Hamiltonian spectral theory and the Maslov index

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Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes. I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Mitchell Curran   31/12/23

Authorship Attribution Statement

Chapter 2 contains the following unaltered manuscript:

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As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

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Contents

Chapter 1

Introduction

1.1 Motivation: Sturmian theory

A central theme underpinning this thesis is "a beautiful connection between analysis, dynamics and topology" [Bec20], which can be found in the 19th century Sturmian theory of oscillations for solutions to a second-order selfadjoint linear differential equation. As Arnol'd [Arn85] so articulately states, Sturm's theory "has a topological nature: it describes the rotation of a straight line in the phase space of the equation," and here I begin with a discussion of that topic. The following is not intended to be a comprehensive treatment, but rather to illustrate the main ideas in this thesis in the context of a simple example.

Consider the following eigenvalue problem for a scalar-valued Schrödinger operator equipped with Dirichlet boundary conditions on a compact interval,

$$y'' + q(x)y = \lambda y, \qquad y(0) = y(\ell) = 0, \tag{1.1}$$

where the *potential* q is a real-valued continuous function on $[0, \ell]$. Sturm-Liouville theory [AHP05, CL55] states that (1.1) has an infinite number of real, simple eigenvalues $\{\lambda_n\}_{n\geq 1}$ that form a monotone decreasing sequence such that $\lambda_n \to -\infty$ as $n \to +\infty$. Moreover, denoting by y_n the eigenfunction for the nth eigenvalue, the following *Sturm oscillation theorem* holds.

Theorem 1.1 ([**CL55, §8, Theorem 2.1**]). *The number of zeros of y_n on the open interval $(0, \ell)$ is equal to $n - 1$.*

In what follows, I will use a dynamical systems approach to prove Theorem 1.1 for the problem (1.1), taking inspiration from the exposition in [CL55, §8] (as well as [Arn92, §27]). The homotopy argument used in the proof here, which appears in much of the literature today, can be seen in as far back as the works of Bott [Bot56], Edwards [Edw64], Arnol'd [Arn67] and Duistermaat [Dui76]. Note that Theorem 1.1 in fact applies to the broader class of (selfadjoint) second-order eigenvalue problems

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda r(x)y,$$

with separated boundary conditions on $[0,\ell]$, where p', r, q are continuous and $p > 0, r > 0$. However, for the purposes of this section, I will restrict to the simpler case of (1.1).

Writing the differential equations in (1.1) as a first order system yields

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \lambda - q(x) & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{1.2}$$

In the following I will regard solutions to (1.2) as paths of radius vectors in the associated phase space $\mathbb{R}^2 \backslash \{0\}$; note the origin is excluded since it corresponds to the trivial solution. With this interpretation, a solution to (1.2) satisfies Dirichlet boundary conditions whenever it aligns with the vertical axis $\{(0,a)^\top : a \in \mathbb{R}\}$ at both $x = 0$ and $x = \ell$,

$$\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}, \qquad \begin{pmatrix} y(\ell) \\ y'(\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}. \tag{1.3}$$

Introducing *Prüfer co-ordinates* [Prü26], i.e. polar co-ordinates in the phase plane,

$$y = r \sin\theta, \qquad y' = r\cos\theta, \tag{1.4}$$

(where θ is the angle made by the solution vector with the positive y' axis, so that clockwise corresponds with the positive direction) alignment of the solution vector with the vertical axis occurs if and only if $\theta = n\pi$, $n \in \mathbb{Z}$. Moreover, for a nontrivial solution, the phase angle θ is well-defined and given by

$$\theta(x;\lambda) = \tan^{-1}\left(\frac{y(x;\lambda)}{y'(x;\lambda)}\right).$$

Let $(\varphi(x;\lambda), \varphi'(x;\lambda))^\top$ be a fundamental solution to (1.2) that satisfies the left boundary condition,

$$\begin{pmatrix} \varphi(0;\lambda) \\ \varphi'(0;\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for all } \lambda \in \mathbb{R}, \tag{1.5}$$

and let $\alpha(x;\lambda) = \tan^{-1}\left(\varphi(x;\lambda)/\varphi'(x;\lambda)\right)$ be the associated polar angle.

Suppose it is known *a priori* that $\varphi(x^*;\lambda^*) = 0$ for some pair $(\lambda^*, x^*) \in \mathbb{R} \times (0,\ell]$. The following calculation shows that $x \mapsto \alpha(x;\lambda^*)$ is increasing through $x = x^*$, and therefore that the path of radius vectors $x \mapsto (\varphi(x;\lambda^*), \varphi'(x;\lambda^*))^\top$ always intersects the vertical axis in a clockwise fashion:

$$\begin{aligned} \frac{d}{dx}\alpha(x;\lambda^*)\Big|_{\alpha=n\pi} &= \frac{d}{dx}\arctan\left(\frac{\varphi(x;\lambda^*)}{\varphi'(x;\lambda^*)}\right)\Big|_{\varphi=0}, \\ &= \frac{\varphi'(x;\lambda^*)^2 - \varphi(x;\lambda^*)\varphi''(x;\lambda^*)}{\varphi(x;\lambda^*)^2 + \varphi'(x;\lambda^*)^2}\Big|_{\varphi=0}, \\ &= 1 > 0. \end{aligned} \tag{1.6}$$

Note that this property holds for *any* solution to (1.2), i.e. regardless of the initial condition.

Now freezing $x = x^*$, I will show that the path $\lambda \mapsto (\varphi(x^*;\lambda), \varphi'(x^*;\lambda))$ always intersects the vertical axis in an *anti*clockwise fashion. Denoting $d/d\lambda$ with a dot, a similar calculation shows:

$$\begin{aligned} \frac{d}{d\lambda}\alpha(x^*;\lambda)\Big|_{\alpha=n\pi} &= \frac{d}{d\lambda}\arctan\left(\frac{\varphi(x^*;\lambda)}{\varphi'(x^*;\lambda)}\right)\Big|_{\varphi=0}, \\ &= \frac{\dot{\varphi}(x^*;\lambda)\varphi'(x^*;\lambda) - \varphi(x^*;\lambda)\dot{\varphi}'(x^*;\lambda)}{y(x^*;\lambda)^2 + y'(x^*;\lambda)^2}\Big|_{\varphi=0}, \\ &= \frac{\dot{\varphi}(x^*;\lambda)}{\varphi'(x^*;\lambda)}. \end{aligned} \tag{1.7}$$

I need to determine $\dot{\varphi}$. Differentiating (1.2) with respect to λ gives the following inhomogeneous equation for $(\dot{\varphi}, \dot{\varphi}')^\top$,

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\varphi}' \end{pmatrix}' - \begin{pmatrix} 0 & 1 \\ \lambda - q(x) & 0 \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{\varphi}' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}. \tag{1.8}$$

The variation of constants formula now yields an expression for $(\dot{\varphi}, \dot{\varphi}')^\top$, the first entry of which is given by

$$\dot{\varphi}(x^*;\lambda) = -\psi(x^*;\lambda)\int_0^{x^*} \varphi(x;\lambda)^2 dx. \tag{1.9}$$

Here, ψ is a second fundamental solution to (1.2) satisfying

$$\begin{pmatrix} \psi(0;\lambda) \\ \psi'(0;\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and I have used that the Wronskian $\psi(x;\lambda)\varphi'(x;\lambda) - \varphi(x;\lambda)\psi(x;\lambda) = 1$ for all $x \in [a,b]$, as well as the fact that $(\dot{\varphi}(0;\lambda), \dot{\varphi}'(0;\lambda))^\top = (0,0)^\top$ for all λ (as seen by differentiating (1.5) with respect to λ). Substituting (1.9) into the last line of (1.7) and again using that the Wronskian equals one yields

$$\frac{d}{d\lambda}\alpha(x^*;\lambda)\Big|_{\alpha=n\pi} = -\psi(x^*;\lambda)^2\int_0^{x^*} \varphi(x;\lambda)^2 dx < 0, \tag{1.10}$$

as required. Just as for (1.6), observe that this property holds for *any* solution to (1.2).

Before continuing with the proof of Theorem 1.1 for (1.1), let me point out that solutions to (1.2) in fact *always* rotate in an anticlockwise fashion for increasing λ (i.e. not just in a neighbourhood of the vertical axis). This fact is a consequence of the following Sturm Comparison theorem.

Theorem 1.2 (**[CL55, §8, Theorem 1.2]**). *Let p_i' and g_i be piecewise continuous on $[0,\ell]$, and suppose*

$$0 < p_2(t) \le p_1(t), \qquad g_2(t) \ge g_1(t) \tag{1.11}$$

on $[0,\ell]$. Suppose $y = \phi_i$ is a solution to

$$(p_i y')' + g_i y = 0,$$

with polar angle in the phase plane $\theta_i = \tan^{-1}\left(\frac{\phi_i}{p\phi_i'}\right)$*, where* $\theta_2(0) \geq \theta_1(0)$*. Then*

$$\theta_2(t) \geq \theta_1(t), \qquad t \in [0, \ell].$$

Moreover, if $g_2(t) > g_1(t)$ *on* $(0, \ell)$*, then*

$$\theta_2(t) > \theta_1(t), \qquad t \in (0, \ell].$$

Indeed, writing the differential equation in (1.1) as

$$y'' + (q(x) - \lambda)y = 0,$$

the potential of which is strictly decreasing in λ, it follows from Theorem 1.2 that for any solution $(y, y')^\top$ to (1.2) with polar angle θ and initial condition independent of λ, the mapping $\lambda \mapsto \theta(c; \lambda)$ (for any $c \in (0, \ell]$) is strictly decreasing. This *global* monotonicity of the winding in λ will be an important idea, namely in Section 1.4, where it is a recurring theme in many of the works reviewed therein.

Returning to the proof of Theorem 1.1, observe the following behaviour of solutions to (1.2) for small x and large λ. First, if τ is small enough, then the radius vector $(\varphi(x;\lambda), \varphi'(x;\lambda))^\top$, initialised at $(0,1)^\top$, will remain in a small neighbourhood of the vertical axis for all $x \in [0, \tau]$, independent of $\lambda \in \mathbb{R}$. It will therefore not complete the minimum half revolution required as x varies over $[0, \tau]$ to satisfy the Dirichlet condition at $x = \tau$. Second, if $\lambda = M \gg 1$ is large enough, then (1.2) will be close to an autonomous system. Roughly speaking, this is because the nonautonomous part of (1.2) – $q(x)$ – will be negligible in comparison to $\lambda = M$. More precisely, the change of variables $z = \sqrt{\lambda}x$ yields an equivalent system to (1.2), the coefficient matrix of which is given by $\begin{pmatrix} 0 & 1 \\ 1 - q(x)/\lambda & 0 \end{pmatrix}$. It can be shown [Was76] that solutions of this equivalent system are asymptotic to solutions of the constant coefficient asymptotic system as $\lambda \to \infty$. It follows that $(\varphi(x; M), \varphi'(x; M))^\top$ will again remain in a small neighbourhood of its initial condition on the vertical axis for all $x \in [0, \ell]$, and will therefore not complete the half revolution required to satisfy the right boundary condition at $x = \ell$. (Recall that the solution vector may only intersect the vertical axis in a clockwise fashion as x increases.)

Now suppose that $\varphi(\ell; \tilde{\lambda}) = 0$, and consider the solid rectangle $R = [\tilde{\lambda}, M] \times [\tau, \ell]$ in the λx-plane, where τ is small and M is large; see Fig. 1.1[1]. On the boundary ∂R of R, note the points (λ^*, x^*) where $\varphi(x^*; \lambda^*) = 0$, which correspond to the *crossings* of $(\varphi, \varphi')^\top$ with the vertical axis. From the arguments above, there are no such crossings on the bottom and right sides of ∂R, where $x = \tau$ and $\lambda = M$ respectively. To each of the crossings on the left and top sides of ∂R, a signature may be assigned based on the local direction of rotation through the vertical axis (i.e. the sign of the derivative of α with respect to the relevant varying parameter). Namely, crossings on the left side of ∂R (x increasing with $\lambda = 0$ fixed) are negative, while crossings on the top side (λ increasing with $x = \ell$ fixed) are positive.

[1] since λ is a real-valued spectral parameter and thus lives on the horizontal axis in the complex plane, it is natural to put λ on the horizontal axis

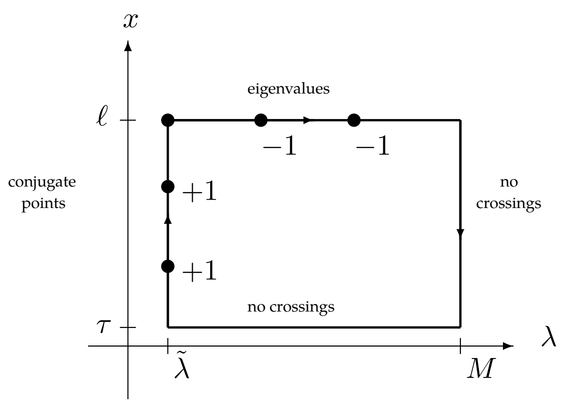


Figure 1.1: The boundary of the rectangle R in the λx-plane. Solid black circles indicate points where the solution vector intersects the vertical axis. In this depiction, $\tilde{\lambda} = \lambda_3$.

To conclude Theorem 1.1, I now use the fact that the sum total of the signatures of the crossings on ∂R is equal to zero. To see this, note that the map $F : R \to \mathbb{R}^2\backslash\{0\}$ taking $(x, \lambda) \in R$ into the radius vector $(\varphi(x;\lambda), \varphi'(x;\lambda))^\top$ is continuous. Because R is contractible (say, to the point $(\tilde{\lambda}, \tau)$), it follows that the map $\tilde{F} : \partial R \to \mathbb{R}^2\backslash\{0\}$ taking $(x, \lambda) \in \partial R$ into the radius vector $(\varphi(x;\lambda), \varphi'(x;\lambda))^\top$ can be continuously deformed into the constant mapping $\hat{F} : \partial R \to \mathbb{R}^2\backslash\{0\}, (x, \lambda) \mapsto (\varphi(\tau; \tilde{\lambda},), \varphi'(\tau; \tilde{\lambda},))^\top$. The latter clearly has no intersections with the vertical axis, and it follows that the image of $\tilde{F}$ also has net zero signed intersections with the vertical axis. Here I am using the fact that the *winding number* of a loop in $\mathbb{R}^2\backslash\{0\}$[2] is a topological invariant: two loops that are homotopic will have the same winding number.

With the sum total of the signatures being zero, it follows that the positive crossings on the top side of ∂R and the negative crossings on the left side of ∂R are in one-to-one correspondence. Now notice that the crossings along the top correspond to eigenvalues of (1.2), since for each crossing there corresponds a solution to (1.2) that satisfies Dirichlet boundary conditions at $x = 0$ and $x = \ell$. Furthermore, each crossing along the left – a point of verticality of the solution vector $(\varphi(\cdot;\tilde{\lambda}), \varphi'(\cdot;\tilde{\lambda}))^\top$ – corresponds to a zero of $\varphi(\cdot;\tilde{\lambda})$, which is the eigenfunction for the eigenvalue $\tilde{\lambda}$. If $\tilde{\lambda} = \lambda_n$, so that there are $n-1$ eigenvalues bigger than λ_n, this shows that $\varphi(\cdot;\lambda_n)$ will have $n-1$ zeros on $(0, \ell)$, proving Theorem 1.1.

The proof given above thus offers a topological interpretation of Sturm's oscillation theorem: it is a consequence of the homotopy invariance of the winding number of a loop of radius vectors in $\mathbb{R}^2\backslash\{0\}$. It is from this perspective that Sturm's theorem may be generalised to Hamiltonian systems. The topological invariant that makes this possible is an intersection index known as the *Maslov index*, which is a signed count of the intersections of a path in the space of *Lagrangian planes* (cf. Section 1.3.3) with a fixed codimension-one set. In the example above, the path of Lagrangian planes is the path of radius vectors, the fixed codimension-one set is the vertical subspace $\{(0, a)^\top : a \in \mathbb{R}\}$, and, loosely speaking, the Maslov index is the winding number of the path of radius vectors, i.e. a signed count of the intersections of the path with the vertical subspace.

[2] i.e. the net number of (half) revolutions made by the loop, computed by summing the signed intersections with the vertical axis

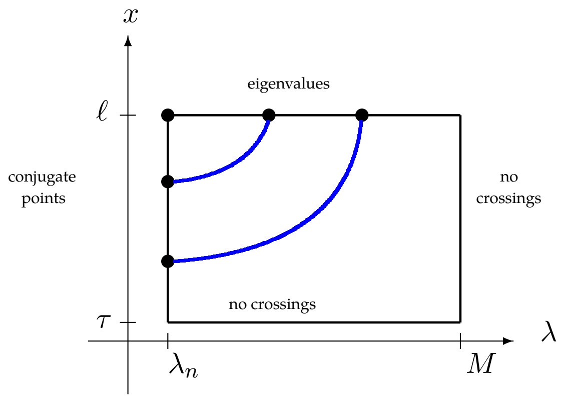


Figure 1.2: Depiction of the eigenvalue curves $x(\lambda)$ satisfying $\varphi(x(\lambda);\lambda) = 0$ contained in the rectangle $[\lambda_n, M] \times [\tau, \ell]$ in the λx-plane, where φ solves (1.2) and satisfies $\varphi(0;\lambda) = 0$.

An alternative way to prove Theorem 1.1 involves proving monotonicity of the *eigenvalue curves*. These are curves in the λx-plane that represent the evolution of the eigenvalues λ of the boundary value problem (1.1) restricted to the subdomain $[0, x]$ (i.e. (1.1) with the right boundary condition replaced by $y(x) = 0$). Thus the eigenvalue curves are the locus of zeros of φ (recall that $\varphi(0;\lambda) = 0$) in the λx-plane, implicitly defined via $\varphi(x;\lambda) = 0$, or, equivalently, via $\alpha(x;\lambda) = 0 \mod \pi$. I showed earlier that $\partial_x \alpha(x;\lambda)|_{\alpha=n\pi} > 0$ and $\partial_x \alpha(x;\lambda)|_{\alpha=n\pi} < 0$; it follows from the implicit function theorem that the eigenvalue curves can locally be written as $x = x(\lambda)$, and moreover

$$\partial_\lambda \alpha(x;\lambda) + \frac{dx}{d\lambda}\partial_x \alpha(x;\lambda) = 0 \qquad \Longrightarrow \qquad \frac{dx}{d\lambda} = \frac{-\partial_\lambda \alpha(x;\lambda)}{\partial_x \alpha(x;\lambda)}\bigg|_{\alpha=n\pi} > 0.$$

Hence the eigenvalue curves are monotone increasing. With this property, as well as the fact that there can be no crossings on the right side of ∂R, the crossings on the left and top sides of ∂R are readily seen to be in one-to-one correspondence: after entering on the left, the eigenvalue curves must leave along the top. See Fig. 1.2.

The key feature in both proofs of Theorem 1.1 that affords the equivalence of the number of crossings on the top and on the left sides of ∂R is the monotonicity of the winding of the radius vector in phase space, near the vertical axis, with respect to each of the parameters x and λ. In the generalisation of Sturm's oscillation theorem to Hamiltonian systems, this monotonicity property does not hold in general: signatures of the crossings on each side of ∂R may offset each other. (In the general setting, crossings correspond to satisfaction of the boundary conditions of a solution to the governing differential equation, as they do above.) Nonetheless, the homotopy argument still applies, so that the sum total of the signatures of the crossings on ∂R, in a suitable two-parameter space, is zero. By analysing the signatures on the left side of ∂R, it is thus possible to determine the existence of eigenvalues (i.e. crossings along the top) for more general Hamiltonian differential operators. Doing so for a particular class of such operators is one of the main goals of this thesis. Regarding the eigenvalue curves, these will no longer be monotone in the general case, with points of both horizontal and vertical tangencies possible. Regardless, analysing the local behaviour of these curves can still be fruitful, as will be shown in Chapter 2.

1.2 Overview of the thesis

This thesis examines the real spectral theory of Hamiltonian differential operators with the *canonical symplectic structure* (see Section 1.5) using the Maslov index. The goal will be to detect real positive eigenvalues of

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \tag{1.12}$$

where $L_\pm$ are selfadjoint differential operators on an interval $\Omega \subseteq \mathbb{R}$. Operators of the form of (1.12) arise, for example, when linearising about standing waves in nonlinear Schrödinger (NLS) type equations. The analysis focuses on two cases. In Chapter 2, $L_\pm$ are arbitrary Schrödinger operators on a compact interval equipped with Dirichlet boundary conditions. In Chapter 3, $L_\pm$ are selfadjoint fourth-order operators on the line, obtained from linearising a fourth-order NLS equation about a soliton solution. In both cases, the eigenvalue equations have a Hamiltonian structure, and induce flows on the *Lagrangian Grassmannian*. For paths in this space, the Maslov index can be defined. Just as in the proof of Theorem 1.1, eigenvalues are encoded as intersections, or *crossings*, of the path with a codimension-one set that encodes the boundary conditions (including those "at infinity" for the fourth-order problem). By exploiting homotopy invariance, a lower bound for the number of positive real eigenvalues of N can be deduced. For explanations of these terms, see Sections 1.3.2, 1.3.3 and 1.4.

The remainder of the current chapter contains preliminary material (Section 1.3) from functional analysis, topology, symplectic geometry and stability theory that will be used throughout. The work done in this thesis will then be contextualised in Sections 1.4 and 1.5 with reviews of two sections of the literature: that pertaining to the Maslov index in dynamical systems, and that pertaining to the spectral theory of operators of the form of (1.12).

Chapters 2 and 3 are entirely self-contained; I apologise in advance for any repetitions in material the reader may encounter across those chapters. The lower bound derived in each problem contains a "correction" term corresponding to a *non-regular* crossing, for which an associated quadratic form, the *crossing form*, is degenerate. Such a crossing represents an atypical contribution to the Maslov index.

The compactness of the domain in Chapter 2 facilitates the use of the eigenvalue curves; analysing their behaviour locally provides a geometric means of handling the non-regularity and determining the correction term. Applications of the theory are made to the spectral stability of standing waves in the classical second-order NLS equation, where the nonlinearity is allowed to depend explicitly on the spatial variable.

In Chapter 3, the approach will instead involve *higher-order crossing forms* to compute the correction term. An interesting feature of the problem is the occurrence of non-regular crossings of varying degrees of degeneracy. In particular, both scenarios where the crossing form is either identically zero, or degenerate with nonzero rank, are encountered. The Maslov index is locally computed in these cases via the *partial signatures* of the higher-order crossing forms. In both Chapters 2 and 3, analysing the lower bound leads to a proof of the *Jones-Grillakis* instability theorem and *Vakhitov-Kolokolov* stability criterion.

Finally, in Chapter 4, some additional results, observations and future directions of study for each of the problems in Chapter 2 and Chapter 3 will be discussed.

1.3 Preliminary material

1.3.1 Functional Analysis

The discussion in this section follows that of [HN01, §10] and [KP13, §2], with additional notes from [Eva10, Bre11, Kat80, TL80]. Familiarity with Hilbert spaces is assumed, as well as basic notions such as closedness, boundedness, and compactness of linear operators. I will always work over the field $\mathbb{R}$.

Throughout the thesis I will often work with certain function spaces known as *Sobolev spaces*; these provide a useful setting in which to study differential operators as they facilitate the use of functional analytic methods. In order to build these spaces, I need to define the notion of a "weak" derivative.

Let $\Omega \subseteq \mathbb{R}$ be an open interval (potentially unbounded) and $u : \Omega \to \mathbb{R}$. The L^p norm of u is given by

$$\|u\|_p := \begin{cases} \left(\int_\Omega |u(x)|^p dx \right)^{1/p} & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| & p = \infty. \end{cases} \tag{1.13}$$

The space $L^p(\mathbb{R})$ is then given by the set of functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_p < \infty$; $L^p(\mathbb{R})$ is a Banach space for all $p \geq 1$. In this thesis I will mainly be interested in the space $L^2(\Omega)$, which is a Hilbert space with the inner product

$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega uv \, dx.$$

I note here that functions in $L^2(\Omega)$ satisfy the Cauchy-Schwarz inequality,

$$\langle u, v \rangle_{L^2(\Omega)} \leq \|u\|_2 \|v\|_2.$$

A *test function* $\varphi : \Omega \to \mathbb{R}$ has continuous derivatives of all orders and compact support. (In the case that Ω is a bounded interval, $C_c^\infty(\Omega)$ is the space of smooth functions that vanish outside a closed interval contained strictly inside Ω.) The set of test functions is denoted by $C_c^\infty(\Omega)$.

