Seminormal forms for cellular algebras

Andrew Mathas

University of Sydney

Algebra Seminar—May 2006

Motivation

Let A be a split semisimple algebra over a field k.

By the Wedderburn Theorem, A is isomorphic to a direct sum of matrix algebras:

$$A \cong \bigoplus_{i \in \mathrm{Irr}(A)} \mathrm{Mat}_{n_i}(k).$$

Question

Can we find an explicit natural basis of matrix units for A?

Every split semisimple algebra is a cellular algebra.

In this talk we answer our question for certain cellular algebras and we discuss applications to non–semisimple cellular algebras.

Cellular algebras

Let R be a commutative ring domain and let K be its field of fractions. Let A be an R–algebra and set $A_K = A \otimes_R K$.

Definition (Graham–Lehrer)

An R–algebra A is a **cellular algebra** if it has a basis

$$\{\, a^\lambda_{\mathrm{st}} : \lambda \in \Lambda, \mathrm{s}, \mathrm{t} \in T(\lambda) \,\},$$

where Λ and $T(\lambda)$ are finite posets, such that

1. The map $* : A \longrightarrow A;\ a^\lambda_{\mathrm{st}} \mapsto a^\lambda_{\mathrm{ts}}$ is an algebra anti–isomorphism.
2. If $x \in A$ and $\mathrm{t} \in T(\lambda)$ then

$$a^\lambda_{\mathrm{st}} x \equiv \sum_{\mathrm{v} \in T(\lambda)} r_{\mathrm{tv}x} a^\lambda_{\mathrm{sv}} \pmod{A^\lambda},$$

where the scalars $r_{\mathrm{tv}x}$ do not depend on s.

Here $A^\lambda = \langle a^\mu_{\mathrm{yz}} \mid \mu > \lambda \text{ and } \mathrm{y}, \mathrm{z} \in T(\mu)\rangle$.

An example

The prototypical example of a cellular algebra is the matrix algebra $\mathrm{Mat}_n(R)$.

The algebra $\mathrm{Mat}_n(R)$ has a basis of matrix units

$$\{\, e_{ij} : 1 \le i, j \le n \,\},$$

where e_{ij} is the matrix with a 1 in row i and column j and zeros elsewhere.

The well–known and much loved formula

$$e_{ij} e_{kl} = \delta_{jk} e_{il}$$

shows that $\mathrm{Mat}_n(R)$ is a cellular algebra.

Corollary

Every split semisimple algebra over a field is cellular.

Why study cellular algebras?

Cellular algebras are most useful for studying non–semisimple algebras.

For each $\lambda \in \Lambda$ define the **cell module** $C(\lambda)$ is the free R–module with basis $\{ a^\lambda_{\mathfrak{t}} : \mathfrak{t} \in T(\lambda) \}$ and with A–action

$$a^\lambda_{\mathfrak{t}} x = \sum_{\mathfrak{v} \in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}x} a^\lambda_{\mathfrak{v}}.$$

The cell module has a natural bilinear form $\langle\ ,\ \rangle$ and

$$\operatorname{Rad} C(\lambda) = \{ x \in C(\lambda) : \langle x, y \rangle = 0 \text{ for all } y \in C(\lambda) \}$$

is an A–submodule of $C(\lambda)$.

Set $D(\lambda) = C(\lambda)/\operatorname{Rad} C(\lambda)$.

Theorem (Graham-Lehrer)

Suppose that R is any field. Then $\{ D(\lambda) \neq 0 : \lambda \in \Lambda \}$ is a complete set of pairwise non–isomorphic absolutely irreducible A–modules.

JM–elements

Definition

A family of **JM–elements** for A is a set $\{L_1, \dots, L_M\}$ of commuting elements of A such that

1. $L_i^* = L_i$, for $i = 1, \dots, M$.
2. For $i = 1, \dots, M$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$

$$a^\lambda_{\mathfrak{s}\mathfrak{t}} L_i \equiv c_{\mathfrak{t}}(i) a^\lambda_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{t}\mathfrak{v}} a^\lambda_{\mathfrak{s}\mathfrak{v}} \pmod{A^\lambda},$$

for some scalars $c_{\mathfrak{t}}(i)$.

Note that, implicitly, the JM–elements depend on the choice of cellular basis for A.

We will see that these elements are not in any sense unique. However, the subalgebra that they generate in A is uniquely determined.

Some examples of JM–elements

Example (Semisimple algebras)

Take $A = \operatorname{Mat}_n(R)$ and set $L_k = e_{kk}$.
Then $e_{ij} L_k = \delta_{jk} e_{ij}$.
$\implies$ $\{L_1, \dots, L_n\}$ is a family of JM–elements for A.

