Rouquier blocks

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Rouquier blocks—applications

Rouquier blocks arose in the study of Broué's abelian defect group conjecture for the symmetric group. My feeling is that these blocks are the "most important" blocks in type A because:

- Chuang–Kessar showed that Brou'e abelian defect group conjecture is true for Rouquier blocks. Chuang–Rouquier built on this result to establish Broué's abelian defect conjecture for the symmetric groups, general linear groups and related algebras.
- Rouquier blocks played a key role in Fayers' proof that the decomposition numbers of symmetric group blocks of weight 3, in characteristic $p > 3$, are always 0 or 1.
- Fayers classified the irreducible Specht modules of the symmetric groups in odd characteristic by showing that the irreducible half of the question could be reduced to Rouquier blocks.

Decomposition numbers of Rouquier blocks

It is a little strange that the Rouquier blocks are so well understood because are the "largest blocks" of a given weight. Nonetheless:

- The Rouquier blocks of the symmetric groups in characteristic p are completely understood through the work of Chuang–Kessar, Chuang–Tan and Turner.
- For Hecke algebras of type A at a root of unity their decomposition numbers are known by Chuang–Tan and Leclerc–Miyachi.
- For the q–Schur algebras at a root of unity their decomposition numbers are known by Chuang–Tan and Leclerc–Miyachi.
- For the general linear groups in characteristic p they are understood through the work of Miyachi
- We will describe a uniform way of computing the decomposition numbers in all of these cases as well as the mixed case.

The Hecke algebra

Fix a ring R and an integer $n \geq 1$ and let q be an *invertible* element of R.

The **Iwahori–Hecke algebra** $\mathscr{H}_n = \mathscr{H}_q(\mathfrak{S}_n)$ is the unital associative R algebra with generators

$$T_1, \dots, T_{n-1}$$

and relations

$$\begin{aligned} (T_i + q)(T_i - 1) &= 0 \\ T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i \\ T_iT_j &= T_jT_i, \qquad \text{when } |i-j| > 1 \end{aligned}$$

So, $\mathscr{H}_n \cong R\mathfrak{S}_n$ when $q = 1$.

Representation theory

- For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n there is an $\mathscr{H}$–module $S(\lambda)$, called a **Specht module**.
- The module $S(\lambda)$ ha a natural bilinear form $\langle\ ,\ \rangle$ and

$$\operatorname{Rad} S(\lambda) = \{\, x \in S(\lambda) : \langle x, y\rangle = 0 \text{ for all } y \in S(\lambda) \,\}$$

  is an $\mathscr{H}$–submodule of $S(\lambda)$.
- The quotient module $\mathbf{D}(\lambda) = S(\lambda)/\operatorname{Rad} S(\lambda)$ is either zero or absolutely irreducible.
- **Define** $e > 1$ to be minimal such that $1 + q + \dots + q^{e-1} = 0$.
- Then $D(\lambda) \neq 0$ if and only if λ is e**–regular**; that is, if no e non–zero parts of λ are equal.
- We want to compute $d_{\lambda\mu} = [S(\lambda) : D(\mu)]$.

Abacus combinatorics

An e**–abacus** is a Chinese abacus with e vertical runners, labelled $0, 1, \dots, e-1$ from left to right.

We label the positions on the abacus $0, 1, 2, \dots$ from left to right, top to bottom. So, taking $e = 3$, the positions on the abacus are numbered as follows:

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \\ \vdots & \vdots & \vdots \end{array}$$

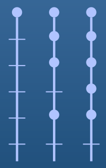
Abacus combinatorics—II

Let λ be a partition and choose and integer $k > 0$ such that $\lambda_{k+1} = 0$. The abacus configuration for λ is the abacus with k beads at positions

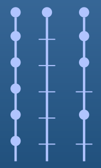
$$\lambda_1 - 1 + k, \lambda_2 - 2 + k, \dots, \lambda_k$$

There is a bijection between the partitions with at most k non–zero parts and the abacus configurations with k beads.