A function $u \in L^2(\Omega)$ has a weak derivative $v \in L^2(\Omega)$ if

$$\int_\Omega v\varphi \, dx = -\int_\Omega u\varphi' \, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In this case, we write $u' = v$. More generally, the kth weak derivative of $u \in L^2(\Omega)$ is a function $u \in L^2(\Omega)$ such that

$$\int_\Omega v\varphi \, dx = (-1)^k \int_\Omega u\varphi^{(k)} \, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega),$$

and again we write $u^{(k)} = v$. The Sobolev space $H^k(\Omega)$ consists of the functions with k weak derivatives in $L^2(\Omega)$,

$$H^k(\Omega) = \left\{ u \in L^2(\Omega) : u, u', \dots, u^{(k)} \in L^2(\Omega) \right\},$$

equipped with the inner product and norm:

$$\begin{aligned} \langle u, v \rangle_{H^k} &= \int_\Omega uv + u'v' + \cdots + u^{(k)} v^{(k)} \, dx, \\ \|u\|_{H^k} &= \left(\int_\Omega u^2 + (u')^2 + \cdots + (u^{(k)})^2 \, dx \right)^{1/2}. \end{aligned}$$

Note that $u \in H^k(\Omega)$ if and only if $\|u\|_{H^k} < \infty$. Also, $H^0(\Omega) = L^2(\Omega)$. Technically speaking, elements of $H^k(\Omega)$ are equivalence classes of functions with k square-integrable weak derivatives that are equal almost everywhere. Thus any $u \in H^k(\Omega)$ is only defined up to a set of measure zero. The spaces $H^k(\Omega)$ have the very desirable property of being Hilbert spaces, making them a convenient choice of function space to work with.

The Sobolev spaces $H^k(\Omega)$ satisfy a number of embeddings, depending on the boundedness of Ω. If $\Omega = \mathbb{R}$, then $C_c^\infty(\mathbb{R})$ is densely embedded in $H^k(\mathbb{R})$ for all nonnegative integers k. Since $H^m(\mathbb{R}) \subset H^k(\mathbb{R})$ for all $m > k$ by definition, it follows that $H^m(\mathbb{R})$ is dense in $H^k(\mathbb{R})$ for all $m > k$. If $\Omega = (0, \ell)$ is bounded, then $C^\infty([0, \ell])$ is dense in $H^k(0, \ell)$ for each nonnegative integer k. Moreover, the embeddings $H^m(0, \ell) \subset H^k(0, \ell)$ are compact for $m > k$ (k, m nonnegative integers), meaning that every bounded sequence in $H^k(0, \ell)$ has a convergent subsequence in $H^m(0, \ell)$. Finally, $H^k(0, \ell) \subset C^{k-1}([0, \ell])$ for $k \geq 1$; thus, given enough weak derivatives, a function on a compact interval will be differentiable in the classical sense.

The space $H_0^k(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in $H^k(\Omega)$,

$$H_0^k(\Omega) \coloneqq \overline{C_c^\infty(\Omega)} \subset H^k(\Omega).$$

If $\Omega = (0, \ell)$ then $H^1(0, \ell) \subset C([0, \ell])$, so for any $u \in H^1(0, \ell)$ the pointwise values $u(0)$ and $u(\ell)$ are well-defined. In this case, $u \in H_0^1(0, \ell)$ if and only if $u(0) = u(\ell) = 0$ [Bre11, Theorem 8.12]. For a second-order differential operator on $(0, \ell)$ equipped with Dirichlet boundary conditions, a convenient choice of domain is therefore given by $H^2(0, \ell) \cap H_0^1(0, \ell)$.

Functions in $H_0^1(0, \ell)$ enjoy the following *Poincaré inequality* [HN01, Theorem 12.77]: for all $u \in H_0^1(\Omega)$ there is a constant C such that

$$\|u\|_2 \leq C \|u'\|_2.$$

The *adjoint* $L^* : \mathrm{dom}(L^*) \subset H \to H$ of L is the operator with domain

$$\mathrm{dom}(L^*) = \{ y \in H : \text{there is a } z \in H \text{ with } \langle Lx, y \rangle = \langle x, z \rangle \text{ for all } x \in \mathrm{dom}(L) \}.$$

If $y \in \mathrm{dom}(L^*)$, then we define $L^* y = z$, where z is the unique element such that $\langle Lx, y \rangle = \langle x, z \rangle$ for all $x \in \mathrm{dom}(L)$. L is called *selfadjoint* if $L = L^*$, i.e. $\mathrm{dom}(L) = \mathrm{dom}(L^*)$ and $Lu = L^* u$ for all $u \in \mathrm{dom}(L)$.

For the rest of this section, H will be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and L will be a closed, densely-defined unbounded linear operator $L : \mathrm{dom}(L) \subset H \to H$ acting in H.

The operator L is called *Fredholm* if its range $\mathrm{Ran}(L)$ is closed and has finite codimension, and its kernel $\ker(L)$ is finite-dimensional. Its *Fredholm index* is the integer $\dim \ker(L) - \mathrm{codim}\, \mathrm{Ran}(L)$. Fredholm operators satisfy the following *Fredholm alternative*. If $f \in H$, then the inhomogeneous equation $Lu = f$ has a unique solution $u \in \mathrm{dom}(L)$ if and only if $f \in \ker(L^*)^{\perp}$. This solvability condition can be seen by applying $\langle \cdot, v \rangle$ to both sides of $Lu = f$ for any $v \in \ker(L^*)$.

The *resolvent set* of L is the set

$$\varrho(L) := \{\lambda \in \mathbb{C} : L - \lambda I \text{ is invertible with bounded inverse}\}, \tag{1.14}$$

and the *spectrum* is given by $\mathrm{Spec}(L) = \mathbb{C} \backslash \varrho(L)$.

The operator $L - \lambda I$ can fail to be invertible with bounded inverse in a number of ways, leading to different types of spectrum. An *eigenvalue* is a $\lambda \in \mathbb{C}$ for which $L - \lambda I$ is not injective. In this case, $Lu = \lambda u$ for some $u \in \mathrm{dom}(L) \backslash \{0\}$, and u is called an *eigenfunction*. Note that λ need not be isolated[3]. If λ is an eigenvalue, its *geometric multiplicity* is $m_g(\lambda) := \dim \ker(L - \lambda I)$, and its *algebraic multiplicity* is $m_a(\lambda) := \dim \ker(L - \lambda I)^{\alpha}$, where α is the *ascent* of L, i.e. the smallest nonnegative integer α such that $\ker(L - \lambda I)^{\alpha} = \ker(L - \lambda I)^{\alpha+1}$. If no such integer exists, then, following [TL80], one sets $\alpha = \infty$. Thus it is possible that the algebraic multiplicity is infinite. If α is finite, the algebraic multiplicity of λ is the dimension of the largest subspace $Y_\lambda \subset H$ which is invariant under the action of L, and such that the restriction $L|_{Y_\lambda}$ satisfies $\mathrm{Spec}(L|_{Y_\lambda}) = \lambda$. Such an invariant subspace is called the *generalised eigenspace* or *generalised kernel* of λ, and denoted $\mathrm{gker}(L - \lambda I) := Y_\lambda$. Note that we always have $m_a(\lambda) \geq m_g(\lambda)$. If $m_a(\lambda) > m_g(\lambda)$ then λ is called *deficient*; if $m_a(\lambda) = m_g(\lambda)$ then λ is called *semisimple*; and if $m_a(\lambda) = 1$ then λ is called *simple*. With these definitions, the *discrete spectrum* $\mathrm{Spec}_{\mathrm{d}}(L)$ is the set of isolated eigenvalues of L with finite algebraic multiplicity.

Suppose $\lambda \in \mathrm{Spec}_{\mathrm{d}}(L)$. Then $L - \lambda I$ is Fredholm, and moreover if P is the orthogonal projection onto $\mathrm{gker}(L - \lambda I)$, then $P(L - \lambda I)P$ acts in a finite-dimensional space, and results from linear algebra apply. In particular, if λ is deficient then the elements of $\mathrm{gker}(L - \lambda I)$ can be organised into Jordan chains. That is, associated with each linearly independent eigenvector u_0 of λ is a family of *generalised eigenvectors* $\{u_i\}_{i=1}^n$ satisfying

$$(L - \lambda I)u_i = u_{i-1}, \qquad u_i \in \mathrm{dom}(L) \backslash \{0\}, \qquad i = 1, \ldots, n.$$

The set $\{u_i\}_{i=0}^n$ is a *Jordan chain*, and $\mathrm{gker}(L - \lambda I)$ is spanned by the collection of all Jordan chains associated with λ. If the geometric multiplicity of λ is equal to one, the algebraic multiplicity is equal to the ascent of L, $m_a(\lambda) = \alpha$, and corresponds to the length of the associated Jordan chain.

Following [KP13, Definition 2.2.3], I will define the *essential spectrum* $\mathrm{Spec}_{\mathrm{ess}}(L)$ to be the set of $\lambda \in \mathbb{C}$ such that either (a) the operator $L - \lambda I$ is not Fredholm, or (b) the operator $L - \lambda I$ is Fredholm but has nonzero Fredholm index. It follows from the arguments in [Kat80, Chapter

[3]that is, it may be that $(B(\lambda, \varepsilon) \backslash \{\lambda\}) \cap \mathrm{Spec}(L) \neq \{0\}$ for every open ball $B(\lambda, \varepsilon) \subset \mathbb{C}$ with $\varepsilon > 0$.

V, §5.6] that

$$\mathrm{Spec_d}(L) = \mathrm{Spec}(L) \setminus \mathrm{Spec_{ess}}(L),$$

provided the following conditions are satisfied:

(i) there are no isolated eigenvalues with infinite algebraic multiplicity;

(ii) the resolvent set $\varrho(L)$ is nonempty;

(iii) there are no open subsets $\Delta \subset \mathbb{C}$ in which both $L - \lambda I$ is Fredholm with index zero, and $L - \lambda I$ is not injective for all $\lambda \in \Delta$.

Equivalently, (cf. [Kat80, Chapter V, §5.6]) condition (ii) states that the points $\lambda \in \mathbb{C}$ where $L - \lambda I$ is Fredholm with index zero but $\dim\ker(L - \lambda I) = \operatorname{codim}\operatorname{Ran}(L - \lambda I) \neq 0$ are isolated. In general, it is possible that the complement of the essential spectrum (i.e. the set of $\lambda \in \mathbb{C}$ where $L - \lambda I$ is Fredholm with index zero) contains open subsets of $\mathbb{C}$ as described by (ii). Conditions (i)–(iii) will always hold in this thesis.

If $\lambda \in \varrho(L)$, the *resolvent* of L is the operator

$$R(\lambda, L) = (L - \lambda I)^{-1}.$$

Suppose there exists $\lambda \in \varrho(L)$ such that $R(\lambda, L)$ is a compact operator. Then $R(\lambda, L)$ is compact for all $\lambda \in \varrho(L)$, and in this case L is said to have *compact resolvent*. An operator L with nonempty resolvent set has compact resolvent if and only if its domain $\mathrm{dom}(L)$ is compactly embedded in H, i.e. the inclusion map $J : \mathrm{dom}(L) \to H$ is compact. If L has compact resolvent then $\mathrm{Spec}(L) = \mathrm{Spec_d}(L)$, and moreover $\mathrm{Spec_d}(L)$ contains at most countably many isolated eigenvalues λ_j, with the only possible accumulation point satisfying $|\lambda_j| \to \infty$ as $j \to \infty$.

Let me now apply some of this machinery to the operators studied in this thesis. I am going to view (1.1) as the eigenvalue problem for the closed, densely defined, unbounded Schrödinger operator

$$L = \partial_{xx} + g(x), \qquad L : \mathrm{dom}(L) := H^2(0,\ell) \cap H^1_0(0,\ell) \subset L^2(0,\ell) \to L^2(0,\ell), \tag{1.15}$$

where $g \in C([0,\ell], \mathbb{R})$. Note that I used that $u \in H^1_0(0,\ell)$ if and only if $u(0) = u(\ell) = 0$. L is selfadjoint; to show this I first need to find the adjoint operator L^*. By definition, if $v \in \mathrm{dom}(L^*)$ then there is a unique element $z \in L^2(0,\ell)$ such that

$$\int_0^\ell (u'' + gu)v\,dx = \int_0^\ell uz\,dx \tag{1.16}$$

for all $u \in \mathrm{dom}(L)$, and $L^*v := z$. Now (1.16) is true, in particular, for all $u \in C_c^\infty(0,\ell)$ (since $C_c^\infty(0,\ell) \subset \mathrm{dom}(L)$), and rearranging yields

$$\int_0^\ell u''v\,dx = \int_0^\ell u(z - gv)\,dx \tag{1.17}$$

for all $u \in C_c^\infty(0,\ell)$. Since $z - gv \in L^2(\Omega)$, from the definition of $H^2(0,\ell)$ we have $v \in H^2(0,\ell)$ and $v'' = z - gv$. Thus $\mathrm{dom}(L^*) \subset H^2(0,\ell)$, and $L^*v = z = v'' + gv$.

Since $v \in H^2(0,\ell)$, integrating by parts twice yields

$$\int_0^\ell (u'' + gu)v\,dx = \int_0^\ell u(v'' + gv)\,dx + u'(\ell)v(\ell) - u'(0)v(0), \tag{1.18}$$

where I used that $u(0) = u(\ell) = 0$. Equating (1.18) with (1.16), I find that $v \in \mathrm{dom}(L^*)$ if and only if $v(\ell) = v(0) = 0$. Therefore $\mathrm{dom}(L) = \mathrm{dom}(L^*)$ and $Lu = L^*u$ for all $u \in \mathrm{dom}(L)$, so that L is selfadjoint.

It follows that $\mathrm{Spec}(L) \subset \mathbb{R}$, and the eigenvalues of L are semisimple [Kat80, §V.3.5]. Furthermore, since $H^2(0,\ell)$ is compactly embedded in $L^2(0,\ell)$, it follows that $\mathrm{dom}(L) = H^2(0,\ell) \cap H_0^1(0,\ell)$ is compactly embedded in $L^2(0,\ell)$, and L has compact resolvent. Thus $\mathrm{Spec}(L) = \mathrm{Spec}_{\mathrm{d}}(L) \subset \mathbb{R}$, and the only possible accumulation points are at $\pm\infty$. In fact, from Sturm-Liouville theory (see the discussion following (1.1)), the spectrum of L is bounded from above and $-\infty$ is the only possible accumulation point. In this case the number of positive eigenvalues of L is a well-defined distinguished quantity. Note that if the second derivative term is negative (i.e. ∂_{xx} is negative in (1.15) as will be the case in Chapter 2), the spectrum is reflected about zero: L is bounded from below, and the spectrum accumulates at $+\infty$. In this case, the number of negative eigenvalues of L is well-defined.

In Chapter 2 the operators $N, L_\pm$ in (1.12) are equipped with the domain

$$\begin{aligned} \mathrm{dom}(N) &:= \big(H^2(0,\ell) \cap H_0^1(0,\ell)\big) \times \big(H^2(0,\ell) \cap H_0^1(0,\ell)\big) \subset L^2(0,\ell) \times L^2(0,\ell), \\ \mathrm{dom}(L_\pm) &:= H^2(0,\ell) \cap H_0^1(0,\ell) \subset L^2(0,\ell), \end{aligned} \tag{1.19}$$

while in Chapter 3,

$$\mathrm{dom}(N) := H^4(\mathbb{R}) \times H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \times L^4(\mathbb{R}), \qquad \mathrm{dom}(L_\pm) := H^4(\mathbb{R}) \subset L^4(\mathbb{R}). \tag{1.20}$$

With these choices, the Schrödinger operators $L_\pm$ in Chapter 2 and the fourth order operators $L_\pm$ in Chapter 3 are selfadjoint. In both cases N is not selfadjoint, and its spectrum may be complex in general. By the same reasoning as for the operator L of (1.15), the operators $N, L_\pm$ under (1.19) have compact resolvent and therefore only have discrete spectrum, $\mathrm{Spec}(N) = \mathrm{Spec}_{\mathrm{d}}(N)$. On the other hand, $N, L_\pm$ under (1.20) will have essential spectrum. Our analysis of the positive real eigenvalues in that case will therefore require certain assumptions on the parameters appearing in the operators $N, L_\pm$, which ensure their essential spectra do not intersect the non-negative real axis.

1.3.2 Topology

Here I follow the discussions in [Hat02, Mun00]. Let X be a topological space. In what follows, the unit interval $[0,1]$ may be replaced by any compact interval. A *path* in X is a continuous map $f : [0,1] \to X$. A (fixed-endpoint) *homotopy* between two paths f and g, which have the same initial and final points, $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, is a continuous map

$F : [0, 1] \times [0, 1] \to X$ such that

$$\begin{aligned} F(t, 0) &= f(t), \qquad \text{and} \qquad F(t, 1) = g(t), \\ F(0, s) &= x_0, \qquad \text{and} \qquad F(1, s) = x_1, \end{aligned} \tag{1.21}$$

for each $s \in [0, 1]$ and each $t \in [0, 1]$. In this case f and g are said to be *homotopic* (with fixed endpoints). The relation of homotopy (on paths with fixed endpoints) is an equivalence relation; the equivalence class of a path f is denoted by $[f]$ and called the *homotopy class* of f.

The product of two paths is defined to be their concatenation. More precisely, suppose $f, g : [0, 1] \to X$ are two paths such that the final point of f is the initial point of g, $f(1) = g(0)$. The product $f \cdot g$ is defined as

$$f \cdot g(t) = \begin{cases} f(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g(2t - 1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \tag{1.22}$$

That is, $f \cdot g$ is the path starting at $f(0)$ and ending at $g(1)$, which first traverses f and then g. The product operation (1.22) induces a product operation on homotopy classes via

$$[f] \cdot [g] := [f \cdot g], \tag{1.23}$$

which is well-defined provided $f(1) = g(0)$. To see this, note that if F is a homotopy of two paths $f, f' \in [f]$, and G is a homotopy of two paths $g, g' \in [g]$, where $f(1) = g(0)$, then

$$H(t, s) = \begin{cases} F(2t, s) & \text{for } t \in [0, \frac{1}{2}] \\ G(2t - 1, s) & \text{for } t \in [\frac{1}{2}, 1], \end{cases}$$

is well-defined and provides a homotopy between $f \cdot g$ and $f' \cdot g'$.

Let me now focus on *loops* i.e. closed paths $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$. Here x_0 is called the base point. The set of homotopy classes of loops based at x_0 forms a group under the product operation in (1.23), and is called the *fundamental group* of X at the base point x_0, denoted $\pi_1(X, x_0)$. The identity element in this group is the set of loops that are homotopic to a fixed point.], while the inverse $\bar{f}$ of a path f is its reverse, $\bar{f}(t) := f(1-t)$. If X is path-connected, then for any two base points x_0 and x_1 the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. In this case it is reasonable to suppress dependence on x_0 and write $\pi_1(X)$.

As an aside, note that if f and g are not loops, then the product operation $[f] \cdot [g]$ of (1.23) is not defined for every pair of classes $[f], [g]$, but only for those for which $f(1) = g(0)$. In this case the operation $[f] \cdot [g]$ is a *partial* operation, and the set of homotopy classes forms a *groupoid*, called the *fundamental groupoid* and again denoted $\pi_1(X)$.

The fundamental group $\pi_1(X)$ contains topological information about the space X. For example, if $\pi_1(X)$ is trivial (contains only the identity element), then the space is *simply connected*, meaning that all loops are homotopic to a fixed point. In contrast, a nontrivial fundamental group indicates the presence of nontrivial loops and the existence of holes or tunnels in X. Perhaps the simplest example of a nontrivial fundamental group is that of the circle S^1, given by

$$\pi_1(S^1) \approx \mathbb{Z}. \tag{1.24}$$

Thus, to each homotopy class of loops in S^1 there corresponds an integer. Noting that two loops in S^1 are homotopic if and only if they have the same winding number (i.e. they complete the same net number of revolutions, in the same direction, around S^1), it follows that the integer assigned to each homotopy class in (1.24) can be interpreted as the winding number of loops in that class. More generally, for any space X with $\pi_1(X) = \mathbb{Z}$, a notion of winding also exists for loops in X.

1.3.3 Symplectic geometry

Here I follow the discussions in [Gos01, Dui04]. Denote by J the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{1.25}$$

where $I = I_{n\times n}$ and $0_{n\times n}$ are the $n \times n$ identity and zero matrices respectively. Note that J satisfies $J = J^\top = -J^{-1}$. The *symplectic form* on $\mathbb{R}^{2n}$ is the nondegenerate, skew-symmetric bilinear form

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \quad \omega(x, y) = \langle Jx, y \rangle. \tag{1.26}$$

(Note that any $2n\times 2n$ matrix $\hat{J}$ satisfying $\hat{J} = \hat{J}^\top = -\hat{J}^{-1}$ induces a symplectic form, but in this thesis I will only consider J given by (1.25).) A *Lagrangian subspace* of $\mathbb{R}^{2n}$ is an n-dimensional subspace upon which the symplectic form vanishes. The *Lagrangian Grassmannian* is the set of all Lagrangian subspaces,

$$\mathcal{L}(n) = \{V \subset \mathbb{R}^{2n} : \dim V = n, \omega(x, y) = 0 \text{ for all } x, y \in V\}.$$

It is proven in [Arn67] that the fundamental group of the Lagrangian Grassmannian is given by

$$\pi_1(\Lambda(n)) \approx \mathbb{Z}$$

for any n (for more details, see Section 1.4). That $\Lambda(n)$ has this topological structure facilitates the definition of the *Maslov index*, a winding number for loops in $\Lambda(n)$, or, more generally, an intersection index for non-closed curves in $\Lambda(n)$ with a certain codimension-one set. More details of this construction will be given in Section 1.4 (see also Section 2.2.1).

A Lagrangian plane can be viewed as a real subspace of complex space in the following sense. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via

$$(x, y) \longleftrightarrow x + iy. \tag{1.27}$$

Set $z_j = x_j + iy_j$, $j = 1, 2$, with $x_j, y_j \in \mathbb{R}^n$. Evaluating the Hermitian inner product in $\mathbb{C}^n$ on z_1, z_2 yields

$$\langle z_1, z_2 \rangle_{\mathbb{C}^n} = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{\mathbb{R}^{2n}} - i \left\langle J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{\mathbb{R}^{2n}}.$$

Thus

$$\operatorname{Im}\langle z_1, z_2\rangle_{\mathbb{C}^n} = -\omega\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right).$$

In this way, a Lagrangian plane can be viewed as a *real* subspace of $\mathbb{C}^n$ via the identification (1.27): the corresponding Hermitian inner product of any two vectors in the Lagrangian plane is always real.

A *frame* for a Lagrangian subspace V is a $2n \times n$ matrix

$$\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad X, Y \in \mathbb{R}^{n\times n},$$

whose columns span V, where

$$X^\top Y = Y^\top X. \tag{1.28}$$

To see (1.28), note that vectors $u, v \in V$ can be written as

$$u = \begin{pmatrix} X \\ Y \end{pmatrix} k_1, \qquad v = \begin{pmatrix} X \\ Y \end{pmatrix} k_2,$$

for some $k_1, k_2 \in \mathbb{R}^n$. If V is Lagrangian then for all $k_1, k_2 \in \mathbb{R}^n$, one has

$$0 = \omega(u, v) = -\langle Xk_1, Yk_2\rangle + \langle Yk_1, Xk_2\rangle = \left\langle \left(X^\top Y - Y^\top X \right) k_1, k_2 \right\rangle.$$

A matrix $M \in \mathbb{R}^{2n\times 2n}$ that preserves the symplectic form,

$$\omega(Mu, Mv) = \omega(u, v) \tag{1.29}$$

for all $u, v \in \mathbb{R}^{2n}$, is called *symplectic*. The set of symplectic matrices forms a Lie group under matrix multiplication, called the *symplectic group* and denoted $Sp(n; \mathbb{R})$, or simply $Sp(n)$,

$$Sp(n) = \{M \in \mathbb{R}^{2n\times 2n} : M^\top J M = J\}. \tag{1.30}$$

The condition in (1.30) follows from (1.29). The associated Lie algebra $\mathfrak{sp}(n)$ of *infinitesimally symplectic* matrices, i.e. the tangent space of $Sp(n)$ at the identity, is the set of $2n \times 2n$ matrices A such that

$$\mathfrak{sp}(n) = \{A \in \mathbb{R}^{2n\times 2n} : A^\top J + JA = 0\}.$$

It can be shown that any $A \in \mathfrak{sp}(n)$ necessarily has the form

$$A = \begin{pmatrix} B & C \\ D & -B^\top \end{pmatrix} = J \begin{pmatrix} D & -B^\top \\ -B & -C \end{pmatrix}, \qquad C = C^\top,\ D = D^\top, \tag{1.31}$$

where $B, C, D \in \mathbb{R}^{n\times n}$. The groups $Sp(n; \mathbb{C})$ and $\mathfrak{sp}(n; \mathbb{C})$ are defined in the obvious way, with conjugate transposes replacing transposes in the above definitions. (Such groups arise in Section 1.4.)

