**Motivation**

Let $A$ be a split semisimple algebra over a field $k$. By the Wedderburn Theorem, $A$ is isomorphic to a direct sum of matrix algebras:

$$A \cong \bigoplus_{i \in \text{Irr}(A)} \text{Mat}_{n_i}(k).$$

**Question**

Can we find an explicit natural basis of matrix units for $A$?

Every split semisimple algebra is a cellular algebra. In this talk we answer our question for certain cellular algebras and we discuss applications to non-semisimple cellular algebras.

**An example**

The prototypical example of a cellular algebra is the matrix algebra $\text{Mat}_n(R)$. The algebra $\text{Mat}_n(R)$ has a basis of matrix units

$$\{ e_{ij} : 1 \leq i, j \leq n \},$$

where $e_{ij}$ is the matrix with a 1 in row $i$ and column $j$ and zeros elsewhere.

The well-known and much loved formula

$$e_{ij} e_{kl} = \delta_{jk} e_{il},$$

shows that $\text{Mat}_n(R)$ is a cellular algebra.

**Corollary**

Every split semisimple algebra over a field is cellular.
Why study cellular algebras?
Cellular algebras are most useful for studying non-semisimple algebras.
For each \( \lambda \in \Lambda \) define the cell module \( C(\lambda) \) is the free \( R \)-module with basis \( \{ a^k_t : t \in T(\lambda) \} \) and with \( A \)-action
\[
a^k_t x = \sum_{v \in T(\lambda)} f_{tvx} a^v_t.
\]
The cell module has a natural bilinear form \( \langle , \rangle \) and
\[
\text{Rad} C(\lambda) = \{ x \in C(\lambda) : \langle x, y \rangle = 0 \text{ for all } y \in C(\lambda) \}
\]
is an \( A \)-submodule of \( C(\lambda) \).
Set \( D(\lambda) = C(\lambda)/\text{Rad} C(\lambda) \).

**Theorem (Graham-Lehrer)**
Suppose that \( R \) is any field. Then \( \{ D(\lambda) \neq 0 : \lambda \in \Lambda \} \) is a complete set of pairwise non-isomorphic absolutely irreducible \( A \)-modules.

Some examples of JM–elements

**Example (Semisimple algebras)**
Take \( A = \text{Mat}_n(R) \) and set \( L_k = e_{kk} \).
Then \( e_{ij}L_k = \delta_{jk} e_{ij} \).
\( \{ L_1, \ldots, L_n \} \) is a family of JM–elements for \( A \).

**Example (A toy example)**
Let \( A = R[x]/(x - c_1) \ldots (x - c_n) \).
Then \( \{ a_i : 0 \leq i < n \} \) is a cellular basis of \( A \), where
\[
a_i = \prod_{j=1}^{i-1} (x - c_j).
\]
Set \( L = L_1 = x \). Then
\[
a_i x = (x - c_1) \ldots (x - c_i-1)x = c_i a_i + a_{i+1},
\]
So \( \{ L \} \) is a family of JM–elements for \( A \).

JM–elements

**Definition**
A family of JM–elements for \( A \) is a set \( \{ L_1, \ldots, L_M \} \) of commuting elements of \( A \) such that
1. \( L_i^t = L_i \) for \( i = 1, \ldots, M \).
2. For \( i = 1, \ldots, M \) and \( s, t \in T(\lambda) \)
\[
a_{st}^i L_i \equiv a_s(i) a_{st}^i + \sum_{v > t} f_{tvx} a_{sv}^i \pmod{A^i},
\]
for some scalars \( a_s(i) \).

Note that, **implicitly**, the JM–elements depend on the choice of cellular basis for \( A \).

We will see that these elements are not in any sense unique. However, the subalgebra that they generate in \( A \) is uniquely determined.

A non–trivial example of JM–elements...

**Example**
Let \( \mathcal{H} \) be the Hecke algebra of the symmetric group.
This has a natural basis \( \{ T_w : w \in \mathfrak{S}_n \} \) indexed by \( \mathfrak{S}_n \).
Let \( \{ m^\lambda_{st} \} \) be the **Murphy basis** of \( \mathcal{H} \).
This is basis indexed by pairs of standard tableaux.
Set \( L_i = q^{-i} T(1,i) + q^{-2} T(2,i) + \cdots + q^{1-i} T(i-1,i) \).
Then
\[
m^\lambda_{st} L_i = a_s(i) m^\lambda_{st} + \text{ more dominant terms}
\]
where \( a_s(i) = [c - r]_q \) if \( i \) appears in row \( r \) and column \( c \) of \( t \)
and \( [k] = q^{k-1} q^{-1} \) (a quantum integer).