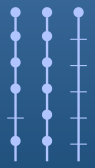
Example If $\lambda = (5^2, 4, 2^2, 1^2)$ then it has abacus configurations



$k = 11$



$k = 12$



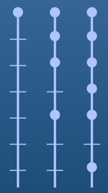
$k = 13$

e–Cores

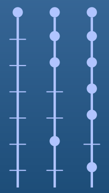
Given an abacus configuration for a partition λ we can form a new abacus configuration by sliding all of the beads on each runner to their highest possible positions. The partition ρ which corresponds to this new abacus configuration is the e**–core** of λ.

The e**–weight** of λ is $\frac{|\rho|-|\lambda|}{e}$.

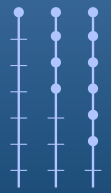
Example Suppose that $e = 3$ and consider the two partitions $\lambda = (10, 5^2, 4, 2^2, 1^2)$ and $\mu = (10, 7, 6, 4, 2^2, 1^2)$. Then both 3–core λ and μ have the same 3–core; namely, $\rho = (7, 5, 3^2, 2^2, 1^2)$.



λ



μ



ρ

Blocks

Theorem

a) Two Specht modules $S(\lambda)$ and $S(\mu)$ belong to the same block if and only if λ and μ have the same weight and the same core.

b) Two simple modules $D(\lambda)$ and $D(\mu)$ belong to the same block if and only if λ and μ have the same weight and the same core.

If ρ is a core and $w \geq 1$ then we let $B_{\rho,w}$ be the corresponding block of $\mathscr{H}$ (or of the q–Schur algebra).

We say that two partitions λ and μ **belong to the same block** if they have the same weight and the same core.

Scopes equivalence

Let $B_{\rho,w}$ be a block. Fix an abacus configuration for ρ with $k \gg 0$ beads. Suppose that there are $\kappa > 0$ more beads on runner i of this abacus than there are on runner $i-1$, for some i with $1 \leq i < e$.

Let $\bar{\rho}$ be the partition which has the abacus configuration obtained by swapping runners $i-1$ and i in the abacus configuration

More generally if $\lambda \in B_{\rho,w}$ let $\bar{\lambda}$ be the partition in $B_{\bar{\rho},w}$ which is obtained by swapping the $(i-1)^{\text{th}}$ and i^{th} runners in the abacus configuration for μ.

Theorem (Scopes) Suppose that $\kappa \geq w$. Then

a) There is a Morita equivalence $B_{\rho,w}\text{–Mod} \cong B_{\bar{\rho},w}\text{–Mod}$.

b) We have $d_{\lambda\mu} = d_{\bar{\lambda}\bar{\mu}}$, for all $\lambda, \mu \in B_{\rho,w}$.

The **Scopes'** equivalence classes of blocks are classified by pyramids.

Pyramids

Definition (Richards) Suppose that $B_{\rho,w}$ is a block. The **e–pyramid** of $B_{\rho,w}$ is the array of numbers $\Delta(\rho,w) = ({}_i\delta_j)_{1\leq i<j\leq e}$, where

$${}_i\delta_j = \max\{0, w-1-\lfloor\tfrac{d_j-d_i}{e}\rfloor\}.$$

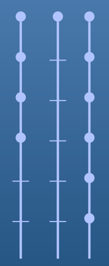
where $d_1 < d_2 < \cdots < d_e$ and ρ has an abacus display with a bead at position d_i but no bead at position $d_i + e$, for $1 \leq i \leq e$.

The pyramid is independent of the choice of abacus configuration for ρ.

Pyramids—II

Theorem (Richards) Suppose that ρ and γ are e–cores and that $w \geq 1$. Then the blocks $B_{\rho,w}$ and $B_{\gamma,w}$ are Scopes equivalent if and only if $\Delta(\rho,w) = \Delta(\gamma,w)$.

Example Take $e=3$ and $\rho = (7,5,3,2^2,1^2)$. Then ρ has abacus



The last occupied bead positions on each runner are 9, 1 and 17.
So $d_1 = 1 < d_2 = 9 < d_3 = 17$.