The differential equations studied in this thesis can always be reduced to first order systems of the form

$$\frac{dx}{dt} = A(t)x, \qquad A(t) \in \mathfrak{sp}(n), \qquad x \in \mathbb{R}^{2n}. \tag{1.32}$$

Indeed in Chapter 3, the central focus is one such system. I will call these systems *Hamiltonian*, on account of being able to write the coefficient matrix as $A(t) = JM(t)$, where $M(t) \in \mathbb{R}^{2n\times 2n}$ is symmetric (see (1.31)). The principal fundamental matrix solution $\Psi(t)$ of (1.32) is symplectic, as the following calculation shows:

$$\begin{aligned} \frac{d}{dt}\left(\Psi(t)^\top J\Psi(t)\right) &= \dot{\Psi}(t)^\top J\Psi(t) + \Psi(t)^\top J\dot{\Psi}(t), \\ &= (A(t)\Psi(t))^\top J\Psi(t) + \Psi(t)^\top JA(t)\Psi(t), \\ &= \Psi(t)^\top \left(A(t)^\top J + JA(t)\right)\Psi(t), \\ &= 0, \end{aligned} \tag{1.33}$$

so that $\Psi(t)^\top J\Psi(t)$ is constant in t. The fact that $\Psi(0) = I$ implies that $\Psi(t)^\top J\Psi(t) = J$ for all t. It follows that (1.32) preserves Lagrangian planes, since if $V \in \mathcal{L}(n)$ and $u, v \in V$, then $\omega(\Psi(t)u, \Psi(t)v) = \omega(u, v) = 0$ for all t.

1.3.4 Stability theory

The issue of the stability of an *equilibrium* or *stationary solution* of an evolutionary partial differential equation concerns its robustness under perturbations. That is, starting with an initial condition that is “close" to the equilibrium, one seeks to determine whether the time evolution of this initial condition remains “close" to the equilibrium for all time.

In this thesis I will be interested in a weak notion of stability known as *spectral stability*. To set this up mathematically, I will follow the discussion in the opening section of [KP13, §4]. Since I will only be interested in the case of a single spatial variable in this thesis, I will restrict the discussion to that case.

Consider a general evolutionary PDE of the form

$$\partial_t\psi = \mathcal{F}(\psi), \quad \psi = \psi(x,t), \quad \psi(x,0) = \psi_0(x), \tag{1.34}$$

where $x \in \Omega \subseteq \mathbb{R}$, posed in a (possibly complex) Hilbert space X. I will assume the nonlinearity $\mathcal{F}$ is smooth and densely defined, and that the initial value problem is locally well-posed in X, i.e. there exists a finite time T for which a unique solution exists in X on the interval $[0, T)$. An example of (1.34) is given by the nonlinear Schrödinger (NLS) equation

$$i\psi_t = \psi_{xx} + f(|\psi|^2)\psi + \omega\psi \tag{1.35}$$

for an analytic function f, which may be put into the form of (1.34) with $\mathcal{F}(\psi) = -i(\psi_{xx} + f(|\psi|^2) + \omega\psi)$. If $f(0) = 0$ and $\psi_0 \in H^s(\mathbb{R})$ for $s \geq 1$, then by [Pel11, §1.3.1, Theorem 1.1] the initial value problem for (1.35) is locally well-posed in $H^s(\mathbb{R})$.

A stationary solution $\widehat{\psi}$ of (1.34) is one whose time derivative is identically zero, i.e. $\mathcal{F}(\widehat{\psi}) = 0$. In order to study the stability of $\widehat{\psi}$, one considers a perturbed solution to (1.34) of the form $\psi(x,t) = \widehat{\psi}(x) + \varepsilon v(x,t)$, where ε is small, with initial condition $\psi(x,0) = \widehat{\psi}(x) + \varepsilon v_0(x)$, $v(x,0) = v_0(x)$. Supposing that ψ starts "close" to $\widehat{\psi}$, one needs to determine the long term behaviour of the function $v(x,t)$. More precisely, we have the following definition.

Definition 1.3. *The stationary solution $\widehat{\psi}$ of* (1.34) *is* Lyapunov stable *or* nonlinearly stable *if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\|\psi_0 - \widehat{\psi}\|_X \leq \delta$ for all solutions ψ to* (1.34) *with $\psi(x,0) = \psi_0(x)$, then $\|\psi(\cdot,t) - \widehat{\psi}\|_X \leq \varepsilon$ for all $t \geq 0$.*

Substituting $\psi(x,t) = \widehat{\psi}(x) + \varepsilon v(x,t)$ into (1.34) yields

$$\partial_t v = \mathcal{F}(\widehat{\psi} + \varepsilon v). \tag{1.36}$$

Taylor expanding $\mathcal{F}$ about $\widehat{\psi}$ (recall $\mathcal{F}$ is smooth) leads to

$$\varepsilon \partial_t v = \mathcal{F}(\widehat{\psi}) + \varepsilon \mathcal{L} v + \mathcal{N}(\varepsilon v), \tag{1.37}$$

where $\mathcal{L}$ is a densely-defined closed linear operator corresponding to the linearisation of $\mathcal{F}$ about $\widehat{\psi}$. The action of $\mathcal{L}$ is given by $\mathcal{L}v = D\mathcal{F}(\widehat{\psi})v$, while its domain is determined by the types of perturbations one is considering. Now $\mathcal{F}(\widehat{\psi}) = 0$, and the nonlinear term $\mathcal{N}(\varepsilon v) = \mathcal{F}(\widehat{\psi} + \varepsilon v) - \mathcal{F}(\widehat{\psi}) - \varepsilon \mathcal{L} v$ contains higher order terms that are at least quadratic in ε. Therefore, *if $\|v(\cdot,t)\|$ indeed remains small enough*, then it is reasonable to expect that nonlinear effects are insignificant, in which case the dynamics of (1.37) are dominated by the linear system

$$\partial_t v = \mathcal{L} v, \qquad v(x,0) = v_0(x). \tag{1.38}$$

If λ is an eigenvalue of $\mathcal{L}$ with eigenfunction u, then $\widehat{v}(x,t) = e^{\lambda t} u(x)$ solves $\partial_t v = \mathcal{L} v$. Now $\|\widehat{v}\|_X = e^{(\operatorname{Re}\lambda)\, t} \|u\|_X$, and the behaviour of solutions to (1.38) is determined by the sign of $\operatorname{Re}\lambda$ for all eigenvalues $\lambda \in \operatorname{Spec}(\mathcal{L})$. In particular: $\operatorname{Re}\lambda > 0$ implies the existence of exponentially growing solutions; $\operatorname{Re}\lambda < 0$ implies the existence of exponentially decaying solutions; and $\operatorname{Re}\lambda = 0$ implies the existence of bounded, nondecaying solutions. If $\lambda \in i\mathbb{R}$ has algebraic multiplicity greater than one, then there may exist solutions that grow algebraically in time. For the purposes of this section, I will not comment on the effect of the essential spectrum.

The above discussion leads to the following notion of stability.

Definition 1.4. *The stationary solution $\widehat{\psi}$ is called* spectrally stable *if the spectrum of the associated linearised operator $\mathcal{L}$ does not intersect the open right half plane, i.e.* $\operatorname{Spec}(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$. *Otherwise, it is called* spectrally unstable.

In many cases, spectral instability leads to nonlinear instability. Sufficient conditions for this to occur are given in [SS00, Theorem 1]. Namely, $\mathcal{L}$ is required to generate a strongly continuous semi-group on X, and the nonlinearity $\mathcal{N}(v)$ in (1.37) needs to satisfy a certain estimate. The opposite direction, i.e. the conditions required for spectral stability to imply nonlinear stability, is more involved. I will not comment on this further.

In Chapter 2 and Chapter 3, separated solutions of the form $\psi(x,t) = e^{i\omega t}\phi(x)$, $\omega \in \mathbb{R}$, $\phi(x) \in \mathbb{R}$ to the NLS equations studied in those chapters are considered. While strictly speaking these are not stationary solutions, the NLS equations in those chapters do not contain the ω term in (1.35). Since the wave profile ϕ of these separated solutions satisfies the same time-independent equation as stationary solutions of the NLS equations *with* the ω term, and linearisation leads to the same linear operator, the notions of stability outlined in this section remain valid.

1.4 Review: the Maslov index in dynamical systems

1.4.1 Part I: Development of the index and early applications

Nearly one hundred years after Sturm, Morse [Mor34] generalised Sturm's oscillation theorem to vector-valued functions. Curiously, it was variational theory that provided the means to do so. To state Morse's result, I will follow the account of Milnor [Mil63, §III]. Let M be a Riemannian manifold, and Ω the space of (piecewise smooth) paths $\omega : [0,1] \to M$ with fixed endpoints. A path γ is a critical point of the energy functional

$$E : \Omega \to \mathbb{R}, \qquad E(\omega) = \int_0^1 \left\| \frac{d\omega}{dt} \right\|^2 dt,$$

if and only if γ is a geodesic. Associated to any such geodesic γ is a symmetric bilinear functional $E_{**} : T\Omega_\gamma \times T\Omega_\gamma \to \mathbb{R}$, the second variation of E along γ. Here, $T\Omega_\gamma$ is the vector space of (piecewise smooth) vector fields along γ which vanish at the endpoints. In the classical variational theory, γ is the solution to the Euler-Lagrange equation, and the second variation determines the nature of the critical point. A *Jacobi field* $J \in T\Omega_\gamma$ is a vector-valued function that satisfies a certain linear second-order differential equation (the Jacobi equation). A point $t^* \in (0,1]$ is then called a *conjugate point* if there exists a nontrivial Jacobi field along γ which vanishes at $t = 0$ and $t = t^*$; the multiplicity of the conjugate point is the dimension of the subspace of $T\Omega_\gamma$ consisting of all such Jacobi fields. A vector field $W \in T\Omega_\gamma$ belongs to the null space of E_{**} if and only if W is a Jacobi field. Hence E_{**} is degenerate if and only if $t = 1$ is a conjugate point. Defining the *index* of E_{**} to be the maximum dimension of a subspace of $T\Omega_\gamma$ upon which E_{**} is negative definite, Morse's index theorem then states that the index of E_{**} is equal to the number of conjugate points in $(0,1)$, counted with multiplicity.

To see that Morse's theorem is really a generalisation of Sturm's oscillation theorem, note that the (bounded from below) symmetric bilinear functional E_{**}, which is densely-defined on an appropriate function space, induces a selfadjoint linear operator $\mathcal{E}$ such that

$$E_{**}(U,V) = \langle \mathcal{E}U, V \rangle \qquad \text{for all } \; U \in \text{dom}(\mathcal{E}), V \in \text{dom}(E_{**}) \tag{1.39}$$

(As per [Kat80, Theorem VI.2.6]). Moreover, $\mathcal{E}$ is the linear operator that determines the Jacobi equation. Evaluating $\mathcal{E}$ on the sum of its negative eigenspaces, one sees that the index of E_{**} equals the number of negative eigenvalues of $\mathcal{E}$. On the other hand, conjugate points are the zeros of a vector-valued solution J to the linear second-order differential equation $\mathcal{E}J = 0$, counted with multiplicity. Here, a "zero" of J is a point $t = t^*$ where the all the entries vanish simultaneously. Thus, Morse's result says that the number of negative eigenvalues of a certain selfadjoint second-order linear differential operator is equal to the number of interior zeros (counted with multiplicity) of the solutions spanning the null space of that operator. In the scalar case, this is Sturm's oscillation theorem stated for the eigenfunction corresponding to the zero eigenvalue. While Morse himself did not explicitly relate the index of the second variation to the negative eigenvalues of a related selfadjoint operator, this was done by both Edwards [Edw64] and Duistermaat [Dui76] years later. Hereafter, the *Morse index* will, depending on context, refer either to the index of a symmetric bilinear functional (as defined above), or the number of negative eigenvalues of a selfadjoint operator[4].

In the mid nineteen-fifties, Bott [Bot56] gave a new perspective on Sturm's oscillation and Morse's index theorems. He developed what he called the *Sturm intersection theory* for second-order selfadjoint systems of ordinary differential equations, which involved a topological interpretation of eigenvalues. Specifically, Bott considers operators of the form

$$Ly := -\frac{d}{dt}\left(P(t)\frac{dy}{dt} + Q(t)y\right) + Q^*(t)\frac{dy}{dt} + R(t)y, \qquad y \in \mathbb{C}^n, \;\; P, Q, R \in \mathbb{C}^{n\times n}, \tag{1.40}$$

equipped with separated selfadjoint boundary conditions B on an interval $t \in [0, \ell]$. Here, P is positive definite and P and R are Hermitian. He reduces the eigenvalue equations $Ly = \lambda y$ to the first order system

$$\frac{du}{dt} = A(t;\lambda)u(t), \qquad A(t;\lambda) := \begin{pmatrix} P(t)^{-1}Q(t) & P(t)^{-1} \\ R(t) - Q(t)^*P(t)Q(t) - \lambda I & Q(t)^*P(t)^{-1} \end{pmatrix}, \tag{1.41}$$

and observes that the principal fundamental matrix solution $\Psi(t;\lambda)$ to (1.41) is symplectic for all t and λ, due to $A(t;\lambda)$ being infinitesimally symplectic, as per (1.33). Thus the map $\lambda \mapsto \Psi(\ell;\lambda)$ gives a curve in $Sp(n;\mathbb{C})$. The selfadjoint boundary conditions B define a certain codimension-one cycle γ_B in $Sp(n;\mathbb{C})$, and, by construction, intersections of the curve $\lambda \mapsto \Psi(\ell;\lambda)$ with this cycle correspond to eigenvalues of L. With this interpretation, the main result of the paper is given in [Bot56, Theorem V]. Bott proves the count of eigenvalues of L (including multiplicity) on an interval $\tau = [\alpha, \beta]$ of the λ-axis is equal to an intersection number, an integer denoted $[\gamma_B : \tau]_{Sp(n;\mathbb{C})}$, counting the (signed) intersections of the positively oriented curve $[\alpha, \beta] \ni \lambda \mapsto \Psi(\ell;\lambda)$ with γ_B. This integer is shown in [Bot56, §5] to be a topological invariant. One of the key steps needed in the proof of [Bot56, Theorem V] is to show that the path $\lambda \mapsto \Psi(\ell;\lambda)$ always intersects γ_B in the positive direction. This follows from [Bot56, Propositions 3.1 and 3.2], in which Bott shows that $\lambda \mapsto \Psi(a;\lambda)$ is a $\oplus$-*curve* ("plus curve") which only intersects γ_B transversally. Here, a $\oplus$-curve $\lambda \mapsto M(\lambda) \in Sp(n)$ is one whose direction vector $\Psi^{-1}(a;\lambda)\,\partial_\lambda\Psi(\ell;\lambda) \in \mathfrak{sp}(n)$ lies in a certain cone – the $\oplus$-cone of $Sp(n;\mathbb{C})$, consisting of those $A \in \mathfrak{sp}(n)$ such that JA is positive definite – for all λ. Such curves are therefore monotone in this sense. In a nutshell, it is again through a certain kind of monotonicity, which, as is the case in the proof of Theorem 1.1, is *global* in the parameter λ, that allows Bott to count the eigenvalues of the second-order selfadjoint operator L.

As a consequence of [Bot56, Theorem V], Bott proves analogues of Sturm's and Morse's theorems. In [Bot56, §8], he considers the map $(\lambda, t) \to \Psi(t;\lambda)$ on the boundary of the rectangle $[\varepsilon, a] \times [\lambda_1, \lambda_2]$ in the $t\lambda$-plane. Homotopy invariance yields that the intersection numbers (with

[4] or the number of positive eigenvalues, as in §1.1, depending on the orientation of the spectrum and whichever is the distinguished quantity

γ_B) along each side of the rectangle sum to zero. This yields a formula relating the count of eigenvalues of L (subject to B on $[0, \ell]$) in the interval $[\lambda_1, \lambda_2]$, to the same count for the operator L (subject to B on $[0, \varepsilon]$, $0 < \varepsilon < \ell$), via the intersection numbers $[\gamma_B : C_{\lambda_{1,2}}]$ of the paths $C_{\lambda_i} : t \to \Psi(t, \lambda_i)$, $t \in [\varepsilon, \ell]$. For suitable choices of ε and λ_1, the formula reduces to show that the number of eigenvalues of L less than λ_2 is equal to the intersection number $[\gamma_B : C_{\lambda_2}]$. The intersection points of the path C_2 are precisely the conjugate points of Morse. In [Bot56, §9], Bott shows that for a certain class of boundary conditions (of which Dirichlet boundary conditions are an example) the intersections of C_2 with γ_B all occur in the same direction. In this case $[\gamma_B : C_{\lambda_2}]$ is an *exact count* of the conjugate points, so that Morse's theorem holds for the operator L.

Edwards [Edw64] extended the work done by Bott for second order operators by developing an oscillation theorem for selfadjoint ordinary differential operators of *arbitrary* even order. Like Bott, this was obtained via a topological intersection theory. Edwards' analysis focuses on Hermitian forms (i.e. quadratic one-forms or sesquilinear two-forms). If E is a finite-dimensional complex vector space and $V[0, \ell]$ the space of sufficiently differentiable functions $x : [0, \ell] \mapsto E$, Edwards shows that to any selfadjoint differential operator L of order 2ν on $V[0, \ell]$, there corresponds a Hermitian form (à la Morse)

$$(\Omega, \beta, [0, \ell]) : V[0, \ell] \to \mathbb{R}, \qquad x \mapsto \int_0^\ell \Omega[x(t)]dt + \beta[x].$$

$(\Omega, \beta, [0, \ell])$ can be viewed as a variational equivalent of the differential equation $Lx = 0$ (with boundary conditions). Here, Ω is a Hermitian form depending on the value of $x(t)$ and its derivatives up to order ν, obtained from $Lx = 0$ by multiplying by a suitable test function and integrating by parts ν times, while β is a Hermitian form depending on the values of x and its derivatives at $t = 0$ and $t = \ell$ that describes the boundary conditions. Solutions to the eigenvalue problem for L correspond to points of nontrivial kernel of the form $x \mapsto \int_0^\ell \Omega[x] - \lambda \|x\|^2 dt + \beta[x]$; through this correspondence Edwards makes the explicit connection between the number of negative eigenvalues of L and the index of the functional $(\Omega, \beta, [0, \ell])$.

Using this framework, Edwards proves in ([Edw64, Theorem 3.1]) that when β encodes the Dirichlet condition at the right endpoint (i.e. derivatives up to order $\nu - 1$ vanish) and an arbitrary selfadjoint condition at the left endpoint, the index of the form $(\Omega, \beta, [0, \ell])$ is equal to the sum total over $t \in (0, \ell)$ of the nullities of $(\Omega, \beta, [0, t])$. The proof makes use of the intersection index of $\oplus$-curves in "U-manifolds" with a codimension-one subvariety Γ_β that corresponds to the boundary form β. A U-manifold (so named on account of being homeomorphic to the unitary group in certain cases) is the set $U(F, \psi)$ of subspaces P of an even-dimensional complex vector space F, with $\dim P = 1/2 \dim F$, upon which a nondegenerate Hermitian form ψ of signature zero vanishes. In this space a $\oplus$ curve is one that is monotone with respect to Γ_β. The properties of the integer-valued intersection index are given in [Edw64, §4.3]. In particular, Axioms 1–5 and [Edw64, Proposition 4.8] show that it is, for example, additive under concatenation and invariant under fixed-endpoint homotopies; any intersections of a $\oplus$-curve with Γ_β contribute $+1$ to its index. [Edw64, Theorem 3.1] then follows from a homotopy argument, similar to that used by Bott in [Bot56, §8]. Once again the Morse index and Sturm oscillation theorems are special cases of [Edw64, Theorem 3.1]. It was later highlighted by Cushman and Duistermaat in [CD77, §3] that the U-manifolds described by Edwards can be identified with the space of Lagrangian subspaces of a real symplectic vector space. In this way, the intersection theory developed by Edwards for curves in U-manifolds can be seen as a Hermitian version of the intersection theory for paths of Lagrangian subspaces.

A year after Edwards, Smale [Sma65] generalised Morse's index theorem to multiple space dimensions for a class of partial differential operators. In that work, Smale studies selfadjoint elliptic operators L of order $2k$ on a smooth manifold M with boundary. Here, L acts on functions u on M vanishing to order $k-1$ on ∂M; if $k=1$ then L is second-order with Dirichlet boundary conditions. Smale extends Morse's notion of a conjugate point to this setting by shrinking M via a smooth family of subdomains $\{M_t\}$ to a small set. A value $t=t^*$ is then *conjugate* if there is a nontrivial u satisfying $Lu=0$ on M_{t^*}, along with the boundary conditions on ∂M_{t^*}. The index theorem then holds: associated with L is a symmetric bilinear form B_L, the index of which (i.e. the number of negative eigenvalues of L) equals the number of conjugate points. Just as in the proof of Sturm's oscillation theorem presented in Section 1.1, the important step in Smale's proof ([Sma65, Lemma 2]) is to show a certain monotonicity of the eigenvalues of the operator L (restricted to M_t) as M_t is shrunk; namely, that the eigenvalues are non-decreasing functions of t. Smale's result can be viewed as extending the Sturmian oscillation theorem to *partial* differential equations. Note that in the case that $M=[0,\ell]$ is a compact interval and u is vector-valued, Smale recovers Morse's result if $k=2$ and Edwards' result for arbitrary k.

In 1967 Arnol'd [Arn67] defined an intersection index for a closed curve in the Lagrangian Grassmannian. His motivation was to put the index introduced by Maslov in [Mas65] for closed curves on a Lagrangian manifold[5] on a rigorous footing. The definition described by him, at least for closed curves in $\Lambda(n)$, came to be known as the Maslov index.

Arnol'd determines the topology of the Lagrangian Grassmannian $\Lambda(n)$ by realising it as the quotient group $U(n)/O(n)$. This follows from the transitive action of the unitary group $U(n)$ (the unitary matrices on $\mathbb{C}^n$) on $\Lambda(n)$, with stabiliser the orthogonal group $O(n)$ (the orthogonal matrices on $\mathbb{R}^n$). Using this fact and by considering higher homotopy groups and long exact sequences of fibre bundles, Arnol'd proves that the fundamental group of $\Lambda(n)$ is free cyclic, $\pi_1(\Lambda(n))\approx\mathbb{Z}$. Fixing $\alpha\in\Lambda(n)$, he then defines $\Lambda_k(n):=\{\beta\in\Lambda(n):\dim(\alpha\cap\beta)=k\}$. It is shown that $\Lambda_k(n)$ is an open manifold of codimension $k(k+1)/2$ in $\Lambda(n)$, and that the closure $\overline{\Lambda_1(n)}$ determines a codimension-one cycle ("the singular cycle") in $\Lambda(n)$. Arnol'd proves that $\overline{\Lambda_1(n)}$ is two-sidedly embedded in $\Lambda(n)$, in the sense that there exists a continuous vector field in $\Lambda(n)$ which is transversal to $\Lambda_1(n)$. One can therefore speak about the positive and negative sides of $\overline{\Lambda_1(n)}$. Using this construction Arnol'd defines in [Arn67, §3.6] an index, $\operatorname{Ind}\gamma$, for oriented closed curves $\gamma: S^1\to\Lambda(n)$ that are transversal to $\overline{\Lambda_1(n)}$, as the intersection index of γ with the cycle $\overline{\Lambda_1(n)}$. In other words, $\operatorname{Ind}\gamma=\nu_+-\nu_-$, where ν_+ (ν_-) is the number of intersections of γ with $\overline{\Lambda_1(n)}$ in which γ crosses from the negative (positive) to the positive (negative) side. $\operatorname{Ind}\gamma$ is precisely the integer identified with the homotopy class $[\gamma]$ in the fundamental group $\pi(\Lambda(n))\approx\mathbb{Z}$.

The index constructed by Arnol'd (but attributed to Maslov) is related to the one defined by Maslov in [Mas65] in the following way. If M is a Lagrangian manifold, Arnol'd shows that M contains a two-sidedly embedded codimension-one singular cycle Σ (distinct from the singular

[5] A Lagrangian manifold is an n-dimensional submanifold of $\mathbb{R}^{2n}$, whose tangent spaces are Lagrangian subspaces of $\mathbb{R}^{2n}$

cycle $\overline{\Lambda_1(n)}$ of $\Lambda(n)$). For non-closed curves $\gamma : [0,1] \to M$ that are transverse to Σ and have non-singular endpoints $\gamma(0), \gamma(1) \notin \Sigma$, Maslov's index $\operatorname{ind}\gamma$ is then defined similarly to Arnol'd's: $\operatorname{ind}\gamma = v_+ - v_-$, where $v_\pm$ are defined relative to Σ as $\nu_\pm$ are relative to $\overline{\Lambda_1(n)}$. In the case when $\gamma : S^1 \to M$ is closed, Arnol'd shows in [Arn67, Lemma 4.1.3] that the index Ind generates the index ind under the tangential mapping $\tau : M \to \Lambda(n)$; that is, that $\operatorname{Ind}\tau\gamma = \operatorname{ind}\gamma$.

