What is now proved was once only imagined.
William Blake

Introduction

The symmetric group $\mathfrak{S}_n$ is the group of permutations on $1, 2, \ldots, n$. The ordinary irreducible representations of $\mathfrak{S}_n$ are very well understood, with everything from their degrees and character formulae, to explicit matrix representations being known for many years. In contrast, very little detailed information is known about the modular representations of the symmetric groups; in fact, in spite of a great deal of effort, not that much progress has been made since James [**84**] classified and constructed the irreducible modular representations of the symmetric groups in 1976.

The aim of this book is to give a self–contained introduction to the modular representation theory of the Iwahori–Hecke algebras of the symmetric groups; this includes the modular representation theory of $\mathfrak{S}_n$ as a special case. In studying the Iwahori–Hecke algebras it is profitable to widen the scope of our investigations to include the q–Schur algebras. The study of these algebras was pioneered by Dipper and James in a series of landmark papers [**37**, **39**–**41**]. Here we recast this theory, taking account of recent advances, with a primary goal of classifying the blocks and the simple modules of both algebras. We have written these notes so as to be accessible to the advanced graduate student and also to be useful to researchers in the field.

Apart from being interesting in and of themselves, the main motivation for studying the Iwahori–Hecke algebras is that they provide a bridge between the representation theory of the symmetric and general linear groups; these connections are even more transparent with the q–Schur algebras. In the classical case (that is, $q = 1$), the Schur algebras were introduced by Schur [**158**] who used them, together with the representation theory of the symmetric groups, to classify the ordinary irreducible polynomial representations of $GL_n(\mathbb{C})$; see also [**74**]. Dipper and James' motivation for introducing the q–Schur algebra was to study the modular representation theory of $GL_n(q)$ over fields of characteristic not dividing q (that is, in non–defining characteristic); they showed that the q–Schur algebras completely determine the decomposition matrix of $GL_n(q)$ in this case [**40**].

Our motivation for studying the q–Schur algebra is more modest in that we view the q–Schur algebra as a tool for studying the Iwahori–Hecke algebra. As we will see, the q–Schur algebras have a rich and beautiful combinatorial representation theory which is closely allied with that of the Iwahori–Hecke algebras. Indeed, full knowledge of the representation theory of one class of these algebras is equivalent to full knowledge of the other. Further, it is often the case that the easiest way to prove a result about one of these algebras is to first prove an analogous result for the other. Throughout the 'classical' theory for the symmetric and general linear groups can be obtained by setting $q = 1$.

These notes adopt the view that the Iwahori–Hecke algebras — rather than the q–Schur algebras — are the objects of central importance. This is partly a matter of personal taste and partly expedience; other authors, such as Donkin [**48**] and Green [**74**], travel in the reverse direction. One consequence of our perspective is that our definition of the Iwahori–Hecke algebra may strike some readers as being contrived; to remedy this we now provide additional motivation.

First recall that, as an abstract group, the symmetric group has a presentation with generators $s_1, \dots, s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, 2, \dots, n-1, \\ s_i s_j &= s_j s_i, && \text{for } 1 \le i < j-1 \le n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && \text{for } i = 1, 2, \dots, n-2. \end{aligned}$$

Identifying s_i with the transposition $(i, i+1)$ shows that $\mathfrak{S}_n$ is a quotient of the group W with the presentation above; a little more work verifies that $W \cong \mathfrak{S}_n$.

Now fix a ring R and let q be an element of R. The Iwahori–Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the associative algebra with generators $T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, && \text{for } i = 1, 2, \dots, n-1, \\ T_i T_j &= T_j T_i, && \text{for } 1 \le i < j-1 \le n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } i = 1, 2, \dots, n-2. \end{aligned}$$

In particular, $\mathscr{H}$ and $R\mathfrak{S}_n$ are isomorphic when $q = 1$. Thus, $\mathscr{H}$ is a *deformation* of the group ring $R\mathfrak{S}_n$ of the symmetric group; that is, the Iwahori–Hecke algebras of $\mathfrak{S}_n$ are a family of algebras which 'look like' the group ring of the symmetric group except that the multiplication is 'deformed' by q.

From the presentation of $\mathscr{H}$ it seems likely that $\mathscr{H}$ will collapse for some choices of the parameter q; in fact, we will show in chapter 1 that, independently of q, the Iwahori–Hecke algebra is always a free R–module of rank $n! = |\mathfrak{S}_n|$. The best explanation of why the rank of $\mathscr{H}$ is independent of q is that the Iwahori–Hecke algebras appear naturally in the representation theory of the general linear groups; we now describe how this comes about.