Hence,

$$\Delta(\rho,3) = \begin{array}{cc} 0 & \\ 0 & 0 \end{array}$$

$$\Delta(\rho,4) = \begin{array}{cc} 0 & \\ 1 & 1 \end{array}$$

$$\Delta(\rho,5) = \begin{array}{cc} 0 & \\ 2 & 2 \end{array}$$

$$\Delta(\rho,7) = \begin{array}{cc} 1 & \\ 4 & 4 \end{array}$$

Rouquier blocks

Definition Suppose that $B_{\rho,w}$ is a block and that $\Delta(\rho,w)=({}_i\delta_j)$ is its pyramid. Then $B_{\rho,w}$ is a **Rouquier block**, or **RoCK block**, if ${}_i\delta_j=0$, for all $1\le i<j\le e$.

This actually defines the Scopes equivalence class of Rouquier blocks.

The point of this definition is that (Scopes equivalence classes of) blocks are naturally ordered by the pyramids: suppose that the blocks $B_{\rho,w}$ and $B_{\rho',w}$ have pyramids $\Delta(\rho,w)=({}_i\delta_i)$ and $\Delta(\rho',w)=({}_i\delta'_i)$. Then

$$B_{\rho,w}\le B_{\rho',w} \quad \text{if} \quad {}_i\delta_j\le {}_i\delta'_j,$$

for all $1\le i<j\le e$.

Observe that the class of Rouquier blocks is minimal with respect to this ordering.

Rouquier blocks II

Looking at the definition of the pyramid, we can give the following (slightly weaker) definition of Rouquier block purely in terms of the abacus: We suppose that ρ has b_i beads on runner i, for $0\le i<e$. Then:

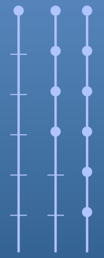
a) ρ is a **Rouquier core** if $b_i\le b_{i+1}$, for $i=0,\dots,e-2$.

b) $B(\rho,w)$ is a **Rouquier block** if ρ is a Rouquier core and $w-1\le\min\{\,b_{i+1}-b_i : 0\le i<e-1\,\}$.

This definition is the one that we work with in practice.

Example

Looking back at our previous examples, the partition $\rho=(7,5,3^2,2^2,1^2)$ has abacus



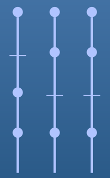
Hence, $B_{\rho,w}$ is a Rouquier block for $0\le w\le 3$, and it is not Rouquier if $w\ge 4$.

Before we can discuss decomposition numbers we need one more piece of notation.

e–Quotients of partitions

Given an e–abacus representation for a partition λ we define its **e–quotient** to be the e–tuple of partitions $(\lambda^0,\lambda^1,\dots,\lambda^{e-1})$, where λ^i is the partition corresponding to runner i of the abacus (with $e=1$).

Example (a) The e–quotient of a core is $(\emptyset,\dots,\emptyset)$.
(b) If $\lambda=(2^3,1^3)$ then it has abacus configuration



Hence, its quotient is $\big((1^2),(1),(1)\big)$.

Partitions are completely determined by the cores and their quotients.

Decomposition numbers for Rouquier blocks

Let $c^{\gamma}_{\sigma\tau}$ be the Littlewood–Richardson coefficient.

Theorem

Suppose that λ and μ are partitions in the Rouquier block $B(\rho, w)$ and that $w < p$. Then

$$d_{\lambda\mu} = \sum_{\substack{\alpha^0,\dots,\alpha^e \\ \beta^0,\dots,\beta^{e-1}}} \prod_{0\le j\le e-1} c^{\mu^j}_{\alpha^j\beta^j} c^{\lambda^j}_{\beta^j(\alpha^{j+1})'}$$

where $|\alpha^i| = \sum_{j=0}^{i-1}\left(|\lambda^j| - |\mu^j|\right)$ and $|\beta^i| = |\mu^i| + \sum_{j=0}^{i-1}\left(|\mu^j| - |\lambda^j|\right)$.

Remarks

The main idea behind the proof

Given two partitions $\lambda = (\lambda^0, \dots, \lambda^{e-1})$ and $\mu = (\mu^0, \dots, \mu^{e-1})$ in a Rouquier block write $\lambda \approx \mu$ if $|\lambda^i| = |\mu^i|$, for $0 \le i < e$.

Proposition Suppose that $w < p$ and that $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{e-1})$ and $\mu = (\mu^0, \mu^1, \dots, \mu^{e-1})$ are Rouquier partitions in $B(\rho, w)$ such that $\lambda \approx \mu$. Then $d_{\lambda\mu} = 0$ unless $\lambda = \mu$.