In the mid seventies Duistermaat [Dui76] defined a Maslov index for non-closed curves of Lagrangian subspaces in a study of the Morse index in variational calculus. Duistermaat considers critical points of the energy functional

$$E : \mathscr{C}_R \to \mathbb{R}, \qquad E(c) = \int_0^T f\left(t, c(t), \frac{dc}{dt}(t)\right) dt,$$

for a smooth f, on the space $\mathscr{C}_R$ of C^1 curves $c : [0,T] \to X$ on a smooth manifold X, where the endpoints $c(0), c(T)$ are confined to a smooth submanifold R of $X \times X$. (This includes fixed endpoints, as Morse considered, as a special case). As observed by Edwards, Duistermaat shows that the Morse index $i_R(\widehat{c})$ associated with the second variation of E at a critical point $\widehat{c}$ is equal to the number of negative eigenvalues (including multiplicity) of a related selfadjoint operator $\mathscr{E}$. The eigenvalue equations for $\mathscr{E}$ are written as a linear first-order Hamiltonian system on $\mathbb{R}^{2n}$, and the accompanying boundary conditions, derived from the manifold R, are encoded in a Lagrangian subspace ρ of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$[6]. The (symplectic) principal fundamental solution matrix $\Phi(\mu, T)$ to the Hamiltonian system induces a path of Lagrangian subspaces $\mu \to \operatorname{graph}\Phi(\mu,T) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$; the parameter μ is related to the eigenvalue parameter λ of $\mathscr{E}$ in such a way that $\mu \in (-1, 0)$ if and only if $\lambda < 0$. The Morse index may therefore be expressed as the sum

$$i_R(\widehat{c}) = \sum_{-1<\mu<0} \dim\left(\operatorname{graph}\Phi(\mu,T) \cap \rho\right). \tag{1.42}$$

To compute the right hand side of (1.42), Duistermaat defines an intersection index in [Dui76, §2] as follows. Fix a reference plane $\alpha \in \Lambda(n)$ and let $\Sigma(\alpha) = \cup_{k\geq 1}\Sigma_k(\alpha)$, where $\Sigma_k(\alpha) := \{\beta \in \Lambda(n) : \dim(\beta \cap \alpha) = k\}$. Duistermaat defines the *Maslov-Arnold index* $[\gamma]$ for a loop $\gamma : S^1 \to \Lambda(n)$, which intersects $\Sigma(\alpha)$ transversally and only in $\Sigma_1(\alpha)$, to be the sum of the signatures of certain quadratic forms defined on the intersection spaces $\widetilde{\gamma}(t) \cap \alpha$, over all t where such intersections are nontrivial. It follows from the results of Arnol'd that $[\gamma]$ is a homotopy invariant. This formula in terms of quadratic forms is based on identifications of the spaces $\Sigma_0(n)$ and $T_\alpha\Lambda(n)$ with the space $S^2\alpha$ of symmetric bilinear forms on α. Duistermaat then defines an index for non-closed curves $\gamma : [0,\ell] \to \Lambda(n)$ with $\gamma(0), \gamma(\ell) \in \Sigma_0(\alpha)$. To do so, he constructs a loop $\widetilde{\gamma}$ by closing γ via a path $\gamma' \in \Sigma_0(\alpha)$ from $\gamma(\ell)$ back to $\gamma(0)$. The *intersection number* $[\gamma : \alpha]$ of γ with α is then defined to be the Maslov-Arnold index of $\widetilde{\gamma}$, i.e. $[\gamma : \alpha] := [\widetilde{\gamma}]$. Finally, Duistermaat defines an index $\operatorname{ind}_D(\gamma)$ for a path $\gamma : [0,\ell] \to \Lambda(n)$ satisfying $\gamma(0), \gamma(\ell) \in \Sigma_0(\alpha)$ as its intersection number $[\gamma : \alpha]$ with α, plus a correction term determined by a quadratic form on the endpoint $\gamma(\ell)$. This makes the index ind_D proposed by Duistermaat independent of the reference plane α, but not additive under concatenations of the path.

[6] Here $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is equipped with the symplectic form $\omega \oplus (-\omega)$, where ω is a symplectic form on $\mathbb{R}^{2n}$, and the associated Lagrangian Grassmannian is denoted by $\Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$.

In the same vein as Bott and Edwards, Duistermaat shows in [Dui76, Proposition 4.1] that the path $\psi : \mu \mapsto \operatorname{graph}\Phi(\mu, T))$ is a *plus-curve*, i.e. a curve in $\Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ for which a certain quadratic form associated with ψ is positive definite *for all* $\mu \in [-1, 0]$. Such curves are the same in spirit as the plus-curves of Bott and Edwards, in that they exhibit a certain global monotonicity in their direction of travel. The key assumption needed to arrive at this result is that the matrix

$$D_v^2 f\left(t, \widehat{c}(t), \frac{d\widehat{c}}{dt}(t)\right)$$

is positive definite for all $t \in [0, T]$; here subscript v denotes differentiation with respect to the third argument. As a consequence of [Dui76, Proposition 4.1], the right hand side of (1.42) is given by $[\psi : \rho]$. Duistermaat then uses homotopy invariance of the intersection number to give a formula in [Dui76, Theorem 4.3] for the Morse index $i_R(c)$ in terms of the *index* $\operatorname{ind}_D(\varphi)$ of the path $\varphi : t \to \operatorname{graph}\Phi(0, t)$. The formula also contains a correction term involving the matrix $\Phi(0, T)$, not found in Morse's original formula, which arises from the arbitrary boundary relation defined by R. When R encodes a fixed endpoint at $t = T$, the boundary condition of the linear problem at $t = T$ is the Dirichlet condition, and the reference plane is given by $\rho = U \times V$, where $U \subseteq \mathbb{R}^{2n}$ and $V = 0 \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ the vertical subspace of $\mathbb{R}^{2n}$. In this case the Lagrangian path $t \to \operatorname{graph}\Phi(0, t)$ is also a plus-curve, and the correction terms in the formula for the Morse index vanish. This agrees with the result of Edwards [Edw64]. If R also encodes a fixed-endpoint at $t = 0$, so that U is also the vertical subspace, Morse's classical theorem equating the Morse index and the number of conjugate points is recovered.

In a second paper published in the mid eighties, Arnol'd [Arn85] used a Maslov index to generalise Sturm's classical theory to general linear Hamiltonian systems. In particular, he showed that the statements regarding the oscillations of solutions in scalar second-order linear ODEs in Sturm's theory generalise to statements about the oscillations of Lagrangian planes evolving in the symplectic phase space of the Hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \qquad q, p \in \mathbb{R}^n, \quad x \in [0, \ell], \tag{1.43}$$

with Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. For example, Arnol'd observed that Sturm's comparison theorem has a Hamiltonian interpretation. The Hamiltonian for the second order equation $(p_i(x)y')' + g_i(x)y = 0$ is given by

$$H_i(y, y') = \frac{1}{2}p_i(x)^{-1}(y')^2 + \frac{1}{2}g_i(x)y^2.$$

Thus, the condition (1.11) (i.e. $0 < p_2(t) \leq p_1(t)$ and $g_2(t) \geq g_1(t)$) is equivalent to $H_2 \geq H_1$. Theorem 1.2 therefore states that radius vectors in the phase plane oscillate faster as the Hamiltonian is increased. Arnol'd generalised this statement as follows. Consider two systems of the form (1.43) with Hamiltonians H_1 and H_2. Suppose that both H_1 and H_2 are positive definite on a Lagrangian plane $\alpha \in \Lambda(n)$, and denote by $\nu(H)$ the number of instances, on any interval contained in $[0, \ell]$, at which any Lagrangian plane is nontransversal to α. If $H_2 \geq H_1$, then $\nu(H_2) \geq \nu(H_1) - n$.

The proofs of Arnol'd's main theorems use a Maslov index defined in [Arn85, §2] for non-closed curves satisfying a transversality condition at the endpoints (similar to Duistermaat). The index is based on a local identification of the space $\Lambda(n)$ with the space of nondegenerate quadratic forms as follows. Again fixing $\alpha \in \Lambda(n)$, Arnol'd calls the set $\Sigma(\alpha) = \{\beta \in \Lambda(n) : \alpha \cap \beta \neq \{0\}\}$ the *train* of α, with α the *vertex* of that train. He notes that any train divides a small neighbourhood of its vertex into $n+1$ subsets, with each subset being identified with the set of nondegenerate quadratic forms on α having fixed positive and negative inertial indices[7]. Since the train is two-sidedly embedded in $\Lambda(n)$ as per [Arn67], its orientation defines an order of those subsets, i.e. the quadratic forms corresponding to subset i have positive inertial index i. Thus there is a distinguished positive region P near the vertex corresponding to positive definite quadratic forms. Arnol'd then defines the Maslov index of $\gamma : [0,1] \to \Lambda(n)$ with $\gamma(0) = \delta, \gamma(1) = \alpha$ and $\delta \cap \alpha = \{0\}$ as follows. It is the increment in the positive inertial index of the curve of quadratic forms associated with a nearby path $\tilde{\gamma}$ that starts at δ and ends in the positive region P of α defined by $\Sigma(\alpha)$. Arnol'd's main theorems thus follow from analysing the change in the positive inertial indices of curves of quadratic forms. For example, one key result used ([Arn85, §4, Lemma 1]) is the following. Suppose a smooth curve of quadratic forms has an isolated degeneracy instant, and its derivative is positive definite on the kernel of the degenerate form. Then, the positive inertial index of the form increases by the dimension of the kernel of the form at the point of degeneracy. Later definitions of the Maslov index given in [RS93] and [GPP04a, GPP04b] (see below) can be seen to be based on generalisations of [Arn85, §4, Lemma 1] to the case when the derivative of the curve of quadratic forms is not positive definite but nondegenerate ([RS93]), and to the case when the derivative is degenerate ([GPP04a, GPP04b]).

In the late eighties, Jones [Jon88] used the Maslov index developed by Arnol'd to analyse an eigenvalue problem for a Hamiltonian differential operator,

$$N\begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \lambda\begin{pmatrix} p \\ q \end{pmatrix}, \qquad \mathrm{dom}(N) = H^1(\mathbb{R}) \times H^1(\mathbb{R}), \tag{1.44}$$

where L_+ and L_- are Schrödinger operators. Jones derived a criterion for the existence of a positive real eigenvalue of N in terms of the Morse indices of the selfadjoint operators L_+ and L_- (see Theorem 1.5 in the next section). The proof involves a shooting argument in the space of Lagrangian planes. Specifically, with a convenient change of variables Jones converts the differential equations in (1.44) to a first order (spatially) Hamiltonian system which preserves Lagrangian planes. Eigenfunctions of N are viewed as trajectories connecting the Lagrangian planes encoding the left and right "boundary" conditions at $\pm\infty$. By shooting forward the subspace of solutions satisfying the condition at $-\infty$, a connecting orbit is shown to exist as the spectral parameter is varied, thus creating an eigenvalue. Jones' problem is different to those of Bott, Edwards and Duistermaat in two critical respects. First, the operator N in (1.44) under consideration is not selfadjoint, so its spectrum is complex in general. The analysis is therefore restricted to real eigenvalues, since the Maslov index is unable to detect complex eigenvalues. Second, the Maslov index is not monotone in the (real) spectral parameter λ. (In fact, nor is it monotone in x; however, as will be shown in Chapter 2, the Maslov index in the spatial parameter can always be accounted for when $\lambda = 0$.) An exact count of the positive real eigenvalues is therefore not afforded by the analysis, but Jones' argument still gives the existence of such eigenvalues.

[7] the *positive* (*negative*) *inertial index* of a quadratic form is the number of positive (negative) squares of the form when diagonalised

In the early nineties, Robbin and Salamon [RS93] gave a definition of the Maslov index for non-closed curves that did away with the assumptions of Arnol'd and Duistermaat of transversality at the endpoints and of only one dimensional intersections. Like Arnol'd ([Arn85]) and Duistermaat, theirs exploits the identification of the tangent space of $\Lambda(n)$ with the space of quadratic forms. This lead to their construction of the *crossing form*, a quadratic form defined at intersections of the path with *any* stratum $\Sigma_k(\alpha)$ of the train $\Sigma(\alpha)$ of a fixed plane α. They define the Maslov index to be the sum of the signatures of the crossing form at each crossing with $\Sigma(\alpha)$, provided crossings are all *regular*, in the sense that the associated crossing form is nondegenerate. A path has only regular crossings if and only if it is transverse to $\Sigma(\alpha)$. Their definition of the Maslov index is shown to obey a set of axioms, including invariance under fixed-endpoint homotopies and additivity under concatenation (see Proposition 3.9). The definition is extended to all continuous Lagrangian paths via homotopy invariance. More details of this approach will be given in Section 2.2.1.

Cappell, Lee and Miller [CLM94] gave an axiomatic characterisation of the Maslov index in terms of six elementary properties. They describe four definitions – two geometric and two analytic in nature, including the geometric intersection index described by Arnol'd – and show that all four definitions obey their set of axioms. Their systematic treatment unified a number of different perspectives on the Maslov index. For example, the intersection indices described by Bott, Edwards, Arnol'd and Robbin and Salamon all obey these axioms, while Duistermaat's index ind_D, for example, does not obey the concatenation axiom.

The Maslov index was defined in *infinite* dimensions by Furutani [Fur04] and Booss-Bavnbek and Furutani [BBF98]. In this case the index is defined for paths in the *Fredholm*-Lagrangian Grassmannian of a symplectic Hilbert space. This Grassmannian consists of subspaces of H that are Lagrangian (maximally isotropic) and which form a Fredholm pair with another fixed subspace; two subspaces form a Fredholm pair if their intersection is finite-dimensional and their sum is closed and finite-codimensional. The Fredholm condition is essential for defining an integer-valued homotopy invariant, since the Lagrangian-Grassmannian of an infinite-dimensional symplectic Hilbert space is contractible, and therefore has trivial fundamental group. In [BBF98], Booss-Bavnbek and Furutani give a definition for the Maslov index in terms of the spectral flow of a family of unitary operators in a related complexified Hilbert space. Theirs is based on a formulation of the spectral flow of a family of selfadjoint Fredholm operators introduced by Phillips in [Phi96].

In two papers [GPP04a, GPP04b], Gambio, Portaluri and Piccione developed a Maslov index that did away with the assumption of regular crossings in [RS93], but required analyticity of the path. Their construction is based on local computations of the spectral flow of a family of symmetric matrices using the notion of *partial signatures*. By doing away with the assumption of regular crossings, they are able to compute contributions to the Maslov index from isolated nontransversal intersections of the Lagrangian path with the train directly, without using homotopy arguments. More details of this approach are given Section 3.3.2.

The above survey of works concerning the early development of the Maslov index in dynamical systems is not an exhaustive one. For instance, I have not mentioned the works by Lidskii [Lid55], Keller [Kel58], Souriau [Sou76], Leray [Ler81], Conley and Zehnder [CZ84] and DeGosson [dG90].

1.4.2 Part II: Further applications

By the early twenty-first century, the Maslov index for a path of Lagrangian planes was well established. The last two decades have seen an abundance of works further developing the relation between the Morse and Maslov indices in various contexts, as well as a number of applications to the stability of nonlinear waves in Hamiltonian systems.

Regarding the multidimensional case, in 2011 Deng and Jones [DJ11] related the Morse index of an elliptic partial differential operator to the Maslov index of a related path in the Fredholm-Lagrangian Grassmannian. They studied scalar-valued Schrödinger operators on bounded star-shaped domains, and the path was given by the space of traces of solutions to the differential equations restricted to a family of shrinking domains. This is the approach taken in Chapter 2 (for details see Section 2.2.2) albeit in the much simpler finite-dimensional setting where traces of solutions form Lagrangian subspaces of $\mathbb{R}^{2n}$. The formulas of Deng and Jones relating the Morse and Maslov indices extend the work of Smale [Sma65] by treating the case of Neumann boundary conditions. In [CJLS16], a formula for the Morse index via the Maslov index was developed to the case of matrix-valued Schrödinger operators on star-shaped domains with Lipschitz boundary, and in [CJM15] these results were extended to general bounded domains with smooth boundary (i.e. with no assumptions on the geometry of the domain). In [CM19], the authors prove a constrained analogue of the Morse index theorem for multidimensional Schrödinger operators restricted to act on a subspace of their domain. In all of the above mentioned works, monotonicity always holds in the spectral parameter, while monotonicity in the spatial parameter only holds in the Dirichlet case. A consequence of this lack of monotonicity in the Neumann case is the appearance of extra terms in the formulas for the Morse index.

In [LS17] the authors derived a *Hadamard-type* formula for the derivative of an eigenvalue of a multidimensional Schrödinger operator with respect to perturbation of the domain. The derivative is given in terms of the Maslov crossing form, an infinite-dimensional analogue of the crossing form of Robbin and Salamon, defined in, for example, [Fur04, CJLS16]. Such a formula can be viewed as an infinitesimal version of the Morse-Maslov theorem. These formulas were extended to a more general class of operators in [LS20a].

The case of a one-dimensional spatial domain (and thus a finite dimensional Maslov index) has seen a number of works. Jones, Latushkin and Marangell [JLM13] treated the case of quasi-periodic boundary conditions for matrix-valued Schrödinger operators with a symmetric periodic potential. They used the approach of Deng of Jones by considering a path of traces of solutions to the differential equations on a family of shrinking domains, which eventually lead to a formula for the Morse index in terms of the Maslov index of such a path. The boundary conditions are encoded in a system of differential equations that is appended to the original system, and with this construction an eigenvalue corresponds to an intersection of the path with a fixed subspace. Periodic boundary conditions were also handled by [JLS17], who considered the same set-up as [JLM13], but used the phase describing the quasi-periodic boundary conditions (and not the spatial variable) as the second parameter in the homotopy argument.

Howard and co-authors [HS16, HLS17, HLS18, HJK18, HS22, How23, How21] have developed an extensive theory of the Maslov index in linear Hamiltonian systems (i.e. equations of the form (1.32)). In these works, the Maslov index is formulated as the spectral flow of a family of unitary matrices, determined by the number of eigenvalues passing through a fixed gauge on the unit circle. (This can be viewed as a finite-dimensional equivalent of the definition given by Booss-Bavnbek and Furutani [BBF98]). The systems of interest typically arise from converting the eigenvalue equations associated with the linearisation about a steady state solution in some nonlinear system to a first order system. Once again, a key assumption of the analyses is monotonicity of the Maslov index in the spectral parameter. In [HLS18], the authors dubbed the boundary of the rectangle in the λx-plane (i.e. the rectangle R in Figs. 1.1 and 1.2), as used by Bott, Edwards, Arnol'd and Duistermaat and many authors since, the *Maslov box*.

The Maslov index has been applied to problems posed on the line, especially in the context of stability of nonlinear waves. In these cases an assumption is needed to ensure the essential spectrum of the linearised operator does not impede the calculation of the Maslov index. In addition to the first analysis of this kind given by Jones [Jon88], Bose and Jones [BJ95] used the Maslov index to study solitary waves in gradient reaction-diffusion systems. The Maslov index of a homoclinic orbit was defined by Chen and Hu in [CH07] in terms of the unstable bundle associated with the orbit and the stable subspace of the asymptotic system. (More details about the construction of Lagrangian paths and the appropriate choice of reference plane in this case will be given in **??**.) In a second paper, Chen and Hu [CH14] used their Maslov index to prove the instability of standing pulses in a doubly-diffusive FitzHugh-Nagumo equation. The Maslov index has been used to determine the existence of real unstable eigenvalues in various nonlinear Schrödinger equations [MSJ10,MSJ12,JMS14], as well as Hamiltonian PDEs in [CDB09b,CDB09a,CDB11]. Morse-Maslov theorems were developed in [HLS18] for matrix-valued Schrödinger operators, and in [How23] for three specific classes of systems, including a fourth-order potential equation.

Beck *et al.* in [BCJ+18] used the Maslov index to prove the instability of pulses in a reaction-diffusion system with gradient nonlinearity, generalising the result that all pulses are unstable in scalar reaction diffusion systems [KP13, §2.3.3.1]. They study the eigenvalue problem for a Schrödinger operator on the real line,

$$Hu := -Du'' + V(x)u = \lambda u, \quad D = \operatorname{diag} d_i > 0, \quad u \in \mathbb{R}^n, \ x \in \mathbb{R}, \tag{1.45}$$

for a symmetric $n \times n$ matrix V (satisfying certain other conditions). They first restrict the problem to the half line $x \in (-\infty, L]$, equipping $x = L$ with the Dirichlet boundary condition, and show that the paths of subspaces given by the stable and unstable bundles, i.e. the exponentially dichotomic subspaces that converge to the stable and unstable subspaces as $x \to \infty$ and $x \to -\infty$ respectively, are Lagrangian. By showing monotonicity of these Lagrangian paths in x and λ, they prove that the Morse index of the restricted operator is equal to the number of conjugate points on the half line $(-\infty, L]$. The result is then extended to the full line by showing that the spectrum of restricted operator converges to that of the operator on the full domain as $L \to +\infty$. As an application of their abstract result, they prove that for pulses in a reaction-diffusion system, which are necessarily even symmetric about some point in their domain due to the (spatial) reversibility of the governing reaction-diffusion equation, at least one conjugate point necessarily exists. Consequently the operator has at least one unstable eigenvalue.

Cornwell and Jones [Cor19,CJ18,CJ20] used the Maslov index to investigate the stability of travelling waves in reaction-diffusion systems of skew-gradient type. In [Cor19], Cornwell gives a lower bound for the number of unstable eigenvalues of the linearised operator associated with the wave. In this case the linearised system is not Hamiltonian, but still preserves Lagrangian planes with respect to a non-standard symplectic form. One of the reasons for obtaining a lower bound, and not an equality, is that the Maslov index is not monotone in the spectral parameter λ (just as in [Jon88]). Furthermore, the linearised operator is not selfadjoint, and its eigenvalues need not be real; since the Maslov index is only able to detect real eigenvalues, an exact count of the unstable eigenvalues is in general not possible. Cornwell then considers travelling pulses in a doubly diffusive FitzHugh-Nagumo system (which is skew-gradient). In this case, unstable eigenvalues are necessarily real, and monotonicity of the Maslov index in λ, as well as semisimplicity of the unstable eigenvalues, permits an exact count of the Morse index via the Maslov index of the travelling wave (as defined for homoclinic orbits by Chen and Hu [CH07]). In [CJ18] Cornwell and Jones prove the stability of travelling pulses which they construct via geometric singular perturbation techniques. In this instance the Maslov index is not monotone in x; nonetheless, by determining all possible conjugate points and computing the signatures of the associated crossing forms, it is shown that the contributions cancel each other out, so that the Maslov index of the pulse is zero. The equality with the Morse index derived in [Cor19] is then used to conclude stability of the wave.

Numerical schemes for computing the Maslov index of homoclinic orbits have been given in [CDB09b, CDB09a, CDB11]. In these works, Chardard, Dias and Bridges define the Maslov index of a homoclinic orbit as the limit as $\lambda \to 0^+$ of the Maslov index of a related path of Lagrangian subspaces. The path is obtained by integrating the unstable subspace of the asymptotic system in x, with λ small and fixed, and the reference plane is given by the stable subspace for the same λ. This definition is different from that of Chen and Hu, but is more suitable for numerical computations. A computable formula is then given in terms of the crossing form of Robbin and Salamon [RS93]. Because integration of the unstable subspace is numerically unstable, the differential equations are represented in the exterior algebra space $\bigwedge^n(\mathbb{R}^{2n})$, and the unstable subspace in $\bigwedge^n(\mathbb{R}^{2n})$ is integrated. This procedure is numerically stable because the exponential growth rate can be factored out. However, since the dimension of $\bigwedge^n(\mathbb{R}^{2n})$ increases exponentially with n, the algorithm is best for small n. Beck and Malham [BM15] addressed this issue by developing a numerical method to compute the Maslov index in Hamiltonian systems for large n. In the context of stability, these methods provide useful tools to practically compute the Maslov index, allowing one to determine the Morse index of steady states of Hamiltonian PDEs that have been determined numerically.

A recent development of the Maslov index was given in [BCC$^+$22], where the Hamiltonian (i.e. Lagrangian) requirement was dropped. In that paper, the authors define a *generalised Maslov index* for loops in a certain subset of the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^{2n})$ of n-dimensional subspaces of $\mathbb{R}^{2n}$. This subset is a component of what the authors call a *Maslov-Arnold space*. The collection of Maslov-Arnold spaces contains the Lagrangian Grassmannian $\Lambda(n)$ as a subset, and, importantly, has the same key topological features as $\Lambda(n)$, allowing for the construction of an integer-valued intersection number. As an application of the theory, the authors examine steady states of reaction diffusion systems which are not gradient. In this case the linearised operator (given by (1.45) with non-symmetric V) is not selfadjoint, and moreover the eigenvalue problem is not Hamiltonian. Nonetheless, they are still able to give a lower bound for the number of positive real eigenvalues.