Other examples of cellular algebras which have a family of JM–elements include the Ariki–Koike algebras, (cyclotomic) \( q \)-Schur algebras, the Brauer algebras and the BMW algebras.
The separated case

We break the study of cellular algebras with JM–elements into two cases, which correspond roughly to the semisimple and the non–semisimple case.

**Definition**

The JM–elements separate $A_K$ if whenever $s, t \in T(\Lambda)$ and $s \triangleright t$ then $\alpha_s(i) \neq \alpha(t(i))$, for some $i$ with $1 \leq i \leq M$.

That is, the content functions $\alpha(i)$ can distinguish between the elements of $T(\Lambda) = \bigcup_\lambda T(\lambda)$.

The separation condition forces $A$ to be semisimple.

**Proposition**

Suppose that the JM–elements separate $A_K$.

Then $A_K$ is a semisimple algebra.

An orthogonal basis of $A_K$

Recall that $a^\lambda_{st}L_i = \alpha_s(i)a^\lambda_{st} +$ more dominant terms

$\implies a^\lambda_{st}\frac{L_i - e}{\alpha(i) - e} = a^\lambda_{st} +$ more dominant terms

$\implies f^\lambda_{st} = F_s a^\lambda_{st}F_t = a^\lambda_{st} +$ more dominant terms.

Consequently, $\{ f^\lambda_{st} : \lambda \in \Lambda, s, t \in T(\lambda) \}$ is a basis of $A_K$.

In fact, we have the following.

**Theorem**

Suppose that $A$ has a family of JM–elements which separate $A_K$.

Then $\{ f^\lambda_{st} : \lambda \in \Lambda, s, t \in T(\lambda) \}$ is a cellular basis of matrix units for $A_K$.

More explicitly, there exist non–zero scalars $\gamma_t \in K$ such that

$f^\lambda_{st}F_u \gamma_t = \begin{cases} f^\lambda_{sv} &, \text{if } \lambda = \mu \text{ and } t = u, \\ 0 &, \text{otherwise.} \end{cases}$

Averaging operators

Until further notice, suppose that $A$ is a cellular algebra with a family of JM–elements which separate $A$.

The key definition which makes everything work is the following:

**Definition**

Suppose that $s, t \in T(\lambda)$, for some $\lambda \in \Lambda$. Define

- $F_1 = \prod_{i=1}^M \prod_{c \neq \alpha(i)} \frac{L_i - c}{\alpha(i) - c}$

- $f^\lambda_{st} = F_s a^\lambda_{st}F_t$.

Thus, $F_1$ and $f^\lambda_{st}$ both belong to $A_K$, but not necessarily to $A_R$.

In practice, many of the terms in $F_1$ can be omitted—but it is hard to say exactly which ones aren’t needed!

Idea of proof

We have already seen that $\{ f^\lambda_{st} \}$ is a basis of $A_K$.

So we need only prove the multiplication formula.

The key result is that the $f^\lambda_{st}$ are a basis of simultaneous eigenvectors for the JM–elements. Explicitly,

- $f^\lambda_{st}L_i = \alpha(i)f^\lambda_{st}$

- $f^\lambda_{st}F_u = \delta_u f^\lambda_{st}$

- $f^\lambda_{st}F_u \gamma_t = f^\lambda_{sv}$

Recall that $f^\lambda_{st} = F_s a^\lambda_{st}F_t$.

To prove the key result consider $f^\lambda_{st} = F_s a^\lambda_{st}F_t N$, where $N > 0$.

Now, $f^\lambda_{st}L_i = F_s a^\lambda_{st} F_t N L_i = F_s a^\lambda_{st} \alpha(i) F_t N$

$= F_s \left( \alpha(i) a^\lambda_{st} + \text{more dominant terms} \right) F_t$

$= \alpha(i) f^\lambda_{st} N = \alpha(i) f^\lambda_{st}$.

Armed with this fact one can show that $f^\lambda_{st} = f^\lambda_{st}$, which then completes the proof of the key result.
Some consequences of the basis theorem

- The cell module \( C(\lambda) \cong (t_s^i | t \in T(\lambda))_k \), for any \( s \in T(\lambda) \).
- The Gram determinant of \( C(\lambda) \) is \( \prod_{t \in T(\lambda)} \gamma_t \).
- \( F_i = \frac{1}{\gamma_i} t_i^i \) is a primitive idempotent in \( A_k \).
- \( F_\lambda = \sum_{t \in T(\lambda)} F_i \) is a primitive central idempotent in \( A_k \).
- \( 1_{A_k} = \sum_{\lambda \in \Lambda} F_\lambda = \sum_{t \in T(\lambda)} F_t \).
- \( L_i = \sum_t a_i(t) F_t \).

Corollary

The JM–elements generate a maximal commutative subalgebra of \( A_k \).

Residue classes

If \( r \in R \) let \( \tau = r + \pi \) be its reduction modulo \( \pi \).
In particular, write \( \eta(i) = \overline{a(i)} \).
If \( a = \sum r_{st} a_s^l \in A_R \) write \( \overline{a} = \sum t_{st} a_s^l \in A_k \).

