}{c} (T_0 - Q_1)\dots(T_0 - Q_r) = 0 \\ (T_i - q)(T_i + 1) = 0 \\ \text{+ braid relations} \end{array} \right\rangle$$

Theorem (Ariki–Koike)

The Ariki–Koike algebra $\mathscr{H}_{r,n}$ is free as an R–module of rank $|W_{r,n}|$

Special cases and significance

Special cases

1. $r = 1$ and $q = 1$ $\Longrightarrow$ $\mathscr{H}_{1,n} \cong R\mathfrak{S}_n$ (recover symmetric groups).
2. $r = 1$ $\Longrightarrow$ $\mathscr{H}_{1,n}$ is the Iwahori–Hecke algebra of type A
3. $r = 2$ $\Longrightarrow$ $\mathscr{H}_{2,n}$ is the Iwahori–Hecke algebra of type B

When $r = 1$ or $r = 2$ the group $W_{r,n}$ is a Weyl group and

$$\mathscr{H}_{r,n} \cong \mathrm{End}_{G(\mathbb{F}_q)}\left(\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \mathbf{1}\right)$$

$\Longrightarrow$ $\mathscr{H}_{r,n}$ explains "generic features" of the principle series

Conjecture (Broué, Malle, Michel, ...)

For suitable parameters the Ariki–Koike algebra $\mathscr{H}_{r,n}$ is isomorphic to the endomorphism algebra of a Deligne–Lusztig representation of a finite group of Lie type.

When $r > 2$ many consequences of the BMM-conjectures are known, but the conjectures are still open

The LLT conjecture

Theorem (Ariki)

Suppose that $R = \mathbb{C}$, $q \neq 1$ and that $Q_i = q^{a_i}$, for some $a_i \in \mathbb{Z}$.

1. *$L(\Lambda) = \bigoplus_n K_0(\mathscr{H}_{r,n})$ is an integrable highest weight module for an affine quantum group $U_v(\widehat{\mathfrak{sl}}_e)$.*
2. *The canonical basis of $L(\Lambda)$ is precisely the basis of projective indecomposable $\mathscr{H}_{r,n}$–modules.*

Theorem (Dipper–M.)

Up to Morita equivalence only the q–orbits of the parameters matter.

Corollary (Ariki–M.)

Suppose that R is an algebraically closed field and that $q \neq 1$. Then the simple modules of the affine Hecke algebra of type A are naturally labelled by content functions on aperiodic multisegments.

Schur elements and generic degrees

When $r = \mathbb{C}(q, Q_i)$ the Ariki–Koike algebra $\mathscr{H}_{r,n}$ is semisimple and the natural trace function τ on $\mathscr{H}_{r,n}$ can be written as $\tau = \sum_{\boldsymbol{\lambda}} \frac{1}{s_{\boldsymbol{\lambda}}} \chi^{\boldsymbol{\lambda}}$.

The rational functions $s_{\boldsymbol{\lambda}}$ are the Schur elements of $\mathscr{H}_{r,n}$

Theorem (Geck–Iancu–Malle, M.)

Suppose that $\boldsymbol{\lambda}$ is a multipartition of n. Then $s_{\boldsymbol{\lambda}}$ is equal to

$$(-1)^{a_{rL}} q^{b_{rL}} \frac{\prod\limits_{1 \le s < t \le r} (Q_s - Q_t)^L \cdot \prod\limits_{1 \le s,t \le r} \prod\limits_{\alpha_s \in B^{(s)}} \prod\limits_{1 \le k \le \alpha_s} (q^k Q_s - Q_t)}{(q-1)^n (Q_1 \dots Q_r)^n \prod\limits_{1 \le s \le t \le r} \prod\limits_{\substack{(\alpha_s,\alpha_t) \in B^{(s)} \times B^{(t)} \\ \alpha_s > \alpha_t \text{ if } s=t}} (q^{\alpha_s} Q_s - q^{\alpha_t} Q_t)},$$

where $a_{rL} = n(r-1) + \binom{r}{2}\binom{L}{2}$ and $b_{rL} = \frac{rL(L-1)(2rL-r-3)}{12}$.

The Schur elements can be used to compute the dimensions of the irreducible constituents of the Deligne–Lusztig representations.