The important features of the analysis in this thesis, as an application of the Maslov index, can be summarised as follows. The operator is not selfadjoint, so has complex spectrum in general. The Maslov index can only detect real spectrum, hence the restriction to this case. (Using the Maslov index to detect complex spectrum is an open problem.) As well, the Maslov index is not monotone in the spectral parameter, as in [Jon88,Cor19]. Consequently, only a lower bound for the number of positive real eigenvalues, and not an exact count, is afforded by the analysis. In certain cases an exact count can be given. In addition, crossings that are non-regular are encountered, corresponding to nontransversal intersections of the Lagrangian path with the train of the reference plane. These degeneracies will be handled via two methods. The first is via perturbative arguments, where the eigenvalue curves are analysed and and homotopy invariance is exploited to compute the contribution to the Maslov index. This will be done in Chapter 2. The second is via the method of partial signatures of [GPP04b]. In this case, the contribution is computed directly via the partial signatures of higher-order crossing forms. This is the approach of Chapter 3.

1.5 Review: Hamiltonian spectral theory

There has been a long line of work aimed at determining the existence of unstable eigenvalues in the spectral problem

$$Nu = \lambda u, \qquad N := JL, \tag{1.46}$$

where J is skew-symmetric with a bounded inverse and L is selfadjoint. (I will assume that both operators are real.) An important case is when J is the standard symplectic matrix and L is diagonal,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \tag{1.47}$$

(where L_+ and L_- are selfadjoint), so that the eigenvalue equations are given by

$$-L_- v = \lambda u, \qquad L_+ u = \lambda v. \tag{1.48}$$

In this case the operator N is said to have the *canonical symplectic structure* [KKS04]. Hereafter I will always assume that N has this form, unless otherwise explicitly stated. The notation $L_\pm$ appears to have been first introduced by Rowlands [Row74], who studied the spectral stability of periodic standing wave solutions to the cubic NLS equation (see equation (1.49) below with $f(x, \phi^2) = \pm\phi^2$) subject to long-wavelength disturbances, for which $L_\pm = \partial_{xx} + 2\phi^2 \pm \phi^2 + \beta$. The eigenvalue problem (1.46)–(1.47) arises when linearising about waves in many nonlinear Hamiltonian PDEs, an example of which includes standing wave solutions $\psi(x,t) = e^{i\beta t}\phi(x)$, $\beta \in \mathbb{R}$, of the spatially inhomogeneous nonlinear Schrödinger (NLS) equation

$$i\psi_t = \psi_{xx} + f(x, |\psi|^2)\psi, \qquad \psi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}, \tag{1.49}$$

in which case $L_\pm$ are the Schrödinger operators

$$L_- = -\partial_{xx} - f(x, \phi^2) - \beta, \qquad L_+ = -\partial_{xx} - f(x, \phi^2) - \partial_2 f(x, \phi^2)\phi^2 - \beta. \tag{1.50}$$

(Here, ∂_2 denotes partial derivative with respect to the second argument). In this context, the domains of N, L_+ and L_- are determined by the type of perturbations one is considering.

When N has the canonical symplectic structure, its spectrum is symmetric with respect to both the real and imaginary axes. To see this, note that if $(u, v)^\top$ is an eigenfunction with eigenvalue λ, then from (1.48) it immediately follows that $-\lambda$ is an eigenvalue with eigenfunction $(u, -v)^\top$. Moreover, taking complex conjugates yields that $\bar{\lambda}$ and $-\bar{\lambda}$ are eigenvalues with eigenfunctions $(\bar{u}, \bar{v})$ and $(\bar{u}, -\bar{v})$ respectively. In this case N is said to have *full Hamiltonian symmetry*. In the context of stability, if the essential spectrum of N is confined to the imaginary axis, then spectral instability follows from the existence of *any* discrete spectrum with nonzero real part.

In practice it is much easier to determine spectral information for the operators L_+ and L_-, the spectra of which are real, than it is for N. For example, if L_+ and L_- are scalar-valued Schrödinger operators, then for each operator Theorem 1.1 affords an exact count of its negative eigenvalues via the nodal count of the eigenfunction for the zero eigenvalue. Since $\mathrm{Spec}(L) = \mathrm{Spec}(L_-) \cup \mathrm{Spec}(L_+)$, it therefore behoves one to determine the spectral properties of $N = JL$ based on those of L. Alas, this is no easy task. Many authors have endeavoured to understand the relation between the spectrum of L and JL, and the following is a brief survey of works in this direction.

In the early seventies, Vakhitov and Kolokolov [VK73] derived a stability criterion for radially symmetric ground state standing waves of the two-dimensional NLS equation,

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = 0. \tag{1.51}$$

Taking $f(\phi^2) = \phi^2/(1 + \phi^2)$ and considering solutions of the form $\psi(r, t) = e^{i\beta t}\phi(r)$, $r = \sqrt{x^2 + y^2}$, the wave profile ϕ satisfies the standing wave equation

$$\phi''(r) + \frac{1}{r}\phi'(r) - \beta\phi + \frac{\phi^3}{1 + \phi^2} = 0, \qquad r \in \mathbb{R}^+. \tag{1.52}$$

The ground state is the standing wave whose amplitude ϕ is a strictly positive solution to (1.52); linearisation about the ground state leads to the eigenvalue problem (1.48), where the linear operators L_+ and L_- satisfy $P = 1$ and $Q = 0$. Defining

$$I_1(\beta) = \int_0^\infty \phi(r; \beta)^2 \, r \, dr,$$

Vakhitov and Kolokolov used variational principles and the method of Lagrange multipliers to show in [VK73, §2] that if $\partial_\beta I_1 \geq 0$, then all eigenvalues of N lie on the imaginary axis, while if $\partial_\beta I_1 < 0$, then a positive real eigenvalue of N exists. This stability criterion, including its cousin for the one-dimensional NLS equation, came to be known as the *Vakhitov-Kolokolov* criterion.

Papers by Jones [Jon88] (discussed in Section 1.4) and Grillakis [Gri88] published at the end of the eighties focused on the existence of purely real eigenvalues of N. Denoting by $n(L)$ the number of negative eigenvalues (the Morse index) of a selfadjoint operator L, and setting

$$P := n(L_+), \qquad Q := n(L_-), \qquad n_+(N) := \#\{\text{positive real eigenvalues of } N\},$$

both Jones and Grillakis gave a lower bound for $n_+(N)$ in terms of P and Q. Jones' result is in the context of standing wave solutions of (1.49), where L_+ and L_- are given by (1.50).

Theorem 1.5 (**[Jon88, Theorem 1]**). *If* $P - Q \neq 0, 1$, *then* $n_+(N) \geq 1$.

Grillakis took a functional analytic approach to prove a slightly stronger result in a more abstract setting. For general selfadjoint operators $L_\pm$ on a Hilbert space, both of which are the sum of a strictly positive operator H and a relatively compact perturbation of H, Grillakis considered the equivalent constrained generalised eigenvalue problem:

$$\left(\Pi L_+ \Pi + \lambda^2 (\Pi L_- \Pi)^{-1}\right) u = 0, \qquad u \in X := \ker(L_-)^\perp. \tag{1.53}$$

(1.53) is obtained by projecting off the kernel of L_- and eliminating v from (1.48), where Π is the orthogonal projection onto X. Defining the Morse index of the constrained operator $\widehat{P} := n(\Pi L_+ \Pi)$, Grillakis proves the following.

Theorem 1.6 (**[Gri88, Theorem 1.1, Corollary 1.1]**). *If* $|\widehat{P} - Q| \geq 1$ *then* $n_+(N) \geq |\widehat{P} - Q|$. *Moreover, if* $Q = 0$ *then* $n_+(N) = \widehat{P}$ *and* $\operatorname{Spec}(N) = \mathbb{R} \cup i\mathbb{R}$.

To prove the first statement of Theorem 1.6, Grillakis considers the (continuous) motion of the eigenvalues of the selfadjoint operator $K(\lambda) = \Pi L_+ \Pi + \lambda^2 (\Pi L_- \Pi)^{-1}$ as a function of $\lambda \in \mathbb{R}$. By considering a contour C in the complex plane surrounding the negative eigenvalues of $K(\lambda)$, Grillakis argues that the dimension of the range of the spectral projection

$$P(\lambda) = \frac{-1}{2\pi i} \int_C (K(\lambda) - zI)^{-1} dz$$

(onto the sum of the negative eigenspaces of $K(\lambda)$) must decrease by at least $|\widehat{P} - Q|$ as λ increases over the interval $[\varepsilon, \Lambda]$, where $0 < \varepsilon \ll 1$ and $\Lambda \gg 1$. From this he concludes that at least $|\widehat{P} - Q|$ of the negative eigenvalues of $K(\lambda)$ must cross zero to become positive as λ increases over $[\varepsilon, \Lambda]$, with each such crossing point corresponding to a pair of eigenvalues $\{\pm\lambda\}$ of N bifurcating onto the real axis.

To see how the theorems of Jones and Grillakis are related, note that for Jones the scalar-valued Schrödinger operators $L_\pm$ have simple eigenvalues, and L_- (given in (1.50)) has a kernel spanned by ϕ. Restricting L_+ to $\ker(L_-)^\perp = \operatorname{span}\{\phi\}^\perp$, a codimension-one subspace of $\operatorname{dom}(L_+)$, will then decrease the Morse index of L_+ by at most one. Jones' result is thus recovered in the following sense: either $\widehat{P} = P$, in which case $P - Q \neq 0$ so that $|P - Q| \geq 1$, and by Theorem 1.6 it follows that $n_+(N) \geq 1$; or $\widehat{P} = P - 1$, in which case $P - Q \neq 1$ implies $|(P-1) - Q| \geq 1$, and by Theorem 1.6 it follows that $n_+(N) \geq 1$.

In order to state the next result, I need to introduce some notation. Suppose L_+ and L_- each have n dimensional kernels with eigenfunctions u_i and v_i respectively, $i = 1, ..., n$. Then the eigenvectors of $0 \in \operatorname{Spec}(N)$ are given by $(u_i, 0)^\top$, $(0, v_i)^\top$, $i = 1, ..., n$. Suppose there exists functions $(\widehat{v}_i, 0)^\top$, $(0, \widehat{u}_i)^\top$, $i = 1, ..., n$, so that

$$L_+ u_i = 0, \quad L_- v_i = 0, \quad -L_- \widehat{v}_i = u_i, \quad L_+ \widehat{u}_i = v_i. \tag{1.54}$$

Define $X_1 \coloneqq \operatorname{Ran} JL$ and the matrix

$$D = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}, \qquad [D_-]_{ij} \coloneqq \langle L_- \widehat{v}_i, \widehat{v}_i \rangle, \qquad [D_+]_{ij} \coloneqq \langle L_+ \widehat{u}_i, \widehat{u}_i \rangle. \tag{1.55}$$

If D is nondegenerate, then $0 \in \operatorname{Spec}(N)$ has geometric multiplicity n and algebraic multiplicity $2n$, where the n functions $(\widehat{v}_i, 0)^\top$, $(0, \widehat{u}_i)^\top$ are generalised eigenvectors of N. The matrix D is that induced from restricting the bilinear form associated with L to the space of generalised eigenvectors of N.

In two papers [GSS87, GSS90] Grillakis, Shatah and Strauss developed the stability theory of abstract Hamiltonian systems with symmetries. Their main results concern the orbital stability of "bound states", and are formulated in terms of certain "charge" and energy functionals, as well as the Hessian of a scalar-valued function d defined in terms of those functionals. When N has the canonical symplectic structure, the Hessian of d is given by $-D$. Grillakis Shatah and Strauss prove the following auxiliary results regarding a constrained eigenvalue count and the spectrum of N, both of which I state in terms of the matrix D for convenience.

Theorem 1.7 (**[GSS90, Theorem 3.1, Corollary 3.2]**). *The operator L restricted to X_1 satisfies*

$$n(L|_{X_1}) = n(L) - n(D) - z(D). \tag{1.56}$$

Consequently, if D is nondegenerate, then $n(L) \geq n(D)$.

Theorem 1.8 (**[GSS90, Theorem 5.1]**). *Suppose D is nondegenerate. If $n(L) - n(D)$ is odd, then $n_+(N) \geq 1$.*

Theorem 1.9 (**[GSS90, Theorem 5.8]**). *The number of eigenvalues of N that have nonnegative imaginary part and positive real part is at most $P + Q$.*

The next development in understanding the spectrum of N came at the turn of the century. Working in the abstract setting of a Hamiltonian system with symmetries, similar to [GSS87, GSS90], Kapitula, Kevrikidis and Sandstede [KKS04, KKS05] (as well as Pelinovsky and co-authors) were able to determine a closed formula relating the number of unstable eigenvalues of N (both real and complex) and the Morse index of L. The tool that made their formula possible was the *Krein signature* of an eigenvalue; let me briefly visit that topic.

The following is given in [Mac86]. Recalling the full Hamiltonian symmetry of N, denote by I_λ the real invariant subspace given by the sum of the generalised eigenspaces of $\{\pm\lambda, \pm\bar{\lambda}\}$. Some examples are as follows. If $\lambda \in \mathbb{R}$ then I_λ is just the sum of the generalised eigenspaces associated with $\{\pm\lambda\}$. If $\lambda \in i\mathbb{R}\{0\}$ is simple, then its eigenfunction can be taken to be of the form (u, iv), where $u, v \in \mathbb{R}$, since $-\lambda = \bar{\lambda}$ is an eigenvalue with eigenfunction $(u, -v) = (\bar{u}, \bar{v})$. In this case, $I_\lambda = \operatorname{span}\{(u, 0), (0, v)\}$. The *Krein signature* of an eigenvalue λ is given by the numbers of positive and negative squares, after diagonalisation, of the quadratic form $\langle L|_{I_\lambda} \cdot, \cdot \rangle$ associated with the restriction of L to I_λ; it is proven in [Mac86, Lemma 4] that $\langle L|_{I_\lambda} \cdot, \cdot \rangle$ is always nondegenerate. λ is said to have positive or negative Krein signature if $\langle L|_{I_\lambda} \cdot, \cdot \rangle$ is positive or negative definite, and has zero Krein signature if $\langle L|_{I_\lambda} \cdot, \cdot \rangle$ is indefinite. It is shown in [Mac86, Lemma 5(ii)] that any eigenvalue with nonzero real part has zero Krein signature, due to the form $\langle L|_{I_\lambda} \cdot, \cdot \rangle$ having an equal number of positive and negative squares. Furthermore, as per [Mac86, Lemma 5(i)], the form for any purely imaginary eigenvalue has an even number of positive and negative squares; it follows from nondegeneracy that if any such eigenvalue is simple it necessarily has positive or negative Krein signature. In fact, using the earlier notation, in this case the signature is given by the sign of $\langle L_+u, u\rangle = \langle L_-v, v\rangle$. As shown by Mackay [Mac86, Theorem p.146], the significance of the Krein signature is that, under perturbation of the parameters in the system, two colliding purely imaginary eigenvalues will bifurcate into the complex plane (in a *Hamiltonian-Hopf* bifurcation) only if they have opposite Krein signature. The proof essentially follows from the continuity of the signature; I will give a rough sketch only. Suppose two pairs of purely imaginary eigenvalues $\{\lambda_1, \bar{\lambda}_1\}$, $\{\lambda_2, \bar{\lambda}_2\}$ collide into one another (away from the origin) and bifurcate into the complex plane, thereby becoming a quartet $\{\pm\lambda, \pm\bar{\lambda}\}$. After collision, the form $\langle L|_{I_\lambda}\cdot, \cdot\rangle$ has an equal number of positive and negative squares; thus, prior to the collision, the form $\langle L|_K\cdot, \cdot\rangle$ evaluated on $K = I_{\lambda_1} \oplus I_{\lambda_2}$ must have the same property. This is only possible if the number of positive squares of one of the forms $\langle L|_{I_{\lambda_1}}\cdot, \cdot\rangle$ and $\langle L|_{I_{\lambda_2}}\cdot, \cdot\rangle$ equals the number of negative squares of the other, and vice versa; i.e. if λ_1 and λ_2 have opposite Krein signature.

The formula given by Kapitula *et al.* follows from a count of the eigenvalues in two related generalised eigenvalue problems satisfying a certain nonpositivity condition. These counts are given in terms of the Morse indices of the constrained selfadjoint operators,

$$\widehat{P} := n(\Pi L_+\Pi), \qquad \widehat{Q} := n\left((\Pi L_-\Pi)^{-1}\right) = n\left(\Pi L_-\Pi\right), \tag{1.57}$$

where Π is the orthogonal projection onto $(\ker(L_-) \oplus \ker(L_+))^\perp$, by virtue of an earlier result given by Grillakis in [Gri90, Theorem 2.3]. Formulas for the indices in (1.57) are given in terms of P, Q and the matrices D_+ and D_- defined in (1.55).

The following constrained eigenvalue counts, in addition to Theorem 1.7, would play a major role in the index theorems concerning the unstable eigenvalues of N. The equations in (1.58) follow from [Mad85, Theorem 2 and Lemma 6].

Lemma 1.10 (**[KKS04, Lemma 3.1]**). *If D is nondegenerate then*

$$n(\Pi L_+\Pi) = P - n(D_+), \qquad n(\Pi L_-\Pi) = Q - n(D_-). \tag{1.58}$$

This leads to the index theorem of Kapitula *et al.* Denote $k_r := n_+(N)$, let k_c be the number of complex quartets of eigenvalues with nonzero real and imaginary parts, and let k_i^- the number of Krein negative eigenvalues. Then:

Theorem 1.11 (**[KKS04, Theorem 3.3]**). *Assume D is invertible. Then*

$$k_r + 2k_c + 2k_i^- = n(L|_{X_1}) = n(L) - n(D) = P + Q - n(D_-) - n(D_+). \tag{1.59}$$

Note the second equality in (1.59) follows from Theorem 1.7. Kapitula *et al.* extended the first equality in (1.59) to the case when $N = JL$ does *not* have the canonical symplectic structure in [KKS05, Theorem 1]. Following the approach of [KP05], the trick was to embed the non-canonical case into the canonical case by setting $L_+ := L$ and $L_- := -JLJ$.

As a corollary to Theorem 1.11, Kapitula *et al.* determine a lower bound for $n_+(N)$.

Corollary 1.12 ([KKS04, **Remark 3.1**]). *Assume D is invertible. Then*

$$n_+(N) \geq |P - Q - n(D_+) + n(D_-)| \tag{1.60}$$

Furthermore, if $Q = n(D_-)$ then $k_c = k_i^- = 0$ and $n_+(N) = P - n(D_+)$.

The parity of the final two terms on the (far) left hand side of (1.59) indicates that if $n(L)-n(D)$ is odd, then $k_r \geq 1$, recovering Theorem 1.8. Formula (1.59) closes the inequality in Theorem 1.9 (which, in the notation introduced above, states that $k_r + k_c \leq P + Q$). Corollary 1.12 includes, as a special case, the earlier results by Jones and Grillakis: since $\widehat{P} = P - n(D_+)$, in the case that $n(D_-) = 0$ Theorem 1.6 is recovered.

Results similar to those found in Kapitula *et al.* were given by Pelinovsky and co-authors at around the same time, but under slightly more restrictive hypotheses. Pelinovksy [Pel05] considered the spectral problem associated with standing wave solutions of a system of N coupled nonlinear Schrödinger equations, for which L_+ and L_- are matrix-valued Schrödinger operators. While he never used the words "Krein signature", he obtained in [Pel05, Theorem 3.8] the index theorem of Kapitula *et al.* (Theorem 1.11) under the assumptions that: (1) the essential spectrum did not contain any semi- or embedded eigenvalues; (2) all nonzero eigenvalues were semisimple; and (3) D is nondegenerate, and $\dim\ker L_+ = 1$. He also obtained in [Pel05, Theorem 3.9] the lower bound of Corollary 1.12, as well as an upper bound on k_c. In [Pel05, Corollary 3.10] he obtained the exact count for $n_+(N)$ in (1.60) in the particular case when $Q = 0$, as well as stability of the standing wave when $Q = 0$, $P = n(D)$ on account of having $n_+(N) = k_c = 0$.

Theorem 1.11 can also be found in a paper by Cuccagna, Pelinovsky and Vougalter ([CPV05, Theorem 2.10, Corollary 2.12]) who studied standing waves in a three-dimensional NLS equation with a potential, as well as a work by Chugunova and Pelinovksy [CP10, Corollary 2.6], who considered the spectral problem (1.46)–(1.47) for abstract selfadjoint operators $L_\pm$.

The question of stability in the case when the matrix D is degenerate was investigated by Comech and Pelinovksy [CP03]. They considered an abstract Hamiltonian system with a one-dimensional symmetry group (as in [GSS87]), for which the linearised operator L has at most one negative eigenvalue and D is a scalar equal to zero. In the case of NLS standing waves, in this instance the VK criterion breaks down because the associated integral condition is zero. Under some additional assumptions to those in [GSS87], they prove the nonlinear instability of the underlying standing wave, despite the wave being spectrally stable, i.e. its spectrum being confined to the imaginary axis. The nonlinear instability results from the zero eigenvalue of N having a higher algebraic multiplicity, and the associated solutions growing algebraically in time. In both Chapters 2 and 3 instances of degeneracy of D corresponding to a higher algebraic multiplicity of $0 \in \mathrm{Spec}(N)$ will be observed. This will be encoded in the Maslov index via degeneracy of the second-order Maslov crossing form. (For definitions of these terms, see Chapter 2).

While the index theorem (1.59) was a significant advancement in the spectral theory of N, providing the first closed formula involving counts of unstable eigenvalues, it does not give a complete answer to the stability question. Indeed, it remains to be able to compute the count $k_r + 2k_c$ of eigenvalues lying off the imaginary axis directly. In special cases (see Corollary 1.12) it might be possible to compute the terms k_r and k_c individually, but in general this remains an open question. For k_r, one has the lower bound (1.60) handy; for stability this is only useful if the right hand side is nonzero. It would be useful to be able to compute k_i^- and use (1.59) to determine $k_r + 2k_c$. Alas, there is no straightforward algorithm to do so, and the appearance of the negative Krein index k_i^- is thus a serious impediment to determining stability. A notable development in this direction was given in [Kap10], where Kapitula constructed an object called the *Krein matrix*, the zeros of the determinant of which coincide with the eigenvalues making up the count $k_c + 2k_i^-$. However, like many of the objects relevant in this theory, the construction is functional analytic in nature and in general this matrix is difficult to compute in practice.

A few years on from Kapitula *et al.*, Hărăguş and Kapitula [HK08] gave a closed formula solely for $k_r = n_+(N)$ in the case when L is invertible with compact inverse. In particular, they showed that

$$k_r + 2k_c + 2k_i^- = n(L), \tag{1.61}$$

where the right hand side is given by $n(L) = P + Q$ when N has the canonical symplectic structure. In this case, formula (1.61) agrees with (1.59) upon noticing that $D_\pm$ is a square matrix of dimension $\dim\ker L_\pm$, so that $n(D_-) = n(D_+) = 0$ if L_+ and L_- are invertible. In fact, it is possible to recover (1.59) from (1.61) in the case when L is *not* invertible. Namely, recognising that nonzero eigenvalues of (1.46)–(1.47) and the equivalent generalised eigenvalue problem

$$\Pi L_+ \Pi u = \lambda v, \qquad -\Pi L_- \Pi v = \lambda u, \tag{1.62}$$

are in one-to-one correspondence, where $\Pi L_+ \Pi$ and $\Pi L_- \Pi$ are selfadjoint and invertible, formula (1.61) then reads

$$k_r + 2k_c + 2k_i^- = n(\Pi L_+ \Pi) + n(\Pi L_- \Pi), \tag{1.63}$$

which is exactly (1.59) upon using (1.58).

In order to give a closed formula for k_r, Hărăguş and Kapitula first gave a closed formula for $k_c + k_i^-$. The *negative cone* of a selfadjoint operator S is the set

$$\mathcal{C}(S) = \{u : \langle Su, u\rangle < 0\} \cup \{0\},$$

and its dimension $\dim \mathcal{C}(S)$ is the dimension of a maximal linear subspace contained in $\mathcal{C}(S)$. Their result was the following.