The basic idea is to reduce the elements \( F_t \) and \( r_{st}^l \) modulo \( \pi \) to obtain elements of \( A_k \) with similar properties.
The basic problem is that this is not possible in general.

Definition

- \( s, t \in T(\Lambda) \) are in the same residue class if \( r_s(i) = \eta_t(i) \),
  for \( 1 \leq t \leq M \). We write \( s \equiv t \).
- \( \lambda, \mu \in \Lambda \) are residually linked if there exist elements \( \lambda_0 = \lambda, \lambda_1, \ldots, \lambda_r = \mu \) and elements \( s_j, t_j \in T(\lambda_j) \) such that \( s_{j-1} \approx t_j \) for \( i = 1, \ldots, r \). We write \( \lambda \sim \mu \).

The non-separated case

We have been assuming that the JM–elements separate \( A_k \).
That is, \( s \triangleright t \implies c_s(i) \neq c_t(i) \), for some \( i \).
We now investigate what happens when we drop this assumption.

We need some technical assumptions, all of which can safely be ignored because they are easy to satisfy in practice.

Let \((R, K, k)\) be a modular system:
- \( R \) is a local ring with maximal ideal \( \pi \);
- \( k = R/\pi \) is the residue field of \( R \); and,
- \( K \) is the field of fractions of \( R \).

Consider the three cellular algebras:

- \( A = A_R \), \( A_k = A_R \otimes_R K \) and \( A_k = A_R \otimes_R K \).
- Assume that \( A_R \) has a family of JM–elements which separate \( A_k \) \( \implies \) \( A_k \) is semisimple.
- Finally, assume that \( c_s(i) \sim c_t(i) \notin \pi \) is invertible in \( R \).

Reduction modulo \( \pi \)

Let \( T \) be a residue class in \( T(\Lambda) \) and define \( F_T = \sum_{t \in T} F_t \).

Lemma

Suppose that \( T \) is a residue class. Then \( F_T \) is an idempotent in \( A_R \).

The proof of this is a clever argument which is due to Murphy. The idea is that, morally,

\[
F_T = \prod_{i=1}^M \prod_{c = c_s(i)} \frac{L_i - c}{a_s^i(i) - c} \in A_R.
\]

In general, \( F_T \neq F_T' \), however using \( F_T' \) we can show that \( F_T \in A_R \).
Let \( G_T = F_T' \). Then, by the Lemma, \( G_T \) is an idempotent in \( A_k \).
If \( s \in S \) and \( t \in T \) define \( g_s^t = G_s G_t \in A_k \).
An almost seminormal basis
We can now prove the following:

1. \( \{ g_{st}^\lambda : s, t \in T(\lambda) \text{ and } \lambda \in \Lambda \} \) is a cellular basis of \( A_k \).

2. Let \( \Gamma \in \Lambda/\sim \). Then \( A_k^\Gamma = G_{\Gamma} A_k G_{\Gamma} \cong \text{End}_{A_k}(A_k G_{\Gamma}) \) is a cellular algebra with cellular basis \( \{ g_{st}^\lambda : s, t \in T(\lambda) \text{ and } \lambda \in \Gamma \} \).

3. The residue linkage classes decompose \( A_k \) into a direct sum of cellular subalgebras; that is, \( A_k = \bigoplus_{\Gamma \in \Lambda/\sim} A_k^\Gamma \).

4. \( \{ G_{\Gamma} : \Gamma \in \Lambda/\sim \} \) and \( \{ G_T : T \in T(\Lambda)/\approx \} \) are complete sets of pairwise orthogonal idempotents of \( A_k \). That is, \( \sum_{\Gamma \in \Lambda/\sim} G_{\Gamma} = \sum_{T \in T(\Lambda)/\approx} G_T \).

Bases for the blocks of several algebras

Theorem
Let \( k \) be a field and suppose that \( A_R \) is one of the following algebras:

- the group algebra \( R S_n \) of the symmetric group;
- the Hecke algebra \( H_{R,q}(S_n) \) of type \( A \);
- the Ariki–Koike algebra \( H_{R,q,u} \) with \( q \neq 1 \);
- the degenerate Ariki–Koike algebra \( H_{R,v} \);
- the \( q \)-Schur algebra \( S_{R,q}(n) \);
- the cyclotomic \( q \)-Schur algebra \( S_{R,t,v}(\Lambda_{m,n}) \) algebra with \( q \neq 1 \);
- the degenerate cyclotomic Schur \( S_{R,v}(\Lambda_{m,n}) \).

Then the basis \( \{ g_{st}^\lambda \} \) is a basis for the block decomposition of \( A_k \) into a direct sum of indecomposable subalgebras.