Example (A toy example)

Let $A = R[x]/(x - c_1)\dots(x - c_n)$.

Then $\{ a_i : 0 \leq i < n \}$ is a cellular basis of A, where $a_i = \prod_{j=1}^{i-1}(x - c_j)$.

Set $L = L_1 = x$. Then

$$a_i x = (x - c_1)\dots(x - c_{i-1})x = c_i a_i + a_{i+1}.$$

So $\{L\}$ is a family of JM–elements for A.

$\implies$ The number of JM–elements is not an invariant of A!

A non–trivial example of JM–elements...

Example

Let $\mathscr{H}$ be the Hecke algebra of the symmetric group.
This has a natural basis $\{ T_w : w \in \mathfrak{S}_n \}$ indexed by $\mathfrak{S}_n$.

Let $\{m^\lambda_{\mathfrak{s}\mathfrak{t}}\}$ be the **Murphy basis** of $\mathscr{H}$.
This is basis indexed by pairs of standard tableaux.

Set $L_i = q^{-1} T_{(1,i)} + q^{-2} T_{(2,i)} + \dots + q^{1-i} T_{(i-1,i)}$.

Then $m^\lambda_{\mathfrak{s}\mathfrak{t}} L_i = c_{\mathfrak{t}}(i) m^\lambda_{\mathfrak{s}\mathfrak{t}} +$ more dominant terms

where $c_{\mathfrak{t}}(i) = [c - r]_q$ if i appears in row r and column c of $\mathfrak{t}$
and $[k] = \frac{q^k - 1}{q - 1}$ (a quantum integer).

Other examples of cellular algebras which have a family of JM–elements include the Ariki–Koike algebras, (cyclotomic) q–Schur algebras, the Brauer algebras and the BMW algebras.

The separated case

We break the study of cellular algebras with JM–elements into two cases, which correspond roughly to the semisimple and the non–semisimple case.

Definition

The JM–elements **separate** A_K if whenever $s, t \in T(\Lambda)$ and $s \rhd t$ then $c_s(i) \neq c_t(i)$, for some i with $1 \le i \le M$.

That is, the content functions $c_t(i)$ can distinguish between the elements of $T(\Lambda) = \bigcup_\lambda T(\lambda)$.

The separation condition forces A to be semisimple.

Proposition

Suppose that the JM–elements separate A_K. Then A_K is a semisimple algebra.

Averaging operators

Until further notice, suppose that A is a cellular algebra with a family of JM–elements which separate A.

The key definition which makes everything work is the following:

Definition

Suppose that $s, t \in T(\lambda)$, for some $\lambda \in \Lambda$. Define

1. $F_t = \prod_{i=1}^{M} \prod_{c \neq c_t(i)} \frac{L_i - c}{c_t(i) - c}$.
2. $f_{st}^\lambda = F_s a_{st}^\lambda F_t$.

Thus, F_t and f_{st} both belong to A_K, but not necessarily to A_R.

In practice, many of the terms in F_t can be omitted—but it is hard to say exactly which ones aren't needed!

An orthogonal basis of A_K

Recall that $a_{st}^\lambda L_i = c_t(i) a_{st}^\lambda +$ more dominant terms

$\Longrightarrow a_{st}^\lambda \frac{L_i - c}{c_t(i) - c} = a_{st}^\lambda +$ more dominant terms

$\Longrightarrow f_{st}^\lambda = F_s a_{st}^\lambda F_t = a_{st}^\lambda +$ more dominant terms.

Consequently, $\{ f_{st}^\lambda : \lambda \in \Lambda, s, t \in T(\lambda) \}$ is a basis of A_K.

In fact, we have the following.

Theorem

Suppose that A has a family of JM–elements which separate A_K. Then $\{ f_{st}^\lambda : \lambda \in \Lambda, s, t \in T(\lambda) \}$ is a cellular basis of matrix units for A_K.

More explicitly, there exist non–zero scalars $\gamma_t \in K$ such that

$$f_{st}^\lambda f_{uv}^\mu = \begin{cases} \gamma_t f_{sv}^\lambda, & \text{if } \lambda = \mu \text{ and } t = u, \\ 0, & \text{otherwise.} \end{cases}$$

Idea of proof

We have already seen that $\{f_{st}^\lambda\}$ is a basis of A_K. So we need only prove the multiplication formula.

The key result is that the f_{st}^λ are a basis of simultaneous eigenvectors for the JM–elements. Explicitly,

$$f_{st}^\lambda L_i = c_t(i) f_{st}^\lambda \quad \Longrightarrow \quad f_{st}^\lambda F_u = \delta_{ut} f_{st}^\lambda \quad \Longrightarrow \quad f_{st}^\lambda f_{uv}^\mu = \delta_{tu} \gamma_{stv} f_{sv}^\lambda$$

Recall that $f_{st}^\lambda = F_s a_{st}^\lambda F_t$.