Theorem 1.13 (**[HK08, Corollary 2.26, Remark 2.27]**). *If all positive real eigenvalues of N are semisimple, i.e. their algebraic and geometric multiplicities coincide, then*

$$k_c + k_i^- = \dim\left(\mathcal{C}(L_+) \cap \mathcal{C}(L_-^{-1})\right), \tag{1.64}$$

which, in conjunction with (1.61) yields

$$k_r = |P - Q| + 2\left(\min\{P, Q\} - \dim\left(\mathcal{C}(L_+) \cap \mathcal{C}(L_-^{-1})\right)\right). \tag{1.65}$$

On the other hand, if all real eigenvalues of N are not semisimple, then one only has the weaker statements

$$k_c + k_i^- \leq \dim\left(\mathcal{C}(L_+) \cap \mathcal{C}(L_-^{-1})\right), \tag{1.66}$$

$$k_r \geq |P - Q| + 2\left(\min\{P, Q\} - \dim\left(\mathcal{C}(L_+) \cap \mathcal{C}(L_-^{-1})\right)\right). \tag{1.67}$$

Noting that $P = \dim \mathcal{C}(L_+)$ and $Q = \dim \mathcal{C}(L_-^{-1})$, an immediate consequence of either (1.65) or (1.67) is that $k_r \geq |P - Q|$, which again recovers the lower bounds of Grillakis and Jones in this case (i.e. when L is invertible). The proof is similar to those presented in [Pel05, CPV05], however does away with the assumption of semisimplicity of the nonzero eigenvalues in [Pel05, Assumption 2.14]. Note as well that the formulas and inequalities in Theorem 1.13 extend to the case when L_+ and L_- are not invertible by projecting off the kernels of L_+ and L_-: for $\lambda \neq 0$ the eigenvalue problem (1.46)–(1.47) is equivalent to the eigenvalue problem

$$\Pi L_+ \Pi u = \lambda v \qquad -\Pi L_- \Pi v = \lambda u. \tag{1.68}$$

In this case, the results of Theorem 1.13 then apply by replacing any instance of L_+ and L_- with $\Pi L_+ \Pi$ and $\Pi L_- \Pi$ respectively.

The exact count for $n_+(N)$ given by Hărăguş and Kapitula, while another development in understanding the spectrum of N, is limited by the semisimplicity assumption of all real positive eigenvalues. In addition, there appears to be no obvious way of numerically computing the dimension of the maximal linear subspace contained in the intersection of the negative cones of $\Pi L_+ \Pi$ and $(\Pi L_- \Pi)^{-1}$.

I shall mention one final work. Kapitula and Promislow [KP12] gave a shortened proof of the following abstract result regarding the Morse index of a constrained operator, the first version of which appeared in [CPV05, Lemma 3.4] for L invertible and then in [CP10, Proposition 2.2].

Theorem 1.14 (**[KP12, Theorem 2.1]**)**.** *Let L be a selfadjoint operator acting in a Hilbert space X, and let S be a finite-dimensional closed subspace of H such that $S \subseteq \ker L^\perp$, $\dim S = m$. Let Π be the orthogonal projection onto the finite codimensional subspace $S^\perp$. Let $\{\phi_1, \ldots, \phi_m\}$ be a basis for S, and define the matrix D via*

$$D_{ij} = \langle \phi_i, L^{-1}\phi_j \rangle. \tag{1.69}$$

Then for the operator $\Pi L \Pi : S^\perp \mapsto S^\perp$, we have

$$n(\Pi L \Pi) = n(L) - n(D) - \dim\ker(D). \tag{1.70}$$

and in addition

$$\dim\ker(\Pi L \Pi) = \dim\ker(L) + \dim\ker(D) \tag{1.71}$$

Theorem 1.14 can be used to recover Theorem 1.7 and Lemma 1.10. Note that the matrix D defined by (1.69) is the same D from (1.55) provided $L = L_+ \oplus L_-$ is diagonal, in which case the sets of functions $\{\phi_i\}$ and $\{u_i, v_i\}$ coincide and are a basis for $\ker(L_-) \oplus \ker(L_+)$. Using Theorem 1.14 and an analysis of the Krein eigenvalues – the eigenvalues of the *Krein matrix* formulated in [Kap10] – Kapitula and Promislow gave a new proof of the lower bound (1.60) for $n_+(N)$.

One of the main outcomes of the analysis of the count $n_+(N) = k_r$ in this thesis is an alternate form of the lower bound (1.60). This new lower bound yields a geometric characterisation of the "correction term" $n(D_-) - n(D_+)$ in (1.60), in terms of the *eigenvalue curves* of the operator N. This latter interpretation offers a straightforward means of computing $n(D_-) - n(D_+)$ numerically, which involves plotting the zero set of a two-dimensional function in the plane. This avoids having to explicitly construct the matrices $D_\pm$, which involve solutions to the inhomogeneous differential equations in (1.54).

Another outcome is an interpretation of the degeneracy of D in terms of the Maslov index. (Note the index formulas of Kapitula *et al.*, as well as the results of Grillakis, Shatah and Strauss, are not restricted to the case when D is degenerate; for example, see Theorem 1.7). Namely, degeneracy of D corresponds to degeneracy of the second-order Maslov crossing form. In such cases, the earlier geometric interpretation yields a way to compute the correction factor appearing in the lower bound (1.60) (which will now contain an extra term involving the dimension of the kernel of D). Alternatively, the machinery of Chapter 3 can be used to compute the correction factor in terms of the signature of higher-order crossing forms.

Author's note: The following two chapters are the result of collaborative work with several co-authors. In light of this, those chapters will be narrated in the first person plural.

Chapter 2

A second-order Hamiltonian system on a compact interval

We use the Maslov index to study the real spectrum of Hamiltonian differential operators of the form

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix},$$

where $L_\pm$ are scalar-valued Schrödinger operators with arbitrary C^2 potentials on a compact interval $[0, \ell]$. In particular, we provide a lower bound on the number of positive real eigenvalues of the operator N (Theorem 2.2).

Our approach is to restrict N to a subinterval $[0, s\ell]$, $s \in (0, 1]$, and, rescaling back to $[0, \ell]$, study the s-dependent spectrum of the one-parameter family of operators in the spatial parameter s. We are thus led to a characterisation of the eigenvalues of the rescaled operators as a locus of points in the λs-plane (with λ the spectral parameter), which we refer to as *eigenvalue curves*. We interpret the eigenvalue curves as loci of intersections, or *crossings*, of a path in the manifold of Lagrangian planes with a certain codimension one subvariety. This affords the use of the Maslov index, a signed count of such crossings. Formulas for the concavity of the eigenvalue curves are given (Theorems 2.9, 2.40 and 2.41), and are used to compute a correction term appearing in the lower bound in Theorem 2.2.

Operators of the form of N arise in the linearisation about a standing wave solution $\widehat{\psi}(x, t) = e^{i\beta t}\phi(x)$ of the nonlinear Schrödinger (NLS) equation

$$i\psi_t = \psi_{xx} + f\left(|\psi|^2\right)\psi, \tag{2.1}$$

where $\psi : [0, \ell] \times [0, \infty) \longrightarrow \mathbb{C}$, the nonlinearity $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a C^3 function and $\beta \in \mathbb{R}$ is the temporal frequency. The wave around which we linearise is said to be *spectrally unstable* if there exists spectrum of N in the open right half plane, and *spectrally stable* otherwise. By applying Theorem 2.2, we establish stability criteria for standing waves in the NLS equation on a compact interval subject to perturbations satisfying Dirichlet boundary conditions. Namely, we derive analogues of the *Jones–Grillakis instability theorem* (Corollary 2.7) and the *Vakhitov–Kolokolov* (*VK*) *criterion* (Theorem 2.11). While Corollary 2.7 is also a consequence of the abstract result of [KP12, Theorem 3.2], Theorem 2.11, which makes use of the concavity formulas of Theorem 2.9, appears to be new for the case of the compact interval. These two stability results actually remain valid for a spatially dependent nonlinearity $f(x, |\psi|^2)$; see Remark 2.6.

Along the way, we find *Hadamard-type* formulas for the slope of the eigenvalue curves as the ratio of certain quadratic forms, called *crossing forms*, whose signatures locally determine the Maslov index (Proposition 2.37 and Corollary 2.39). Variational formulas for the eigenvalues of boundary value problems with respect to perturbation of the domain are classical and go back to the work of Hadamard [Had68], Rayleigh [Ray45] and Rellich [Rel69]; see also [Hen05, Gri10] and [Kat80, §VII.6.5]. Recently such formulas have been given in terms of the (Maslov) crossing form for families of Schrödinger [LS17, LS20b] and abstract selfadjoint operators [LS20a]. Our formulas agree with and build on those found therein.

We also encounter a *non-regular* crossing when $\lambda = 0$, corresponding to a degeneracy of the associated crossing form and points of zero slope for the eigenvalue curves. Geometrically, this corresponds to the Lagrangian path tangentially intersecting the relevant codimension one subvariety. Some care is then required in order to compute the Maslov index, and it is a key feature of the current work that we are able to do so (Theorem 2.49). In particular, it is sufficient to know the concavity of the eigenvalue curve through the non-regular crossing, as well as whether or not the operators L_+ and L_- have a nontrivial kernel. To the best of our knowledge, no such computation has previously been made in the literature. Analysing the non-regular crossing in the context of the NLS equation leads to stability criteria that resemble the VK criterion in certain cases, furnishing an interesting connection between the concavity of the eigenvalue curve at the non-regular crossing, the Maslov index there, and the classical VK result; see Section 2.4.

In the case when the spatial domain is the entire real line, if zero is a hyperbolic fixed point of the standing wave equation

$$\phi_{xx} + f(\phi^2)\phi + \beta\phi = 0 \tag{2.2}$$

and there exists an orbit that is homoclinic to it in the phase plane, a localised solution to (2.1) exists and belongs to $L^2(\mathbb{R})$ for all time. In this case L_+ and L_-, which are unbounded operators on $L^2(\mathbb{R})$, both have a nontrivial kernel. Indeed, the stationary state ϕ and its derivative ϕ_x satisfy $L_-\phi = 0$ (the stationary equation (2.2)) and $L_+\phi_x = 0$ (the associated variational equation) respectively, and decay exponentially as $x \to \pm\infty$. By the results of Jones [Jon88] and Grillakis [Gri88], one then has that if $P - Q \neq 0, 1$, where P and Q are the numbers of negative eigenvalues (or *Morse indices*) of L_+ and L_-, then N has at least one positive real eigenvalue, and hence the standing wave solution to (2.1) is unstable. In the edge case when $P = 1$ and $Q = 0$, the results of Vakhitov and Kolokolov [VK73] and Grillakis, Shatah and Strauss [GSS87, GSS90] dictate that the wave is spectrally (and orbitally) stable if the β-derivative of the mass of the wave

$$\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} \phi^2\, dx, \tag{2.3}$$

is negative, and spectrally unstable if (2.3) is positive (see [Pel11, Theorem 4.4, p.215]).

One of the key differences upon passing from the real line to the compact interval is that, generically, the operators L_+ and L_- (equipped with Dirichlet boundary conditions) do not simultaneously have a nontrivial kernel. Depending on the boundary conditions satisfied by the wave profile ϕ, typically zero will lie in the spectrum of either L_+ or L_- (or neither). A physical reason for this is the loss of translational invariance, which manifests in the failure of the relevant boundary conditions of arbitrary translates of ϕ. As a consequence, our stability results (Corollary 2.7 and Theorem 2.11) will differ depending on which of the operators $L_\pm$ has a nontrivial kernel. In the case that L_- has a nontrivial kernel, we can recover the integral expression (2.3) appearing in the classical VK criterion. Such a recovery is not possible when L_+ has a nontrivial kernel; for details, see the discussion in Section 2.4.3.2.

There is a large body of work relating the Morse index of a selfadjoint operator and its number of conjugate points (which was later interpreted as the Maslov index of an associated Lagrangian path), going back to the middle of last century [Arn67, Arn85, Bot56, Dui76, Edw64, Sma65]. Most of these theorems can be viewed as generalisations of the classical Sturmian theory, and indeed in [Bot56, Edw64, Sma65] they are framed as such, where the nodal count of an eigenfunction indicates where in the sequence of eigenvalues the corresponding eigenvalue sits. Following on from Jones' seminal work [Jon88], the idea of using the Maslov index for spatially Hamiltonian systems to extrapolate temporal spectral information has proven quite fruitful in the ensuing years (see, for example, [JLM13, CJLS16, CJM15, HS16, HLS18, LS18] and the references therein for a partial list of results).

In more recent times, Deng and Jones in [DJ11] (see also [CJLS16, CJM15]), used the Maslov index to analyse second-order elliptic eigenvalue problems on bounded domains. An important feature of this analysis, as well as that of [BCJ+18, HS16, HLS18, HS22, HJK18], is monotonicity of the Maslov index in the spectral parameter. Monotonicity also holds in the spatial parameter under certain boundary conditions [CJLS16, HLS17, JLM13]. This property is convenient since it enables an *equality* of the Morse index with the Maslov index of the Lagrangian path corresponding to $\lambda = 0$. Importantly, as in [Jon88], we do *not* have monotonicity in either the spatial or the spectral parameter. However, the signature of crossings in the s-direction when $\lambda = 0$ can always be accounted for, and, consequently, a nonzero Maslov index can nonetheless be used to detect a real, unstable eigenvalue, just as in [MSJ10, MSJ12, JMS14, RMS20]. This lack of monotonicity thus leads to the *inequality* in Theorem 2.2.

Another feature in the aforementioned references, as well as in [BJ95, CH07, CH14, CDB09a, CDB09b, CDB11, Cor19, CJ18, CJ20, How23] is a dynamical systems approach to eigenvalue problems. In these works, the eigenvalue equations associated with the linearised operators are Hamiltonian, or can be made Hamiltonian under a suitable change of variables. The critical feature of such systems is that they induce a symplectically invariant flow and hence preserve the manifold of Lagrangian planes, which affords the application of the Maslov index. For recent works where the Hamiltonian requirement is relaxed, see [Cor19, CJ18, CJ20]. In [CJ18, CJ20], a change of variables is used to recover the Hamiltonian structure, and in [Cor19] the system, while not Hamiltonian, still preserves the space of Lagrangian planes. For an example of where the Hamiltonian requirement is dropped altogether, see [BCC+22].

Existing results on the stability of standing wave solutions of (2.1) on a compact spatial interval have been given for periodic solutions of (2.2), with (quasi)periodic perturbations, and predominantly for cubic focusing ($f(\phi^2) = \phi^2$) or defocusing ($f(\phi^2) = -\phi^2$) NLS. Rowlands in [Row74] studied the spectral stability of spatially periodic elliptic solutions to the cubic NLS, subject to long wavelength disturbances. Pava [Pav07] showed that the Jacobi dnoidal solutions to cubic focusing NLS were orbitally stable with respect to co-periodic perturbations. In [GH07a], Gallay and Hărăgus showed the orbital stability of spatially periodic and quasiperiodic travelling waves with complex-valued profile for small amplitude solutions in both the focusing and defocusing case. They extended this result to waves of arbitrary amplitude in [GH07b]. For the real-valued (cnoidal) waves, their orbital stability result is restricted to perturbations that are anti-periodic on a half period. This latter condition was done away with in [IL08], wherein Ivey and Lafortune undertook a spectral stability analysis of the cnoidal travelling wave solutions of the focusing NLS, showing stability with respect to co-periodic perturbations. In [BDN11, GP15] the authors extend the orbital stability results for both real- and complex-valued wave profiles to the class of subharmonic perturbations (i.e. perturbations with period an integer multiple of the period of the wave profile) in the defocusing case. In [DS17, DU20] the authors examine the spectral stability of the elliptic solutions with respect to subharmonic perturbations in the focusing case. Unlike the above works, we are interested in the spectral stability of real-valued solutions of (2.2), for an arbitrary C^3 nonlinearity f, that are subject to perturbations satisfying Dirichlet boundary conditions. Moreover, as previously stated, many of our results hold for a spatially dependent f.

Our theory can be extended in several possible directions. In particular, our theory should hold for the case of quasi-periodic boundary conditions on the perturbations, which is natural to consider given that many of the solutions ϕ to (2.2) that satisfy Dirichlet boundary conditions are periodic. The Maslov index has already been used to develop eigenvalue counts for selfadjoint matrix-valued Schrödinger operators with such boundary conditions in [JLM13, JLS17]. Our theory should also hold when the Schrödinger operators $L_\pm$ are selfadjoint and matrix-valued, and indeed in Sections 2.2 and 2.3 many of our results are stated for the operator N with an n-dimensional kernel to accommodate this scenario. Finally, while the analysis is significantly more involved, it should be possible to extend to the case where the spatial domain is multidimensional, as in [CJM15, CJLS16, CM19].

The paper is organised as follows. In Section 2.1 we set up the eigenvalue problem and state the main results. In Section 2.2 we provide background material on the Maslov index, interpret the (real) eigenvalue problem symplectically and prove Theorem 2.2. In Section 2.3 we analyse the eigenvalue curves. After computing formulas for their derivatives and relating these to the Maslov crossing forms (Proposition 2.37 and Corollary 2.39), we compute their concavities at the zero eigenvalue (Theorems 2.40 and 2.41), facilitating the computation of the Maslov index at the non-regular crossing (Theorem 2.49). We conclude the section by confirming that the signature of the *second*-order Maslov crossing form provides the correct contribution to the Maslov index at this crossing, which is consistent with [DJ11]. In Section 2.4 we provide some applications of Theorems 2.2 and 2.9. In particular, we prove Corollaries 2.7 and 2.8 and Theorem 2.11. We also compute expressions for the concavity (at $s = 1$) of the eigenvalue curve passing through $(\lambda, s) = (0, 1)$ for linearised NLS, in each of the cases when L_+ and L_- has a nontrivial kernel (Propositions 2.53 and 2.57). In the latter case, we recover a compact-interval analogue of the classical VK criterion. We conclude the paper with a comparison of the lower bound in Theorem 2.2 with existing results which make use of constrained eigenvalue counts. We find that the "correction" terms appearing in our lower bound and others in the literature are equivalent (Proposition 2.61), applying our formulas to provide new versions of the Hamiltonian–Krein index theorem in terms of the Maslov index (Proposition 2.62).

Notation: We let I_n and 0_n denote the $n \times n$ identity and zero matrices respectively. We denote the canonical $2n \times 2n$ symplectic matrix and the first Pauli matrix by

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.4}$$

respectively. We let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the L^2 inner product and norm, respectively. Subscripts s or λ will indicate dependence of a quantity on these parameters (not derivatives). The spectrum of a linear operator T will be denoted by $\mathrm{Spec}(T)$, and its kernel by $\ker(T)$.

2.1 Set-up and statement of main results

The basic set-up is an eigenvalue problem of the form

$$N \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u(\ell) \\ v(\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.5}$$

where N is given by

$$N := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix} \tag{2.6}$$

and $L_\pm$ are the Schrödinger operators

$$L_+ = -\partial_{xx} - g(x), \qquad L_- = -\partial_{xx} - h(x), \tag{2.7}$$

with g and h arbitrary functions in $C^2([0, \ell], \mathbb{R})$. To be precise, we consider N as an unbounded operator in $L^2(0, \ell) \times L^2(0, \ell)$ with dense domain

$$\mathrm{dom}(N) = \big(H^2(0, \ell) \cap H_0^1(0, \ell)\big) \times \big(H^2(0, \ell) \cap H_0^1(0, \ell)\big) \subset L^2(0, \ell) \times L^2(0, \ell). \tag{2.8}$$

Hereafter, we drop the product notation on the relevant spaces; it will be clear from the context whether the functions are scalar- or vector-valued. An *eigenvalue* of N is thus a value of $\lambda \in \mathbb{C}$ for which there exists a nontrivial solution $\mathbf{u} := (u, v)^\top$ to the boundary value problem (2.5). Eigenvalues for the unbounded operators $L_\pm$, with dense domains

$$\mathrm{dom}(L_\pm) = H^2(0, \ell) \cap H_0^1(0, \ell) \subset L^2(0, \ell), \tag{2.9}$$

are similarly defined. Note that the unbounded operators $L_\pm = L_\pm^*$ with domain (2.9) are selfadjoint, while N is not.

Remark 2.1. Notationally, we will not distinguish between the formal differential expressions N and $L_\pm$ and the unbounded operators with domains (2.8) and (2.9) whose spectra we wish to study. It will be clear from the context in what sense we refer to these objects.

While it is possible for N to have complex eigenvalues, we will restrict our analysis of (2.5) to the case when λ is real and positive. The existence of such an eigenvalue implies instability. On the other hand, there are cases where the spectrum of N lies entirely on the real and imaginary axes, in which case the absence of a real positive eigenvalue implies stability; see Theorem 2.11 for an example.

Our first result is a lower bound for the number of positive real eigenvalues of N. It follows from an application of the Maslov index. The idea is to study the spectral problem in (2.5) via a rescaling of the domain. We restrict (2.5) to a family of subdomains $[0, s\ell]$ using a parameter $s \in (0, 1]$,

$$N\mathbf{u} = \lambda\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}(s\ell) = 0, \tag{2.10}$$

and define a *conjugate point* to be a value of s for which there exists a nontrivial solution to (2.10) with $\lambda = 0$. We then deduce the existence of unstable eigenvalues of (2.5) by counting conjugate points (via the Maslov index) as s varies from 0 to 1. Defining the quantities

$$\begin{aligned} P &:= \#\{\text{negative eigenvalues of } L_+\}, \\ Q &:= \#\{\text{negative eigenvalues of } L_-\}, \\ n_+(N) &:= \#\{\text{positive real eigenvalues of } N\}, \end{aligned}$$

we have:

Theorem 2.2. *Let N be an operator as in (2.6)–(2.7). The number of positive real eigenvalues of N satisfies*

$$n_+(N) \geq |P - Q - \mathfrak{c}|, \tag{2.11}$$

where $\mathfrak{c}$ (given in Definition 2.26) is the total contribution to the Maslov index in the s and λ directions from the conjugate point at $s = 1$. (If there is no such conjugate point, $\mathfrak{c} = 0$.)

Remark 2.3. One of the main results of this paper is that we are able to give explicit formulas for this so-called "corner term" $\mathfrak{c}$ which has the property that $\mathfrak{c} \in \{-1, 0, 1\}$. The name derives from the location of the associated crossing in terms of the so-called *Maslov box*. For precise statements see Sections 2.2 and 2.3, in particular Theorem 2.49.

Remark 2.4. In (2.10) the symbol N denotes a differential expression. For the associated unbounded operator we define

$$N|_{[0,s\ell]}\mathbf{u} := N\mathbf{u}, \qquad \mathbf{u} \in \operatorname{dom}(N|_{[0,s\ell]}) = H^2(0, s\ell) \cap H_0^1(0, s\ell) \subset L^2(0, s\ell), \tag{2.12}$$

so that $\lambda \in \operatorname{Spec}(N|_{[0,s\ell]})$ if and only if (2.10) has a non-trivial solution.

Theorem 2.2 (the proof of which is given in Section 2.2.4) is in the spirit of a number of lower bounds in the literature. In contrast to [HK08, Assumption 2.1(b)], we do not assume that the operators $L_\pm$ are invertible. If both L_+ and L_- are invertible, it will follow that there is no conjugate point at $s = 1$, and therefore $\mathfrak{c} = 0$. In this case we recover the inequality in [HK08, Theorem 2.25]. The lower bound for $n_+(N)$ in the case when one or both of L_+ and L_- has a nontrivial kernel has been studied in [KP12, Thm 3.2], [KM14, Thm 5.6], [LZ22, Thm 2.3] and [Gri88, Thm 1.2], to name a few; see also [KP13, §7.1.3]. In these works, the authors typically project off the kernels of L_+ and L_-, and give the lower bound in terms of the associated constrained eigenvalue counts for L_+ and L_-. By contrast, we require no such projections. The constrained counts for L_+ and L_- (given in the current work in (2.170)) involve the number of negative eigenvalues of certain matrices denoted $D_\pm$. In Section 2.4.4, we will show that our "correction" factor – given by the corner term $\mathfrak{c}$ – is equivalent to the "correction" factor in [KP13, Theorem 7.1.16], given by the difference $n_-(D_+) - n_-(D_-)$ of negative indices of D_+ and D_- (see Proposition 2.61). Thus, Theorem 2.2 together with Proposition 2.61 recovers [KP13, Theorem 7.1.16]. The Maslov index interpretation afforded by $\mathfrak{c}$ is convenient because it provides a way of *computing* the difference $n_-(D_+) - n_-(D_-)$. Namely, (2.177) shows that the signs of $D_\pm$ (which in our set-up are scalars) are given by the signs of the concavities of the eigenvalue curves at $(\lambda, s) = (0, 1)$.

Our main application will be to the linearisation of (2.1) about a *standing wave* solution. This is a solution to (2.1) of the form $\widehat{\psi}(x,t) = e^{i\beta t}\phi(x)$ for some $\beta \in \mathbb{R}$, where the real-valued wave profile or *stationary state* $\phi : [0, \ell] \to \mathbb{R}$ solves the time-independent equation

$$\phi_{xx} + f(\phi^2)\phi + \beta\phi = 0. \tag{2.13}$$

The results of this paper hold under fairly general boundary conditions on ϕ. Two examples that we will often focus on are Dirichlet conditions

$$\phi(0) = \phi(\ell) = 0, \tag{2.14}$$

or Neumann conditions

$$\phi'(0) = \phi'(\ell) = 0. \tag{2.15}$$

In these cases, one possible choice for the interval length ℓ is to fix a T-periodic solution to (2.13), and to set $\ell = kT/2$ for some $k \in \mathbb{N}$. Some example phase portraits for (2.13) featuring periodic orbits are given in Fig. 2.1. As an aside, note that the homoclinic orbits in Fig. 2.1a correspond to strictly positive or negative localised solutions on $\mathbb{R}$.