Proof Apply the Jantzen sum formula.

An example, weight 3 with $e = 3$

Some consequences

1. **Corollary** James' conjecture is true for the Rouquier blocks of Hecke algebras and q–Schur algebras.

Roughly speaking, James' conjecture says if $w < p$ then the decomposition matrices of blocks of the Hecke algebras and q–Schur algebras should agree in characteristic zero and in characteristic p.

2. Given a partition λ **define** λ^+ be the partition obtained by adding an empty runner to the abacus of λ. From the definitions, if $B_{\rho,w}$ is a Rouquier block then so is $B_{\rho^+,w}$.

Corollary Suppose that $w < p$. Then the decomposition matrix of $B(\rho, w)$ is the submatrix of the decomposition matrix of $B(\rho^+, w)$ with rows and columns indexed by the $(e + 1)$-regular partitions in $B(\rho^+, w)$.

Some consequences–II

3. Significantly, we can also say something about decomposition numbers for Rouquier blocks when $w \geq p$—the "non–abelian defect group case".

Given two partitions λ and μ let:

$$\begin{aligned} d_{\lambda\mu}(e) &= [S(\lambda) : D(\mu)], \quad \text{when } q = \exp(\tfrac{2\pi i}{e}) \in \mathbb{C}, \\ d_{\lambda\mu}(e,p) &= [S(\lambda) : D(\mu)], \quad \text{when } q = \sqrt[e]{1} \in R, \operatorname{char} R = p. \end{aligned}$$

Proposition Suppose λ and μ are Rouquier partitions in $B(\rho, w)$. Then

$$d_{\lambda\mu}(e,p) = \sum_{\substack{\tau \in B(\rho,w) \\ \tau \approx \mu}} d_{\lambda\tau}(e) d_{\tau\mu}(e,p).$$

Some consequences—III

Corollary Suppose that $q = 1$ and $\tau = (\tau^0, \dots, \tau^{e-1})$ are partitions in $B(\rho, w)$ such that $\mu \approx \tau$. Then

$$d_{(\mu^0,\dots,\mu^{e-1})(\tau^0,\dots,\tau^{e-1})}(e,p) = \prod_{i=0}^{e-1} d_{\mu^i\tau^i}(e,p).$$

This follows by combining the Proposition with results of Turner.

Application to irreducible Specht modules

Definition A partition λ is (e,p)-**reducible** if there exist nodes (a,i), (a,j) and (b,i) in $[\lambda]$ such that

$$\nu_{e,p}(h^\lambda_{ai}) > 0 \quad \text{and} \quad \nu_{e,p}(h^\lambda_{aj}) \neq \nu_{e,p}(h^\lambda_{ai}) \neq \nu_{e,p}(h^\lambda_{bi}).$$

If λ is not (e,p)-reducible then λ is (e,p)-**irreducible**.

Conjecture (James–M.) Suppose that $q \neq -1$. Then the Specht module $S(\lambda)$ is irreducible if and only if λ is an (e,p)-irreducible partition.

This conjecture has been proved by Fayers for the symmetric groups.

Irreducible Specht modules

Theorem Suppose that $q \neq -1$ and that λ is an (e,p)-irreducible partition. Then $S(\lambda)$ is irreducible.

Idea of proof:

- Fayers has shown that it is enough to consider the case where λ belongs to a Rouquier block.
- Fayers also showed that if $\lambda = (\lambda^0, \dots, \lambda^{e-1})$ is in a Rouquier block then λ is (e,p)–irreducible if and only if $(\lambda^0)'$ and λ^{e-1} are p–regular (p,p)-irreducible partitions and $\lambda^i = \emptyset$, for $0 < i < e-1$.
- We know the decomposition numbers $d_{\lambda\mu}(e)$ explicitly in characteristic zero.
- As $d_{\lambda\mu}(e,p) = \sum_{\tau\approx\mu} d_{\lambda\tau}(e) d_{\tau\mu}(e,p)$ we can complete the proof.

Lyle has recently proved the other half of the irreduciblity conjecture.