Assignment 2

Due: Monday May 31, 2010 at the start of the lecture.

Bonus marks may be awarded for particularly elegant solutions! To balance this, penalty marks may be deducted for particularly inefficient solutions or for horrendous mistakes.

Throughout, $A$ is a commutative ring with one.

1. Suppose that $A$ is a unique factorization domain and suppose that $a \in A$. Show that every principal ideal of $A$ has a minimal primary decomposition.
   [Hint: factorize!]

2. Suppose that $A$ is a ring and that $x$ is an indeterminate over $A$. If $I$ is an ideal of $A$ let $I[x]$ be the set of polynomials in $A[x]$ which have coefficients in $I$.
   a) Prove that $I[x] = I^e$, where extension is with respect to inclusion $A \hookrightarrow A[x]$.
   b) Show that $p[x]$ is a prime ideal of $A[x]$ whenever $p$ is a prime ideal of $A$.
   c) Suppose that $q$ is a $p$-primary ideal of $A$. Show that $\text{rad} \ q[x] = p[x]$ and that $q[x]$ is a primary ideal of $A[x]$.
   d) Suppose that $I = q_1 \cap \cdots \cap q_n$ is a minimal primary decomposition of the ideal $I$ of $A$. Show that $I[x] = q_1[x] \cap \cdots \cap q_n[x]$ a minimal primary decomposition of $I[x]$.
   e) Suppose that $I$ is an ideal of $A$ with a minimal primary decomposition. Show that $p[x]$ is an isolated prime ideal of $I[x]$ whenever $p$ is an isolated prime of $I$.

3. Suppose that $B$ is integrally closed over its subring $A$.
   a) Suppose that $x \in A$ and that $x$ is a unit in $B$. Show that $x$ is a unit in $A$.
   b) Let $\text{Jac}(A)$ and $\text{Jac}(B)$ be the Jacobson radicals of $A$ and $B$, respectively. Show that $\text{Jac}(A)$ is the contraction of $\text{Jac}(B)$ with respect to the embedding $A \hookrightarrow B$.

4. Suppose $A$ is a subring of $B$ and that $S = B - A$ is multiplicatively closed. Show that $B$ is integral over $A$.

5. Let $F$ be the field of fractions of $A$ and suppose that $f(x)$ is a monic polynomial in $A[x]$, where $x$ is an indeterminate over $F$. Prove that $f(x)$ is reducible in $A[x]$ if and only if $f(x)$ is reducible in $F[x]$.

6. The Krull dimension of a commutative ring $R$ is

$$\dim R = \sup \{ r \mid p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r \text{ is a chain of prime ideals in } R \}.$$ 

Suppose that $A \subset B$ with $B$ integral over $A$. Prove that $\dim A = \dim B$. 

Fourth year: Assignment 2

A.M. 14/5/2010