Assume that q is a prime power and let $G = GL_n(q)$ be the general linear group over the field of q elements. Let $B = B(q)$ be a Borel subgroup of G; thus, B is conjugate to the subgroup of upper triangular matrices in G. Let $\operatorname{Ind}_B^G(1)$ be the induced RG–representation on the right cosets of B in G and let $H_q = \operatorname{End}_{RG}\big(\operatorname{Ind}_B^G(1)\big)$ be the endomorphism algebra of this module. Amazingly, the algebras H_q and $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ are canonically isomorphic; so, $\mathscr{H}$ is also a deformation of the endomorphism algebra H_q!

Here is a rough proof of the isomorphism $\mathscr{H}_{R,q}(\mathfrak{S}_n) \cong H_q$. For any subset X of G let $[X] = \sum_{x \in X} x$, an element of RG. Then $\operatorname{Ind}_B^G(1)$ is free as an RG–module with basis $[Bg]$, where g runs over a set of right coset representatives of B in G. Therefore, H_q has basis $[BgB]$, where g runs over the (B, B)–double coset representatives. Now $\mathfrak{S}_n$ is the Weyl group of G, so $G = \coprod_{w \in \mathfrak{S}_n} BwB$ by the Bruhat decomposition; here we identify $\mathfrak{S}_n$ with the subgroup of permutation matrices in G. Therefore, H_q is R–free with basis $\{\, [BwB] \mid w \in \mathfrak{S}_n \,\}$.

As above, let $s_1, \dots, s_{n-1}$ be generators of $\mathfrak{S}_n$ and write $\ell(w) = k$ if k is minimal such that $w = s_{i_1} \dots s_{i_k}$ for some $1 \le i_j < n$. Then for all $w \in \mathfrak{S}_n$ and all i with $1 \leq i < n$, one can show that

$$[BwB][Bs_iB] = \begin{cases} q[Bws_iB] + (q-1)[BwB], & \text{if } \ell(ws_i) < \ell(w), \\ [Bws_iB], & \text{if } \ell(ws_i) > \ell(w). \end{cases}$$

This implies that the R–linear map $\pi : \mathscr{H}_{R,q}(\mathfrak{S}_n) \longrightarrow H_q$ given by $\pi(T_i) = [Bs_iB]$, for $1 \leq i < n$, is a surjective algebra homomorphism. A counting argument shows that π is an isomorphism.

As there are an infinite number of primes, the isomorphisms $H_q \cong \mathscr{H}_{R,q}$ can also be used to show that $\mathscr{H}_{R,q}$ is free of rank $n!$ for any q. See Note 1.7 on page 13.

To get a little more mileage out of this discussion, suppose that the base ring R is the field of complex numbers. Then H_q is semisimple and, therefore, so is $\mathscr{H}_{\mathbb{C},q}$. Let $e_B = \frac{1}{|B|}[B] \in \mathbb{C}G$; then $e_B^2 = e_B$, so e_B is idempotent. Furthermore, it is not hard to check that $\operatorname{Ind}_B^G(1) \cong e_B\mathbb{C}G$ and that $H_q \cong e_B\mathbb{C}Ge_B$. Thus, H_q is what is known as a *Hecke algebra*. (More generally, a Hecke algebra is any subalgebra of an algebra A of the form eAe for some idempotent $e \in A$.)

Using the elementary theory of Hecke algebras (see [**31**, §12]), the irreducible constituents of $\operatorname{Ind}_B^G(1)$ are in canonical one–to–one correspondence with the irreducible representations of H_q. As we are in the semisimple case, $H_q \cong \mathscr{H}_{\mathbb{C},q} \cong \mathbb{C}\mathfrak{S}_n$, so this shows that the irreducible constituents of $\operatorname{Ind}_B^G(1)$ are indexed by the ordinary irreducible representations of $\mathfrak{S}_n$. Thus, for every irreducible representation χ of $\mathfrak{S}_n$ there exists a family of representations $\{\, \chi_q \mid q \text{ a prime power} \,\}$ such that χ_q is an irreducible $\mathbb{C}GL_n(q)$–module which is a direct summand of $\operatorname{Ind}_B^G(1)$. Moreover, with a little more work, it is possible to prove the astounding fact that there exists a polynomial $D_\chi(x)$, which depends only on χ, such that $D_\chi(q)$ is the dimension of χ_q for any q (and $D_\chi(1)$ is the dimension of χ). More generally, the characters of the representations χ_q are also polynomials in q (!). The representations χ_q are the *unipotent principal series* representations of G. Proofs of these results can be found in [**22**, **31**, **68**].