To prove the key result consider $f'_{st} = F_s^N a_{st}^\lambda F_t^N$, where $N \gg 0$.

$$\begin{aligned} \text{Now, } f'_{st} L_i &= F_s^N a_{st}^\lambda F_t^N L_i = F_s^N a_{st}^\lambda L_i F_t^N \\ &= F_s^N \Big(c_t(i) a_{st}^\lambda + \text{more dominant terms} \Big) F_t^N \\ &= c_t(i) F_s^N a_{st}^\lambda F_t^N = c_t(i) f'_{st}. \end{aligned}$$

Armed with this fact one can show that $f_{st}^\lambda = f'_{st}$, which then completes the proof of the key result

Some consequences of the basis theorem

1. The cell module $C(\lambda) \cong \langle f^{\lambda}_{\mathfrak{st}} \mid \mathfrak{t} \in T(\lambda)\rangle_K$, for any $\mathfrak{s} \in T(\lambda)$.
2. The Gram determinant of $C(\lambda)$ is $\displaystyle\prod_{t \in T(\lambda)} \gamma_{\mathfrak{t}}$.
3. $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f^{\lambda}_{\mathfrak{tt}}$ is a primitive idempotent in A_K.
4. $F_{\lambda} = \displaystyle\sum_{\mathfrak{t} \in T(\lambda)} F_{\mathfrak{t}}$ is a primitive central idempotent in A_K
5. $1_{A_K} = \displaystyle\sum_{\lambda \in \Lambda} F_{\lambda} = \sum_{\mathfrak{t} \in T(\Lambda)} F_{\mathfrak{t}}$
6. $L_i = \displaystyle\sum_{\mathfrak{t}} c_{\mathfrak{t}}(i) F_{\mathfrak{t}}$

Corollary

The JM–elements generate a maximal commutative subalgebra of A_K.

The non-separated case

We have been assuming that the JM–elements separate A_K.

That is, $\mathfrak{s} \rhd \mathfrak{t} \implies c_{\mathfrak{s}}(i) \neq c_{\mathfrak{t}}(i)$, for some i.

We now investigate what happens when we drop this assumption.

We need some technical assumptions, all of which can safely be ignored because they are easy to satisfy in practice.

Let (R, K, k) be a modular system:

- *R is a local ring with maximal ideal π;*
- *$k = R/\pi$ is the residue field of R; and,*
- *K is the field of fractions of R.*

Consider the three cellular algebras:

$$A = A_R, \quad A_K = A_R \otimes_R K \quad \text{and} \quad A_k = A_R \otimes_R k.$$

Assume that A_R has a family of JM–elements which separate $A_K \implies A_K$ is semisimple.

Finally, assume that $c_{\mathfrak{s}}(i) - c_{\mathfrak{t}}(j) \notin \pi$ is invertible in R.

Residue classes

If $r \in R$ let $\overline{r} = r + \pi$ be its reduction modulo π.
In particular, write $r_{\mathfrak{t}}(i) = \overline{c_{\mathfrak{t}}(i)}$.

If $a = \sum r_{\mathfrak{st}} a^{\lambda}_{\mathfrak{st}} \in A_R$ write $\overline{a} = \sum \overline{r_{\mathfrak{st}}} a^{\lambda}_{\mathfrak{st}} \in A_k$.

The basic idea is to reduce the elements $F_{\mathfrak{t}}$ and $f^{\lambda}_{\mathfrak{st}}$ modulo π to obtain elements of A_k with similar properties.

The basic problem is that this is not possible in general.

Definition

1. $\mathfrak{s}, \mathfrak{t} \in T(\Lambda)$ are in the same **residue class** if $r_{\mathfrak{s}}(i) = r_{\mathfrak{t}}(i)$, for $1 \le i \le M$. We write $\mathfrak{s} \approx \mathfrak{t}$.
2. $\lambda, \mu \in \Lambda$ are **residually linked** if there exist elements $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_r = \mu$ and elements $\mathfrak{s}_j, \mathfrak{t}_j \in T(\lambda_j)$ such that $\mathfrak{s}_{j-1} \approx \mathfrak{t}_j$, for $i = 1, \dots, r$. We write $\lambda \sim \mu$.

Reduction modulo π

Let $\mathbb{T}$ be a residue class in $T(\Lambda)$ and define $F_{\mathbb{T}} = \displaystyle\sum_{\mathfrak{t} \in \mathbb{T}} F_{\mathfrak{t}}$.

Lemma

Suppose that $\mathbb{T}$ is a residue class. Then $F_{\mathbb{T}}$ is an idempotent in A_R.