A natural question to ask is whether the standing wave $\widehat{\psi}$ is stable in time with respect to small perturbations in ϕ. Substituting the perturbative solution

$$\psi(x,t) = e^{i\beta t}\left[\phi(x) + \varepsilon e^{\lambda t}(u(x) + iv(x))\right]$$

into (2.1) and collecting $O(\varepsilon)$ terms, we arrive at the differential equations in (2.5), where

$$\begin{aligned} g(x) &= 2f'(\phi^2(x))\phi^2(x) + f(\phi^2(x)) + \beta, \\ h(x) &= f(\phi^2(x)) + \beta. \end{aligned} \tag{2.16}$$

Then, subject to the class of perturbations $\mathbf{u} = (u, v)^\top$ that vanish at both endpoints, the standing wave $\widehat{\psi}$ is spectrally stable if the spectrum of the linearised operator N is contained in the imaginary axis, since the eigenvalues of N are symmetric with respect to the real and imaginary axes.

When $\lambda = 0$ the differential equations in (2.5) decouple into two independent equations: $N\mathbf{u} = 0$ if and only if $L_+u = 0$ and $L_-v = 0$. Thus $\ker(N) = \ker(L_+) \oplus \ker(L_-)$, and $0 \in \operatorname{Spec}(N)$ if and

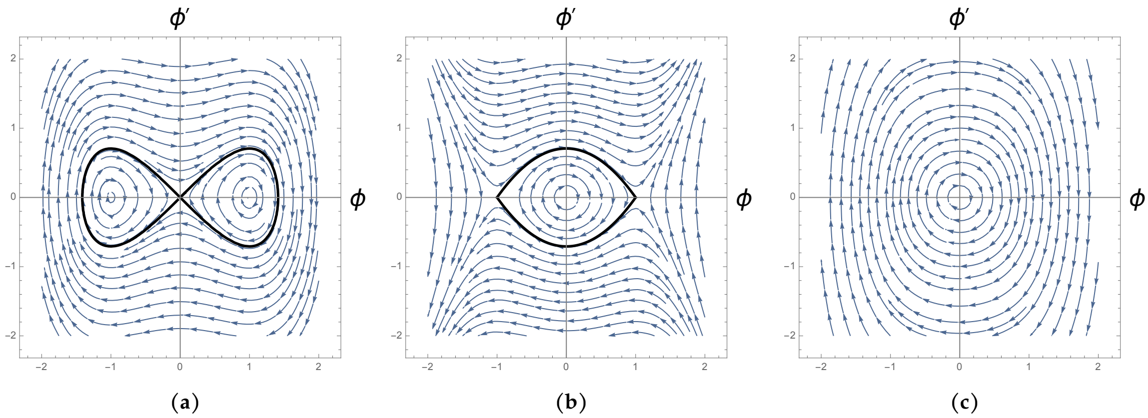


Figure 2.1: Examples of phase portraits for equation (2.13). In (a) we have cubic focusing nonlinearity $f(\phi^2) = \phi^2$ and $\beta < 0$. The homoclinic orbits in black, representing localised solutions on $\mathbb{R}$, separate those inside (nonzero Jacobi dnoidal functions) and those outside (Jacobi cnoidal functions that oscillate evenly about $\phi = 0$). In (b) we have cubic defocusing nonlinearity $f(\phi^2) = -\phi^2$ and $\beta > 0$, with periodic orbits existing only inside the heteroclinic cycle in black. In (c) we have $f(\phi^2) = \phi^2$ and $\beta > 0$.

only if $0 \in \operatorname{Spec}(L_+) \cup \operatorname{Spec}(L_-)$. Furthermore, because the eigenvalues of the Sturm-Liouville operators $L_\pm$ are simple,

$$\begin{aligned} \dim\ker(N) = 1 &\iff 0 \in \operatorname{Spec}(L_-) \triangle \operatorname{Spec}(L_+), \\ \dim\ker(N) = 2 &\iff 0 \in \operatorname{Spec}(L_-) \cap \operatorname{Spec}(L_+), \end{aligned} \tag{2.17}$$

where $A \triangle B := A \cup B \setminus A \cap B$ denotes the symmetric difference. In our application to the stability of standing waves of (2.1), note that (2.13) is equivalent to $L_-\phi = 0$, while autonomy of this equation yields $L_+\phi' = 0$. The boundary conditions satisfied by ϕ therefore influence whether $0 \in \operatorname{Spec}(L_\pm)$. For instance, if ϕ satisfies the Dirichlet conditions (2.14), then $0 \in \operatorname{Spec}(L_-)$ with eigenfunction ϕ, whereas if ϕ satisfies the Neumann conditions (2.15), then $0 \in \operatorname{Spec}(L_+)$ with eigenfunction ϕ', provided ϕ is nonconstant. It is also possible that $0 \notin \operatorname{Spec}(L_+) \cup \operatorname{Spec}(L_-)$ if, for example, more general Robin boundary conditions are imposed on ϕ.

In any of these cases, that L_+ and L_- have nontrivial kernel simultaneously is nongeneric, and so we make this an assumption when studying the stability of NLS standing waves. We stress that the general set-up of the paper is given by (2.5)–(2.7), and the following hypothesis is *not* assumed throughout; we will explicitly state whenever we make use of it.

Hypothesis 2.5. N is of the form (2.6)–(2.7), where

(i) the potentials g and h come from the linearisation of the NLS equation (2.1) about a standing wave $\widehat{\psi}$ (and hence are given by (2.16)), and

(ii) $0 \notin \operatorname{Spec}(L_-) \cap \operatorname{Spec}(L_+)$.

Remark 2.6. With g and h arbitrary functions of x in general, the results of this paper concerning the stability of NLS standing waves are valid for a spatially dependent nonlinearity $f(x, |\psi|^2)$ as appearing in, for example, [Jon88, Gri88]. In this case, the loss of autonomy in the standing wave equation (2.13) means that $L_+\phi' \neq 0$; thus, only the results which rely on ϕ' being an eigenfunction for L_+ (Corollary 2.8, Proposition 2.53 and Corollary 2.55) do not generalise to the non-autonomous case.

Under the assumptions of Hypothesis 2.5, our analogue of the Jones–Grillakis instability theorem will follow from both Theorem 2.2 and a computation of the values of $\mathfrak{c}$ given in Theorem 2.49.

Corollary 2.7. *Let N be an operator as in (2.6)–(2.7). If $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ and $P - Q \neq -1, 0$, or $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ and $P - Q \neq 0, 1$, then $n_+(N) \geq 1$. Under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable in these cases.*

(The proof is given in Section 2.4.1.) This criterion leads to the following instability result. The waves described correspond, for example, to the periodic orbits represented by the phase curves that are contained inside either of the orbits homoclinic to $(0, 0)$ in Fig. 2.1a.

Corollary 2.8. *Assume Hypothesis 2.5. Standing waves satisfying the Neumann boundary conditions (2.15) that are nonconstant and nonvanishing over $[0, \ell]$, and have one or more critical points in $(0, \ell)$, are unstable.*

(The proof is given in Section 2.4.1.) To effectively use Theorem 2.2, we need to understand the quantity $\mathfrak{c}$ appearing in (2.11). Its definition involves the Maslov index at a potentially degenerate crossing, and hence requires some work to calculate. We do this by analysing the curves in the λs-plane that describe the evolution of the real eigenvalues λ of the restricted problem (2.10) as s is varied. As will be seen in Theorem 2.49, $\mathfrak{c}$ is determined by the concavity of these curves. Below, dot denotes $d/d\lambda$. The proof of the following theorem is given in Section 2.3.2.

Theorem 2.9. *Let N be an operator as in (2.6)–(2.7). If $\dim \ker(N) = 1$, then there exists a smooth function $s(\lambda)$, defined for $|\lambda| \ll 1$, such that $s(0) = 1$ and λ is an eigenvalue of (2.10) on $[0, s(\lambda)\ell]$. Moreover, $\dot{s}(0) = 0$ and the concavity of $s(\lambda)$ can be determined as follows:*

1. *If $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ with eigenfunction $v \in \ker(L_-)$, then*

$$\ddot{s}(0) = \frac{2}{\ell} \frac{\langle \widehat{u}, v \rangle}{(v'(\ell))^2} \tag{2.18}$$

   where $\widehat{u} \in H^2(0, \ell) \cap H_0^1(0, \ell)$ is the unique solution to $L_+ \widehat{u} = v$.

2. *If $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ with eigenfunction $u \in \ker(L_+)$, then*

$$\ddot{s}(0) = -\frac{2}{\ell} \frac{\langle \widehat{v}, u \rangle}{(u'(\ell))^2} \tag{2.19}$$

   where $\widehat{v} \in H^2(0, \ell) \cap H_0^1(0, \ell)$ is the unique solution to $-L_- \widehat{v} = u$.

Remark 2.10. In applications, we will primarily be interested in the *sign* of $\ddot{s}(0)$, for which (2.18) and (2.19) give

$$\operatorname{sign} \ddot{s}(0) = \operatorname{sign} \int_0^\ell \widehat{u}\, v \, dx \qquad \text{and} \qquad \operatorname{sign} \ddot{s}(0) = -\operatorname{sign} \int_0^\ell \widehat{v}\, u \, dx, \tag{2.20}$$

respectively. The integrals in (2.20) can be rewritten as

$$\int_0^\ell \widehat{u}\, v \, dx = \int_0^\ell \widehat{u} \, (L_+ \widehat{u}) \, dx \qquad \text{and} \qquad \int_0^\ell \widehat{v}\, u \, dx = \int_0^\ell \widehat{v} \, (L_- \widehat{v}) \, dx. \tag{2.21}$$

Consequently, $\ddot{s}(0) > 0$ if $0 \in \mathrm{Spec}(L_-)$ and L_+ is a strictly positive operator, or if $0 \in \mathrm{Spec}(L_+)$ and L_- is strictly positive.

In Section 2.3 we will prove a more general version of Theorem 2.9; see Theorem 2.40. An analogous result for the case when $\dim\ker(N) = 2$ is given in Theorem 2.41. Using these results, we give a computation of the Maslov index at the non-regular crossing in Theorem 2.49.

As an application of our theory, working under Hypothesis 2.5, we provide a new formula for the sign of $\ddot{s}(0)$ by evaluating the integral expression in (2.19) for stationary states satisfying (2.15); see Proposition 2.53. In the edge cases when $P - Q = 1$ and $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$, or $P - Q = -1$ and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$, we show (see Theorem 2.11) that spectral stability of the standing wave $\widehat{\psi}$ is determined by the sign of $\ddot{s}(0)$. This suggests that on a bounded interval, the integrals $\langle \cdot, \cdot \rangle$ in (2.18) and (2.19) play the same role that (2.3) plays in the well known VK criterion on the real line. We thus refer to the two integral expressions in (2.20) as *VK-type integrals*. In Section 2.4.3.2 we show that it is possible to recover the classical VK criterion on a compact interval using the numerator in (2.18) (but *not* (2.19)).

Theorem 2.11. *Let N be an operator as in (2.6)–(2.7). Consider the case when $P = 1$, $Q = 0$, and $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$. If the associated VK-type integral in (2.18) is positive, then $n_+(N) = 1$, while if the integral is negative, then $\mathrm{Spec}(N) \subset i\mathbb{R}$. In particular, under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable if (2.18) is positive, and spectrally stable if (2.18) is negative.*

Similarly, consider the case when $Q = 1$, $P = 0$, and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$. If the VK-type integral in (2.19) is negative, then $n_+(N) = 1$, while if the integral is positive, then $\mathrm{Spec}(N) \subset i\mathbb{R}$. In particular, under Hypothesis 2.5, $\widehat{\psi}$ is spectrally unstable if (2.19) is positive, and spectrally stable if (2.19) is negative.

(The proof is given in Section 2.4.2.) The proofs that $n_+(N) = 1$ rely on an argument that allows the replacement of the inequality in (2.11) with an equality, as well as a computation of $\mathfrak{c}$ that yields 1 on the right hand side of (2.11). The former comes from the fact that the Maslov index is monotone in λ *provided either P or Q is zero* (see Lemma 2.52). On the other hand, to prove $\mathrm{Spec}(N) \subset i\mathbb{R}$ in the cases described in Theorem 2.11, it will be shown (see Lemma 2.51) that the nonnegativity of L_+ or L_- forces the spectrum of N to be confined to the real and imaginary axes. It will then follow from monotonicity in λ (i.e. Lemma 2.52) that $n_+(N) = 0$ (and therefore that $\mathrm{Spec}(N) \subset i\mathbb{R}$).

Remark 2.12. In Theorem 2.11 we recover the equality in [HK08, Theorem 2.25] without the assumption that the operators $L_\pm$ are invertible (albeit in the case when $P = 0$ or $Q = 0$). Recovering the equality (when L_+ and L_- are invertible) in cases when both P and Q are nonzero via our Maslov index calculations remains an open question.

2.2 A symplectic approach to the eigenvalue problem

In this section we review the definition of the Maslov index and give a symplectic formulation of the eigenvalue problem (2.5), culminating in the proof of Theorem 2.2.

2.2.1 The Maslov index

We begin with some background material on the Maslov index [Mas65]. We follow the definition given by Robbin and Salamon [RS93], wherein the Maslov index is first defined for regular paths, and then extended to arbitrary continuous paths by a homotopy argument. For more on the topological properties of the spaces discussed, see [Arn67]. For a systematic and unified treatment of the Maslov index, featuring an axiomatic description and four equivalent definitions, see [CLM94].

The starting point is $\mathbb{R}^{2n}$ equipped with the nondegenerate, skew-symmetric bilinear form

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \qquad \omega(x,y) = Jx \cdot y \tag{2.22}$$

called a *symplectic form*, where "$\cdot$" is the dot product in $\mathbb{R}^{2n}$ and J is given in (2.4). A *Lagrangian subspace* or *plane* Λ of $\mathbb{R}^{2n}$ is an n-dimensional subspace on which the symplectic form vanishes. The *Lagrangian Grassmannian* is the set of all Lagrangian subspaces, $\mathcal{L}(n) = \big\{\Lambda \subset \mathbb{R}^{2n} : \dim(\Lambda) = n, \quad \omega(x,y) = 0, \forall\, x,y \in \Lambda\big\}$. This space has infinite cyclic fundamental group, i.e. $\pi_1(\mathcal{L}(n)) = \mathbb{Z}$. A notion of winding therefore exists for paths in $\mathcal{L}(n)$; this is the Maslov index. Namely, the Maslov index of a loop in $\mathcal{L}(n)$ is its equivalence class in the fundamental group. Poincaré duality [Hat02, §3.3] affords an interpretation of this winding number as the (signed) number of intersections with a distinguished codimension one submanifold, and this allows one to extend the definition to *any* path in $\mathcal{L}(n)$. This is the approach of Arnol'd, which we briefly review.

Fix a reference plane $\Lambda_0 \in \mathcal{L}(n)$. The distinguished codimension one submanifold of $\mathcal{L}(n)$ is given by the top stratum $\mathcal{T}_1(\Lambda_0)$ of the *train* of Λ_0,

$$\mathcal{T}(\Lambda_0) = \big\{\Lambda \in \mathcal{L}(n) : \Lambda \cap \Lambda_0 \neq \{0\}\big\} = \bigcup_{k=1}^{n} \mathcal{T}_k(\Lambda_0),$$

where $\mathcal{T}_k(\Lambda_0) = \{\Lambda \in \mathcal{L}(n) : \dim(\Lambda \cap \Lambda_0) = k\}$. As the fundamental lemma of [Arn67] states, $\mathcal{T}_1(\Lambda_0)$ is two sidedly embedded in $\mathcal{L}(n)$. This means there exists a continuous vector field transverse to $\mathcal{T}_1(\Lambda_0)$ and tangent to $\mathcal{L}(n)$. One can therefore assign a signature to each transverse intersection of a path in $\mathcal{L}(n)$ with $\mathcal{T}_1(\Lambda_0)$. Any Lagrangian path with endpoints not in $\mathcal{T}(\Lambda_0)$ can be perturbed to one that only intersects the top stratum $\mathcal{T}_1(\Lambda_0)$ of the train, and only does so transversally; the Maslov index is then defined to be the sum of the signatures of all such intersections.

We next recall the approach of Robbin and Salamon [RS93], which requires additional regularity but applies to paths whose endpoints are in the train, and also allows for intersections with $\mathcal{T}_k(\Lambda_0)$ when $k \geq 2$. This approach, while less geometric than the above interpretation of the Maslov index as an intersection number, is more suited to practical computations.

Given a smooth path $\Lambda : [a,b] \longrightarrow \mathcal{L}(n)$, a *crossing* is a point $t = t_0$ where $\Lambda(t_0) \in \mathcal{T}(\Lambda_0)$. Let $\Lambda_0^\perp \subset \mathbb{R}^{2n}$ be a subspace transverse to $\Lambda(t_0)$. Then $\Lambda_0^\perp$ is transverse to $\Lambda(t)$ for all $t \in [t_0-\varepsilon, t_0+\varepsilon]$ for ε small enough. Thus, there exists a smooth family of matrices $R_t : \Lambda(t_0) \to \Lambda_0^\perp$ such that

$$\Lambda(t) = \operatorname{graph}(R_t) = \{q + R_t q : q \in \Lambda(t_0)\} \tag{2.23}$$

for $|t-t_0| \le \varepsilon$, where $R_{t_0}|_{\Lambda(t_0)} \equiv 0$. At a crossing t_0, the *crossing form* is the quadratic form

$$\mathfrak{m}_{t_0}(q) = \frac{d}{dt}\omega(q, q + R_t q)\Big|_{t=t_0} = \omega(q, \dot{R}_{t_0} q), \qquad q \in \Lambda(t_0) \cap \Lambda_0, \tag{2.24}$$

on the intersection $\Lambda(t_0) \cap \Lambda_0$. The full symmetric bilinear form associated with the quadratic form (2.24) may be recovered using the polarisation identity; see, for example, the proof of Corollary 2.22. A crossing is called *regular* if the form (2.24) is nondegenerate, and *simple* if $\Lambda(t_0) \in \mathcal{T}_1(\Lambda_0)$. Since $\mathfrak{m}_{t_0}$ is quadratic, it may be diagonalised; we let $n_+(\mathfrak{m}_{t_0})$ and $n_-(\mathfrak{m}_{t_0})$ be the number of positive and negative squares obtained in so doing. The signature of $\mathfrak{m}_{t_0}$ is the integer $\operatorname{sign}(\mathfrak{m}_{t_0}) = n_+(\mathfrak{m}_{t_0}) - n_-(\mathfrak{m}_{t_0})$. We then define the Maslov index as follows.

Definition 2.13. *The Maslov index for a path $\Lambda : [a,b] \longrightarrow \mathcal{L}(n)$ having only regular crossings is given by*

$$\operatorname{Mas}(\Lambda(t), \Lambda_0; [a,b]) := -n_-(\mathfrak{m}_a) + \sum_{a<t_0<b} \operatorname{sign}(\mathfrak{m}_{t_0}) + n_+(\mathfrak{m}_b), \tag{2.25}$$

where the sum is taken over all crossings $t_0 \in (a,b)$.

One can show that regular crossings are isolated and therefore the sum is well-defined. Note the convention at the endpoints: at $t = a$ only the negative squares contribute to the Maslov index, while at $t = b$ only the positive squares contribute. Other conventions are possible, see e.g. [RS93, §2], but we choose the above in order to ensure the Maslov index is an integer.

The Maslov index of an arbitrary continuous path $\Lambda_1 : [a,b] \longrightarrow \mathcal{L}(n)$ is then defined to be $\operatorname{Mas}(\Lambda_2(t), \Lambda_0; [a,b])$, where Λ_2 is any path that is homotopic (with fixed endpoints) to Λ_1 and has only regular crossings. It is guaranteed by [RS93, Lemmas 2.1 and 2.2] that such a path exists, and any two such paths have the same index, so the Maslov index of Λ_1 is well defined.

The essential properties of the Maslov index that we will use are given in the following proposition, see [RS93, Theorem 2.3].

Proposition 2.14. *The Maslov index enjoys*

1. *Homotopy invariance: if two paths $\Lambda_1, \Lambda_2 : [a,b] \longrightarrow \mathcal{L}(n)$ are homotopic with fixed endpoints, then*

$$\operatorname{Mas}(\Lambda_1(t), \Lambda_0; [a,b]) = \operatorname{Mas}(\Lambda_2(t), \Lambda_0; [a,b]). \tag{2.26}$$

2. *Additivity under concatenation: for $\Lambda(t) : [a,c] \longrightarrow \mathcal{L}(n)$ and $a < b < c$,*

$$\operatorname{Mas}(\Lambda(t), \Lambda_0; [a,c]) = \operatorname{Mas}(\Lambda(t), \Lambda_0; [a,b]) + \operatorname{Mas}(\Lambda(t), \Lambda_0; [b,c]). \tag{2.27}$$

To conclude our discussion of the Maslov index, we expound the notion of a non-regular crossing, that is, a crossing with degenerate crossing form. Consider a Lagrangian path $\Lambda : [a,b] \longrightarrow \mathcal{L}(n)$ with a non-regular crossing $t = t_0$. In the case that $\mathfrak{m}_{t_0}$ is identically zero, in [DJ11, Proposition 3.10] the authors state that the contribution to the Maslov index is determined by the second-order crossing form

$$\mathfrak{m}^{(2)}_{t_0}(q) := \frac{d^2}{dt^2}\omega(q, q + R_t q)\Big|_{t=t_0} = \omega(q, \ddot{R}_{t_0} q), \qquad q \in \Lambda(t_0) \cap \Lambda_0, \tag{2.28}$$

provided it is nondegenerate. Such a crossing can only contribute to the Maslov index if it occurs at one of the endpoints: if $t_0 = a$ then it contributes $-n_-(\mathfrak{m}_a^{(2)})$, and if $t_0 = b$ then it contributes $n_+(\mathfrak{m}_b^{(2)})$.

As an example, consider the case of a simple crossing with $\mathfrak{m}_{t_0} = 0$ but $\mathfrak{m}_{t_0}^{(2)} \neq 0$. In the Lagrangian Grassmannian, this corresponds to our path Λ tangentially intersecting the train $\mathcal{T}(\Lambda_0)$ of the fixed reference plane to quadratic order; i.e. Λ "bounces off" the train as t passes through t_0. Provided t_0 lies in the interior of $[a, b]$, the contribution to the Maslov index will be zero: clearly the path can locally be homotoped to one with no crossings at all. If $t_0 = a$, the contribution is -1 provided the path leaves in the negative direction (and zero otherwise), while if $t_0 = b$, the contribution is $+1$ provided the path arrives in the positive direction (and zero otherwise). If the second order form is degenerate, i.e. $\mathfrak{m}_{t_0}^{(2)} = 0$, higher order derivatives are needed in order to determine the local behaviour of the path Λ.

In the present setting, with the spectral parameter λ acting as the independent variable, we will observe that a non-regular crossing occurs at $\lambda = 0$. To determine the contribution to the Maslov index of this non-regular crossing, we use a homotopy argument, made possible by our analysis of the local behaviour of the eigenvalue curves in Section 2.3.4. We confirm that our computation agrees with the number of negative squares of the second order form (2.28) used in [DJ11]. For a further discussion of non-regular crossings and an alternate way to compute the Maslov index at such points, see [GPP04a, GPP04b].

2.2.2 Spatial rescaling and construction of the Lagrangian path

We now view the problem through the lens of the Lagrangian formalism by interpreting eigenvalues as nontrivial intersections of Lagrangian planes. Following the approach of [DJ11], we restrict the eigenvalue problem to a family of subintervals $[0, s\ell]$ for $s \in (0, 1]$. Rescaling the equations to the full domain $[0, \ell]$, we construct a two-parameter family of Lagrangian subspaces in s and λ via rescaled boundary traces of solutions to the system of differential equations without any boundary conditions at all. An eigenvalue is produced when this family of subspaces nontrivially intersects a fixed reference plane that encodes Dirichlet boundary conditions. Identifying a Lagrangian structure boils down to a judicious choice of both the symplectic form and the definition of the trace map: if we employ the standard symplectic form ω in (2.22), then we need to carefully define the trace map (2.31) such that the space of boundary traces is Lagrangian with respect to ω. We begin by introducing some notation.