Cyclotomic q–Schur algebras

A quasi–hereditary algebra is an algebra whose module category is a highest weight category

Theorem (Dipper–James–M.)

The Ariki–Koike algebra $\mathscr{H}_{r,n}$ has a quasi–hereditary cover $\mathscr{S}_{r,n}$.

The algebra $\mathscr{S}_{r,n}$ is called the **cyclotomic q–Schur algebra**.

Properties

1. $\mathscr{S}_{r,n}$ is explicitly described in terms of a combinatorial basis
2. $\mathscr{S}_{r,n}$ is quasi–hereditary and cellular (in sense of Graham–Lehrer)
3. The simple $\mathscr{S}_{r,n}$–modules are indexed by the multipartitions of n.
4. There is bimodule M such that
$$\mathscr{S}_{r,n} \cong \operatorname{End}_{\mathscr{H}_{r,n}}(M) \text{ and } \mathscr{H}_{r,n} \cong \operatorname{End}_{\mathscr{S}_{r,n}}(M).$$
5. $\Longrightarrow$ there is an exact ‘Schur functor’ $\Phi_\omega : \mathscr{S}_{r,n}\text{-Mod} \longrightarrow \mathscr{H}_{r,n}\text{-Mod}$.

Theorem (M.)

The tilting modules of $\mathscr{S}_{r,n}$ are known.

The Jantzen sum formula

- In the semisimple case, the irreducible $\mathscr{S}_{r,n}$-modules are know as the Weyl modules $W^{\boldsymbol{\lambda}}$.
- The Weyl modules have a Jantzen filtration
$$W^{\boldsymbol{\lambda}} = W_0^{\boldsymbol{\lambda}} \supset W_1^{\boldsymbol{\lambda}} \supseteq W_2^{\boldsymbol{\lambda}} \supseteq \dots$$

where, morally, $W_i^{\boldsymbol{\lambda}} = \{\, x \in W^{\boldsymbol{\lambda}} : \mathfrak{p}^i \mid \langle x, y \rangle \text{ for all } y \in W^{\boldsymbol{\lambda}} \,\}$

- In general, the Weyl module $W^{\boldsymbol{\lambda}}$ has a simple head $L^{\boldsymbol{\lambda}}$
$$\Longrightarrow \quad L^{\boldsymbol{\lambda}} \cong W_0^{\boldsymbol{\lambda}} / W_1^{\boldsymbol{\lambda}}$$

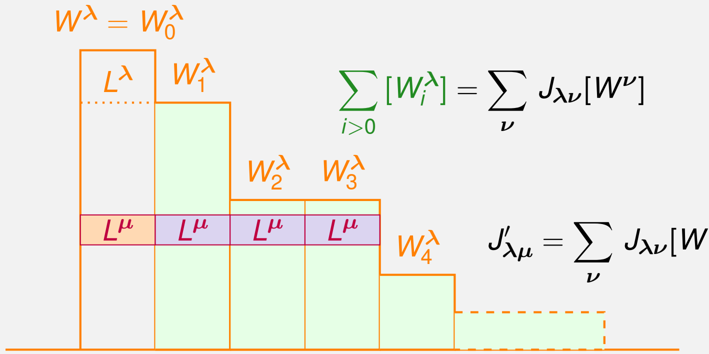
Theorem (James–M.)

There are explicitly known integers $J_{\boldsymbol{\lambda}\boldsymbol{\nu}}$ such that

$$\sum_{i>0} [W_i^{\boldsymbol{\lambda}}] = \sum_{\boldsymbol{\nu}} J_{\boldsymbol{\lambda}\boldsymbol{\nu}} [W^{\boldsymbol{\nu}}]$$

in the Grothendieck group of $\mathscr{S}_{r,n}$.