The proof of this is a clever argument which is due to Murphy. The idea is that, morally,

$$F_{\mathbb{T}} = \prod_{i=1}^{M} \prod_{\substack{c = c_{\mathfrak{s}}(j) \\ \overline{c} \neq r_{\mathfrak{t}}(i)}} \frac{L_i - c}{c_{\mathfrak{t}}(i) - c} \in A_R.$$

In general, $F_{\mathbb{T}} \neq F'_{\mathbb{T}}$, however using $F'_{\mathbb{T}}$ we can show that $F_{\mathbb{T}} \in A_R$.

Let $G_{\mathbb{T}} = \overline{F_{\mathbb{T}}}$. Then, by the Lemma, $G_{\mathbb{T}}$ is an idempotent in A_k.

If $\mathfrak{s} \in \mathbb{S}$ and $\mathfrak{t} \in \mathbb{T}$ define $g^{\lambda}_{\mathfrak{st}} = G_{\mathbb{S}} a^{\lambda}_{\mathfrak{st}} G_{\mathbb{T}} \in A_k$.

An almost seminormal basis

We can now prove the following:

1. $\{\, g^\lambda_{\mathfrak{st}} : \mathfrak{s},\mathfrak{t}\in T(\lambda) \text{ and } \lambda\in\Lambda \,\}$ is a cellular basis of A_k.
2. Let $\Gamma\in\Lambda/\sim$. Then $A^\Gamma_k = G_\Gamma A_k G_\Gamma \cong \mathrm{End}_{A_k}(A_k G_\Gamma)$ is a cellular algebra with cellular basis
$$\{\, g^\lambda_{\mathfrak{st}} : \mathfrak{s},\mathfrak{t}\in T(\lambda) \text{ and } \lambda\in\Gamma \,\}.$$
3. The residue linkage classes decompose A_k into a direct sum of cellular subalgebras; that is,
$$A_k = \bigoplus_{\Gamma\in\Lambda/\sim} A^\Gamma_k.$$
4. $\{\, G_\Gamma : \Gamma\in\Lambda/\sim \,\}$ and $\{\, G_{\mathbb{T}} : \mathbb{T}\in T(\Lambda)/\approx \,\}$ are complete sets of pairwise orthogonal idempotents of A_k. That is,
$$1_{A_k} = \sum_{\Gamma\in\Lambda/\sim} G_\Gamma = \sum_{\mathbb{T}\in T(\Lambda)/\approx} G_{\mathbb{T}}.$$

A new basis for the cell modules

Fix $\lambda\in\Lambda$ and some $\mathfrak{s}\in T(\lambda)$.

For all $\mathfrak{t}\in T(\lambda)$ define $g^\lambda_{\mathfrak{t}} = g^\lambda_{\mathfrak{st}} + A^\lambda_k$.

Proposition

Suppose that $\lambda\in\Lambda$. Then

1. $C(\lambda)_k \cong \langle g^\lambda_{\mathfrak{t}} \mid \mathfrak{t}\in T(\lambda)\rangle_k$.
2. *If* $\mathfrak{t},\mathfrak{u}\in T(\lambda)$ *then*
$$\langle g^\lambda_{\mathfrak{t}}, g^\lambda_{\mathfrak{u}}\rangle_\lambda = \begin{cases} \langle a^\lambda_{\mathfrak{t}}, g^\lambda_{\mathfrak{u}}\rangle_\lambda, & \text{if } \mathfrak{t}\approx\mathfrak{u},\\ 0, & \text{if } \mathfrak{t}\not\approx\mathfrak{u}.\end{cases}$$

In the semisimple case the seminormal basis diagonalizes the Gram matrix of $C(\lambda)$.

In the non–semisimple case this result give a diagonal block decomposition of the Gram matrix.

Bases for the blocks of several algebras

Theorem

Let k be a field and suppose that A_R is one of the following algebras:

1. *the group algebra $R\mathfrak{S}_n$ of the symmetric group;*
2. *the Hecke algebra $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ of type A;*
3. *the Ariki–Koike algebra $\mathscr{H}_{R,q,\mathbf{u}}$ with $q\neq 1$;*
4. *the degenerate Ariki–Koike algebra $\mathcal{H}_{R,\mathbf{v}}$;*
5. *the q–Schur algebra $S_{R,q}(n)$;*
6. *the cyclotomic q–Schur $S_{R,t,\mathbf{v}}(\Lambda_{m,n})$ algebra with $q\neq 1$.*
7. *the degenerate cyclotomic Schur $S_{R,\mathbf{v}}(\Lambda_{m,n})$*

Then the basis $\{g^\lambda_{\mathfrak{st}}\}$ is a basis for the block decomposition of A_k into a direct sum of indecomposable subalgebras.