The theory we have just sketched connecting the Iwahori–Hecke algebras of the symmetric groups with the general linear groups applies more generally to the Iwahori–Hecke algebras of arbitrary Weyl groups and the corresponding groups of Lie type. Further, Iwahori–Hecke algebras may be defined for any Coxeter group and, more recently, for any complex reflection group [**12**]. In addition to these links with the groups of Lie type, the Iwahori–Hecke algebras play a rôle in the representation theory of quantum groups and affine Hecke algebras and have applications to knot theory and statistical mechanics. At best, we touch only briefly on these matters here; the interested reader is referred to [**29**, **64**, **102**, **103**, **131**, **138**, **168**] and the references therein.

The representation theory of the Iwahori–Hecke algebras and the q–Schur algebras is a very rich and beautiful subject. Our approach is largely combinatorial, involving generalizations of well–known concepts such as tableaux from the representation theory of the symmetric group. Here is a broad outline of the book.

Chapter 1 begins by establishing some basic properties of the symmetric group and its Iwahori–Hecke algebra $\mathscr{H}$. In fact, this chapter is really a chapter about Coxeter groups and their Iwahori–Hecke algebras in disguise because everything we do — including all but one of the proofs — extends to this more general situation.

The second chapter develops Graham and Lehrer's [**72**] theory of *cellular algebras.* These are a class of algebras which come equipped with a distinguished basis which is particularly well–adapted to the representation theory of the algebra. For example, given a cellular basis one can immediately write down a collection of modules which contains all of the simple modules of the algebra. Cellular algebras are one of the unifying threads running through these notes as we construct the simple modules of the Iwahori–Hecke algebras and the q–Schur algebras by first showing that these algebras are cellular.

The third chapter embarks upon the study of the representation theory of $\mathscr{H}$. Following Murphy [**150**] we show that $\mathscr{H}$ has a natural basis indexed by pairs of standard tableaux; importantly, Murphy's basis is cellular. As a consequence, for each partition λ we obtain a Specht module S^λ; this module is a q–analogue of the usual Specht module of $\mathfrak{S}_n$. By the theory of chapter 2, there is an intrinsically defined bilinear form on each Specht module; furthermore, S^λ modulo the radical of its form is either zero or absolutely irreducible. The last two sections of this chapter are a detailed study of the Specht modules, culminating in the classification of the simple $\mathscr{H}$–modules. All of this theory closely parallels the modular representation of the symmetric group.

Apart from being cellular, Murphy's basis has another marvelous property in that it is possible to 'lift' this basis to give a cellular basis for the q–Schur algebra; this basis is indexed by pairs of semistandard tableaux, and each basis element is essentially a sum of Murphy basis elements (a semistandard tableaux can be thought of as an orbit of standard tableaux and the sums are over these orbits). In this way we obtain a very clean and very elegant construction of the simple modules of the q–Schur algebras. For free, we discover that the q–Schur algebras are quasi–hereditary. All of these results are proved in chapter 4.

Chapter 5 is devoted to classifying the blocks of the q–Schur algebras; as a corollary this yields the blocks of $\mathscr{H}$. In order to classify the blocks we first prove an analogue of the Jantzen sum formula [**98**] for the Weyl modules of the q–Schur algebras. The Jantzen sum formula is a strong result which gives information about the composition factors of Weyl modules and Specht modules; it is proved by computing the determinants of the Gram matrices of the Weyl modules (that we can compute these determinants in general is in itself surprising). This calculation requires a heavy dose of combinatorics; it is by far the most technical part of these notes.

The final chapter is a survey of some recent and important results and conjectures in the field; here we abandon our claims of being self–contained. The chapter begins with a reasonably thorough account of the LLT algorithm which computes the decomposition matrices of the Iwahori–Hecke algebras defined over the complex field. We then discuss adjustment matrices, the Kleshchev–Brundan branching rules, the theory of Dipper and James [**40**] connecting the q–Schur algebras and the finite general linear groups, rules for computing decomposition matrices and the Ariki–Koike algebras and cyclotomic q–Schur algebras.

In addition, the book contains three appendices. The first of these provides a quick treatment of the assumed representation theory of finite dimensional algebras over a field. This appendix is intended as a primer for those new to the subject; although not necessary, previous exposure to the ordinary representation theory of finite groups would be advantageous. The second appendix contains tables of the crystallized decomposition matrices and adjustment matrices for $n \leq 10$ (see chapter 6), and the third appendix contains tables of the elementary divisors of the Gram matrices of the integral Specht modules for $n \leq 12$.

There are few new results in this book; however, many of the arguments either do not appear in the literature or, when they do, their simplicity is obscured by a more general framework. Throughout we have tried to attribute the main results to their rightful owners; we hope that we have succeeded — at the very least we have provided an extensive bibliography. Each chapter, except for the last, ends with a series of exercises (covering material we have not had time for in the text), together with some historical notes and references.

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