What the Jantzen sum formula tells us



$$\sum_{i>0} [W_i^\lambda] = \sum_{\nu} J_{\lambda\nu}[W^\nu]$$

$$J'_{\lambda\mu} = \sum_{\nu} J_{\lambda\nu}[W^\nu : L^\mu]$$

$\Longrightarrow$ $[W^\lambda : L^\mu] \leq J'_{\lambda\mu}$ and $[W^\lambda : L^\mu] > 0$ only if $J'_{\lambda\mu} > 0$

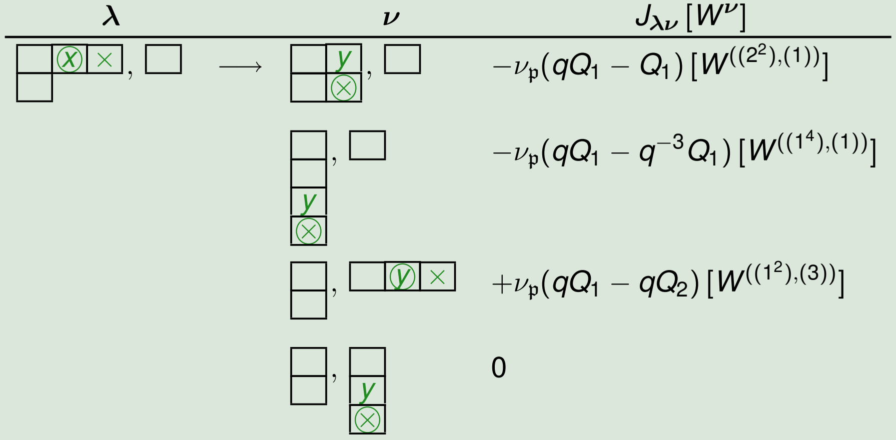
So the Jantzen sum formula detects composition factors but it over counts composition multiplicity.

There are analogous statements for the Specht modules of the Ariki–Koike algebras

Combinatorics of the Janzten coefficients

Example

If $\lambda = ((3,1),(1))$ then some of the terms appearing in the Jantzen sum formula for W^λ are:



λ		ν	$J_{\lambda\nu}\,[W^\nu]$
	$\longrightarrow$		$-\nu_{\mathfrak{p}}(qQ_1 - Q_1)\,[W^{((2^2),(1))}]$
			$-\nu_{\mathfrak{p}}(qQ_1 - q^{-3}Q_1)\,[W^{((1^4),(1))}]$
			$+\nu_{\mathfrak{p}}(qQ_1 - qQ_2)\,[W^{((1^2),(3))}]$
			0

The blocks of $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$

There is a natural surjection of the affine Hecke algebra $\mathscr{H}^{\text{aff}}$ onto the Ariki–Koike algebra $\mathscr{H}_{r,n}$

$\Longrightarrow$ If M and N are in the same block as $\mathscr{H}_{r,n}$-modules then they are in the same block as $\mathscr{H}^{\text{aff}}$-modules

Theorem (Lyle–M.)

Suppose that R is an algebraically closed field and that $q \neq 1$.

① *S^λ and S^μ are in the same block as $\mathscr{H}_{r,n}$-modules*
② *S^λ and S^μ are in the same block as $\mathscr{H}^{\text{aff}}$-modules*
③ *W^λ and W^μ are in the same block as $\mathscr{S}_{r,n}$-modules*
④ *λ and μ have the same multiset of contents*

The last condition is an easily checked combinatorial condition

Sketch of the block classification

- By Schur-Weyl duality, $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ have the ‘same’ blocks
- Standard facts about cellular algebras imply that it is enough to determine when two Weyl modules belong to the same block
- Define $\lambda \sim_J \mu$ if $J_{\lambda\mu} \neq 0$ or $J_{\mu\lambda} \neq 0$ (and take transitive closure)

Proposition

The Weyl modules W^λ and W^μ are in the same block $\iff \lambda \sim_J \mu$.

$(\Longrightarrow)$ Enough to show that $\lambda \sim_J \mu$ whenever $d_{\lambda\mu} = [W^\lambda : L^\mu] \neq 0$

$d_{\lambda\mu} \neq 0 \implies J'_{\lambda\mu} \neq 0 \implies \exists \nu_1$ with $J_{\lambda\nu_1} \neq 0$ and $d_{\nu_1\mu} \neq 0$

$\implies \lambda \sim_J \nu_1$ and $\lambda \rhd \nu_1 \rhd \mu$

If $\nu_1 \neq \mu$ then $J_{\nu_1\mu} \neq 0 \implies \exists \nu_2$ with $J_{\nu_1\nu_2} \neq 0$ and $d_{\nu_2\mu} \neq 0$

$\implies \lambda \sim_J \nu_2$ and $\lambda \rhd \nu_1 \rhd \nu_2 \rhd \mu$ ...

Now use (detailed) combinatorial arguments to describe $\sim_J$