We let

$$N = D + B(x), \qquad D := \begin{pmatrix} 0 & \partial_{xx} \\ -\partial_{xx} & 0 \end{pmatrix}, \qquad B(x) := \begin{pmatrix} 0 & h(x) \\ -g(x) & 0 \end{pmatrix}, \tag{2.29}$$

and introduce the s-dependent operators acting on functions on $[0, \ell]$,

$$B_s(x) := s^2 B(sx), \qquad N_s := \begin{pmatrix} 0 & -L_-^s \\ L_+^s & 0 \end{pmatrix}, \qquad \begin{cases} L_+^s := -\partial_{xx} - s^2 g(sx) \\ L_-^s := -\partial_{xx} - s^2 h(sx) \end{cases} \tag{2.30}$$

so that $N_s = D + B_s(x)$. We define the *rescaled trace* of $\mathbf{u} = (u, v)^\top \in H^2(0, \ell)$ as the vector

$$\operatorname{Tr}_s \mathbf{u} := \left(u(0), v(0), u(\ell), v(\ell), -\frac{1}{s}u'(0), \frac{1}{s}v'(0), \frac{1}{s}u'(\ell), -\frac{1}{s}v'(\ell)\right)^\top \in \mathbb{R}^8, \tag{2.31}$$

and denote the vertical subspace of $\mathbb{R}^8$ by $\mathcal{D} := \{0\} \times \mathbb{R}^4$. Using the above notation, we may rewrite the restricted problem (2.10) as a boundary value problem on $[0, \ell]$. Indeed, if $\mathbf{u}(x) \in H^2(0, s\ell) \cap H_0^1(0, s\ell)$ then $\mathbf{u}_s(x) := \mathbf{u}(sx) \in H^2(0, \ell) \cap H_0^1(0, \ell)$. It follows from (2.31) that $\mathbf{u}(0) = \mathbf{u}(s\ell) = 0$ if and only if $\operatorname{Tr}_s \mathbf{u}_s \in \mathcal{D}$. Thus, rescaled to $[0, \ell]$, (2.10) reads

$$N_s \mathbf{u}_s = s^2 \lambda \mathbf{u}_s, \quad \operatorname{Tr}_s \mathbf{u}_s \in \mathcal{D}. \tag{2.32}$$

Note that the solution spaces of the boundary value problems (2.10) and (2.32) are isomorphic: $\mathbf{u} = (u, v)^\top \in \operatorname{dom}(N|_{[0,s\ell]})$ solves (2.10) if and only if $\mathbf{u}_s = (u_s, v_s)^\top \in \operatorname{dom}(N_s)$ solves (2.32). Consequently, λ is an eigenvalue of $N|_{[0,s\ell]}$ if and only if $s^2\lambda$ is an eigenvalue of N_s.

Remark 2.15. The rescaled problem (2.32) is well-defined for $s > 1$ provided the potentials g and h are defined for $x > \ell$. In this case the "restricted" eigenvalue problem (2.10) corresponds to a *stretching* of the domain.

Remark 2.16. As per Remark 2.1, notationally we will not distinguish between N_s and $L_\pm^s$ as differential expressions and as unbounded operators with dense domains given by (2.8) and (2.9), respectively. Thus, when we write $s^2\lambda \in \operatorname{Spec}(N_s)$ or $\mathbf{u}_s \in \ker(N_s - s^2\lambda)$, we mean that (2.32) is solved for some eigenfunction $\mathbf{u}_s$; similar statements hold when $\lambda \in \operatorname{Spec}(L_\pm^s)$.

That the formulation (2.32) lends itself to a symplectic interpretation can be seen via the following modified version of Green's second identity. Using our definition of the rescaled trace map (2.31) and the symplectic form (2.22), one can verify that for each $s \in (0, 1]$ and all $\mathbf{u}, \mathbf{v} \in H^2(0, \ell)$,

$$\langle S(N_s - s^2\lambda)\mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, S(N_s - s^2\lambda)\mathbf{v}\rangle = s\omega(\operatorname{Tr}_s \mathbf{u}, \operatorname{Tr}_s \mathbf{v}), \tag{2.33}$$

where S is defined in (2.4). Now define the space

$$\mathcal{K}_{\lambda,s} := \left\{\mathbf{u} \in H^2(0, \ell) : (N_s - s^2\lambda)\mathbf{u} = 0 \text{ in } L^2(0, \ell)\right\} \tag{2.34}$$

of all solutions to the homogeneous differential equation $N_s\mathbf{u} = s^2\lambda\mathbf{u}$ *without* any reference to the boundary conditions, so that $\ker(N_s - s^2\lambda) = \mathcal{K}_{\lambda,s} \cap H_0^1(0, \ell)$.

Remark 2.17. The trace map is an injective linear operator on the space $\mathcal{K}_{\lambda,s}$. If $\mathbf{u}_s \in \mathcal{K}_{\lambda_0,s}$, then $\operatorname{Tr}_s \mathbf{u}_s = 0$ implies $\mathbf{u}_s = 0$, since $\mathbf{u}_s$ solves a system of second order equations.

Taking the (rescaled) boundary trace leads to the desired family of Lagrangian subspaces, with respect to the form ω in (2.22).

Lemma 2.18. *The space*

$$\Lambda(\lambda, s) := \operatorname{Tr}_s(\mathcal{K}_{\lambda,s}) = \{\operatorname{Tr}_s(\mathbf{u}) : \mathbf{u} \in \mathcal{K}_{\lambda,s}\} \tag{2.35}$$

is a Lagrangian subspace of $\mathbb{R}^8$ *for all* $s \in (0, 1]$ *and all* $\lambda \in \mathbb{R}$.

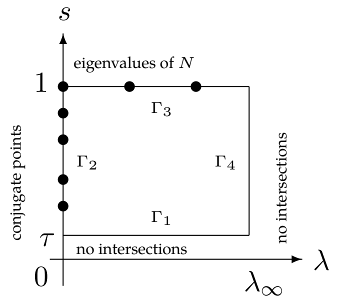


Figure 2.2: Maslov box in the λs-plane.

Proof. Fix $\lambda \in \mathbb{R}$ and $s \in (0,1]$. From (2.33), for $\mathbf{u}, \mathbf{v} \in \mathcal{K}_{\lambda,s}$ we have $\omega(\mathrm{Tr}_s\, \mathbf{u}, \mathrm{Tr}_s\, \mathbf{v}) = 0$. Since $\mathcal{K}_{\lambda,s}$ is the space of solutions to a system of two second-order differential equations, $\dim \mathcal{K}_{\lambda,s} = 4$. Hence $\dim \mathrm{Tr}_s(\mathcal{K}_{\lambda,s}) = 4$, and $\mathrm{Tr}_s(\mathcal{K}_{\lambda,s}) \in \mathcal{L}(4)$ is Lagrangian. □

We now have the desired interpretation of eigenvalues as nontrivial intersections of Lagrangian subspaces.

Proposition 2.19. $s^2\lambda \in \mathrm{Spec}(N_s)$ *if and only if* $\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\}$. *Moreover, the geometric multiplicity of the eigenvalue is equal to the dimension of the Lagrangian intersection,*

$$\dim \ker(N_s - s^2\lambda) = \dim \Lambda(\lambda, s) \cap \mathcal{D}. \tag{2.36}$$

Proof. The first statement follows from the definition of Λ. Equality (2.36) follows from the injectivity (and thus bijectivity) of the trace map acting between the finite dimensional spaces $\ker(N_{s_0} - s_0^2\lambda_0) = \mathcal{K}_{\lambda_0,s_0} \cap H_0^1(0,\ell)$ and $\mathrm{Tr}_{s_0}(\mathcal{K}_{\lambda_0,s_0} \cap H_0^1(0,\ell)) = \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. □

Hereafter, a *crossing* refers to a pair $(\lambda, s) = (\lambda_0, s_0)$ such that $\Lambda(\lambda_0, s_0) \cap \mathcal{D} \neq \{0\}$, while a *conjugate point* refers to a crossing for which $\lambda_0 = 0$. It follows from Proposition 2.19 that crossings where $s_0 = 1$ correspond to eigenvalues of the operator N on $[0, \ell]$.

To prove Theorem 2.2, our goal then is to bound from below the number of crossings for which $s_0 = 1, \lambda_0 > 0$. To do so we use a homotopy argument that involves appropriately counting conjugate points. In order to set this argument up, we introduce in Fig. 2.2 the so-called *Maslov box*, given by the boundary Γ of the rectangle $[0, \lambda_\infty] \times [\tau, 1]$ in the λs-plane, where $\tau > 0$ is small and $\lambda_\infty > 0$ is large.

Since $\Lambda : [0, \lambda_\infty] \times [\tau, 1] \longrightarrow \mathcal{L}(4)$ is a continuous map, the image $\Lambda(\Gamma)$ of the Maslov box is null homotopic, and so

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma) = 0. \tag{2.37}$$

We partition Γ into its constituent sides such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\begin{aligned} &\Gamma_1 : s = \tau, \quad 0 \le \lambda \le \lambda_\infty \qquad &&\Gamma_3 : s = 1, \quad 0 \le \lambda \le \lambda_\infty \\ &\Gamma_2 : \lambda = 0, \quad \tau \le s \le 1 \qquad &&\Gamma_4 : \lambda = \lambda_\infty, \quad \tau \le s \le 1 \end{aligned} \tag{2.38}$$

(see Fig. 2.2) and assign a direction to each of these intervals such that the entirety of the Maslov box is oriented in a clockwise fashion. We then appeal to the concatenation property in Proposition 2.14 to rewrite (2.37) as

$$\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_1) + \operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3) + \operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0. \tag{2.39}$$

Taking $\lambda = \lambda_\infty$ large enough and $s = \tau$ small enough, it will follow (see Lemma 2.35) that there are no crossings along Γ_1 and Γ_4, and therefore that the Maslov indices of these pieces are zero. The crossing forms needed to analyse $\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_2)$ and $\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3)$ are given in the next section.

2.2.3 Crossing forms

Our next task is the calculation of the crossing forms (2.24) associated with the trajectories through the crossing (λ_0, s_0) where $\lambda = \lambda_0$ is held constant and s increases, and vice versa. The key ingredient will be the Green's-type identity (2.33). The approach is inspired by Lemma 4.18 and the proof of Theorem 4.19 in [LS20a], as well as the crossing form calculation in [CJLS16, Lemma 5.2]. Before proceeding, we set some notation that will be useful in this section and throughout the rest of the paper.

Remark 2.20. We denote by $\mathbf{u}_{s_0}$ any eigenfunction $\mathbf{u}_{s_0} \in \ker(N_{s_0} - s_0^2\lambda_0)$, and when $s_0 = 1$ we drop the subscript. If $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$, we denote a basis for this space by $\{\mathbf{u}_{s_0}^{(1)}, \dots, \mathbf{u}_{s_0}^{(n)}\}$, where $\mathbf{u}_{s_0}^{(i)} = \left(u_{s_0}^{(i)}, v_{s_0}^{(i)}\right)^\top$. The set $\{S\mathbf{u}_{s_0}^{(1)}, \dots, S\mathbf{u}_{s_0}^{(n)}\}$ is then a basis for the kernel of the adjoint operator, $\ker(N_{s_0}^* - s_0^2\lambda_0)$, since λ_0 is real. Note that S (given in (2.4)) merely swaps the entries of the vector it acts on. When $s_0 = 1$ we denote:

$$\mathbf{u}_i := \mathbf{u}_1^{(i)}, \qquad u_i := u_1^{(i)},\ v_i := v_1^{(i)}. \tag{2.40}$$

Because $\ker(N_{s_0}) = \ker(L_+^{s_0}) \oplus \ker(L_-^{s_0})$, when $\lambda_0 = 0$ and $\dim\ker(N_{s_0}) = 1$ we have

$$\mathbf{u}_{s_0} = \begin{cases} (u_{s_0}, 0)^\top, & 0 \in \operatorname{Spec}(L_+^{s_0}) \setminus \operatorname{Spec}(L_-^{s_0}), \quad \ker(L_+^{s_0}) = \operatorname{span}\{u_{s_0}\}, \\ (0, v_{s_0})^\top, & 0 \in \operatorname{Spec}(L_-^{s_0}) \setminus \operatorname{Spec}(L_+^{s_0}), \quad \ker(L_-^{s_0}) = \operatorname{span}\{v_{s_0}\}. \end{cases} \tag{2.41}$$

When $\lambda_0 = 0$ and $\dim\ker(N_{s_0}) = 2$, we denote

$$\mathbf{u}_{s_0}^{(1)} = \begin{pmatrix} u_{s_0}^{(1)} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_{s_0}^{(2)} = \begin{pmatrix} 0 \\ v_{s_0}^{(2)} \end{pmatrix}, \tag{2.42}$$

where $\ker(L_+^{s_0}) = \operatorname{span}\{u_{s_0}^{(1)}\}$ and $\ker(L_-^{s_0}) = \operatorname{span}\{v_{s_0}^{(2)}\}$.

In the current paper where the potentials g and h from (2.7) are scalar-valued, we will always have $n \leq 2$. However, if g and h are matrix-valued (and symmetric), so that $L_\pm$ are systems of selfadjoint Schrödinger operators, or if the operator N acts on functions on a multidimensional domain, then we may have $n > 2$. The results in this section and Section 2.3 have been stated for a general n to indicate how the theory extends to these cases.

Returning to our computation of crossing forms, we first compute the crossing form (2.24) for the path of Lagrangian planes $s \mapsto \Lambda(\lambda_0, s)$, holding $\lambda = \lambda_0$ fixed. Recall that $N_s = D + B_s$, as in (2.30), and that $S = S^\top$.

Lemma 2.21. *Let (λ_0, s_0) be a crossing and fix any nonzero $q \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. Then there exists a unique $\mathbf{u}_{s_0} \in \mathcal{K}_{\lambda_0, s_0}$ such that $q = \operatorname{Tr}_{s_0} \mathbf{u}_{s_0}$, and the crossing form for the Lagrangian path $s \mapsto \Lambda(\lambda_0, s)$ at $s = s_0$ is given by*

$$\mathfrak{m}_{s_0}(q) = \frac{1}{s_0} \left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \right\rangle, \tag{2.43}$$

where $\partial_s B_s = 2sB(sx) + s^2 B'(sx)x$. In particular, along Γ_2 where $\lambda_0 = 0$, we have

$$\mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2} \left[-\left(u_{s_0}'(\ell)\right)^2 + \left(v_{s_0}'(\ell)\right)^2 \right]. \tag{2.44}$$

In this case, if the crossing $(0, s_0)$ is simple, then the form (2.44) is non-degenerate.

Proof. Consider a C^1 family of vectors $s \mapsto \mathbf{w}_s \in \mathcal{K}_{\lambda_0, s}$ satisfying

$$N_s \mathbf{w}_s = s^2 \lambda_0 \mathbf{w}_s, \qquad x \in [0, \ell], \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon), \tag{2.45a}$$
$$\operatorname{Tr}_s \mathbf{w}_s = \operatorname{Tr}_{s_0} \mathbf{u}_{s_0} + R_s \operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \qquad \mathbf{w}_{s_0} = \mathbf{u}_{s_0}, \tag{2.45b}$$

where $R_s : \Lambda(\lambda_0, s_0) \to \mathcal{D}^\perp$ is the smooth family of matrices such that $\Lambda(\lambda_0, s) = \operatorname{graph}(R_s)$, cf. (2.23). To prove the existence of such a family $s \mapsto \mathbf{w}_s$, consider the smooth family of vectors $h_s := q + R_s q \in \Lambda(\lambda_0, s)$, where $h_{s_0} = q$ since $R_{s_0} q = 0$ for all $q \in \Lambda(\lambda_0, s_0)$. The injectivity (and thus bijectivity) of the linear map

$$\operatorname{Tr}_s : \mathcal{K}_{\lambda_0, s} \longrightarrow \operatorname{Tr}_s(\mathcal{K}_{\lambda_0, s}) = \Lambda(\lambda_0, s)$$

(see Remark 2.17) then implies that for each $h_s \in \Lambda(\lambda_0, s)$ there exists a unique $\mathbf{w}_s \in \mathcal{K}_{\lambda_0, s}$ such that $\operatorname{Tr}_s \mathbf{w}_s = h_s$, and in particular $\operatorname{Tr}_{s_0} \mathbf{w}_{s_0} = h_{s_0} = q$.

We now turn to the computation of (2.24). We have

$$\begin{aligned}
\mathfrak{m}_{s_0}(q) &= \frac{d}{ds} \omega(q, R_s q) \Big|_{s=s_0} \\
&= \frac{d}{ds} \omega(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \operatorname{Tr}_s \mathbf{w}_s) \Big|_{s=s_0} \\
&= \omega \left(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \frac{d}{ds} \operatorname{Tr}_s \big|_{s=s_0} \mathbf{u}_{s_0} \right) + \omega \left(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \operatorname{Tr}_{s_0} \frac{d}{ds} \mathbf{w}_s \big|_{s=s_0} \right).
\end{aligned}$$

The first term is zero since $\operatorname{Tr}_{s_0} \mathbf{u}_{s_0} \in \mathcal{D}$ implies $\operatorname{Tr}_{s_0} \mathbf{u}_{s_0} = \left(0, s_0^{-1} \gamma_N \mathbf{u}_{s_0}\right)$ and $\frac{d}{ds} \operatorname{Tr}_s \big|_{s=s_0} \mathbf{u}_{s_0} = \left(0, -s_0^{-2} \gamma_N \mathbf{u}_{s_0}\right)$, where $\gamma_N \mathbf{u} := (-u'(0), v'(0), u'(\ell), -v'(\ell))^\top$. For the second term, we differentiate the equation in (2.45a) with respect to s and apply $\langle \cdot, S\mathbf{w}_s \rangle$,

$$\langle (\partial_s B_s - 2s\lambda_0)\mathbf{w}_s, S\mathbf{w}_s \rangle + \langle (N_s - s^2\lambda_0)\, \partial_s \mathbf{w}_s, S\mathbf{w}_s \rangle = 0. \tag{2.46}$$

From the Green's-type identity (2.33) with $\mathbf{u} = \mathbf{w}_s$ and $\mathbf{v} = \partial_s \mathbf{w}_s$, we have

$$s\, \omega(\operatorname{Tr}_s \mathbf{w}_s, \operatorname{Tr}_s \partial_s \mathbf{w}_s) = \langle (N_s - s^2\lambda_0)\mathbf{w}_s, S\, \partial_s \mathbf{w}_s \rangle - \langle S\mathbf{w}_s, (N_s - s^2\lambda_0)\partial_s \mathbf{w}_s \rangle,$$

and using (2.45a) and (2.46) this reduces to

$$s\,\omega(\mathrm{Tr}_s\,\mathbf{w}_s, \mathrm{Tr}_s\,\partial_s\mathbf{w}_s) = \langle(\partial_s B_s - 2s\lambda_0)\mathbf{w}_s, S\mathbf{w}_s\rangle. \tag{2.47}$$

Evaluating (2.47) at $s = s_0$ and dividing by s_0, (2.43) follows. When $\lambda_0 = 0$, substituting the stated expression for $\partial_s B_{s_0}$ in (2.43) gives

$$\begin{aligned}\mathfrak{m}_{s_0}(q) &= \left\langle \left(2B(s_0x) + s_0B'(s_0x)x\right)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\right\rangle \\ &= \int_0^\ell \left\{ \left[2h(s_0x) + s_0xh'(s_0x)\right] v_{s_0}^2(x) - \left[2g(s_0x) + s_0xg'(s_0x)\right] u_{s_0}^2(x)\right\} dx.\end{aligned}$$

A direct calculation using the equation $L_-^{s_0}v_{s_0} = 0$, i.e. $v_{s_0}''(x) + s_0^2h(s_0x)v_{s_0}(x) = 0$, gives

$$\frac{d}{dx}\left[\frac{1}{s_0^2}x\left(v_{s_0}'(x)\right)^2 + xv_{s_0}^2(x)h(s_0x) - \frac{1}{s_0^2}v_{s_0}(x)v_{s_0}'(x)\right] = \left[2h(s_0x) + s_0xh'(s_0x)\right] v_{s_0}^2(x).$$

Integrating and using the fact that $v_{s_0}(0) = v_{s_0}(\ell) = 0$, we get

$$\int_0^\ell \left[2h(s_0x) + s_0xh'(s_0x)\right] v_{s_0}^2(x)dx = \frac{\ell}{s_0^2}\left(v_{s_0}'(\ell)\right)^2.$$

Computing similarly for the second term, we arrive at (2.44). That the form is nondegenerate in the simple case follows from (2.41): if $\dim\ker(N_{s_0}) = 1$ then exactly one of the entries of $\mathbf{u}_s = (u_s, v_s)^\top \in \ker(N_{s_0})$ is nontrivial. Since this function satisfies a second order differential equation with Dirichlet boundary conditions, its derivative is nonzero at $x = \ell$, and therefore (2.44) is nonzero. □

Corollary 2.22. *Assume* $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$ *and let* $\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}, \dots, \mathbf{u}_{s_0}^{(n)}\}$ *be a basis for* $\ker(N_{s_0} - s_0^2\lambda_0)$. *The* $n \times n$ *symmetric matrix* $\mathfrak{M}_{s_0}$ *induced from the quadratic form* (2.43) *is given by*

$$[\mathfrak{M}_{s_0}]_{ij} = \frac{1}{s_0}\left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle, \qquad i, j = 1, \dots, n. \tag{2.48}$$

Consequently, when $\lambda_0 = 0$ *and* $n = 2$, *the form* $\mathfrak{m}_{s_0}$ *is nondegenerate.*

Proof. Letting $q_i := \mathrm{Tr}_{s_0}\,\mathbf{u}_{s_0}^{(i)}$, it follows from the linearity and injectivity of the trace map that $\{q_i\}_{i=1}^n$ is a basis for $\Lambda(\lambda_0, s_0) \cap \mathcal{D}$. To construct the symmetric bilinear form associated with the quadratic form (2.43), we compute the off-diagonal terms $\mathfrak{m}_{s_0}(q_i, q_j)$ via the real polarisation identity

$$\mathfrak{m}_{s_0}(q_i, q_j) = \frac{1}{4}\left[\mathfrak{m}_{s_0}(q_i + q_j) - \mathfrak{m}_{s_0}(q_i - q_j)\right]. \tag{2.49}$$

Since both S and $S\,(\partial_s B_{s_0})$ are symmetric, we obtain

$$\begin{aligned}\mathfrak{m}_{s_0}(q_i, q_j) &= \frac{1}{4}\left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\left(\mathbf{u}_{s_0}^{(i)} + \mathbf{u}_{s_0}^{(j)}\right), S\left(\mathbf{u}_{s_0}^{(i)} + \mathbf{u}_{s_0}^{(j)}\right)\right\rangle \\ &\quad - \frac{1}{4}\left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\left(\mathbf{u}_{s_0}^{(i)} - \mathbf{u}_{s_0}^{(j)}\right), S\left(\mathbf{u}_{s_0}^{(i)} - \mathbf{u}_{s_0}^{(j)}\right)\right\rangle \\ &= \left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle.\end{aligned}$$

The corresponding matrix elements with respect to the basis $\{q_i\}$ are $[\mathfrak{M}_{s_0}]_{ij} = \mathfrak{m}_{s_0}(q_i, q_j)$, and the first statement of the corollary follows. In the case $\lambda_0 = 0$ and $n = 2$, using (2.44) and recalling (2.42), the matrix (2.48) reduces to

$$\mathfrak{M}_{s_0} = \frac{\ell}{s_0^2} \begin{pmatrix} -\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 & 0 \\ 0 & \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2 \end{pmatrix}, \tag{2.50}$$

which clearly has full rank. Nondegeneracy of the quadratic form $\mathfrak{m}_{s_0}$ follows. □

We now move to the λ-direction. Holding $s = s_0$ fixed, we compute the crossing form (2.24) with respect to λ. We denote $d/d\lambda$ with a dot.

Lemma 2.23. *Let (λ_0, s_0) be a crossing and fix any nonzero $q \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. Then there exists a unique $\mathbf{u}_{s_0} \in \mathcal{K}_{\lambda_0, s_0}$ such that $q = \operatorname{Tr}_{s_0} \mathbf{u}_{s_0}$, and the crossing form for the Lagrangian path $\lambda \mapsto \Lambda(\lambda, s_0)$ at $\lambda = \lambda_0$ is given by*

$$\mathfrak{m}_{\lambda_0}(q) = -s_0 \left\langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \right\rangle = -2s_0 \left\langle u_{s_0}, v_{s_0} \right\rangle. \tag{2.51}$$

Proof. The argument is almost identical to that of the s direction. Fixing $s = s_0$, we consider a C^1 family of vectors $\lambda \mapsto \mathbf{w}_\lambda \in \mathcal{K}_{\lambda, s_0}$ satisfying

$$N_{s_0}\mathbf{w}_\lambda = s_0^2 \lambda \mathbf{w}_\lambda, \qquad x \in [0, \ell], \ \ \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \tag{2.52a}$$

$$\operatorname{Tr}_{s_0} \mathbf{w}_\lambda = \operatorname{Tr}_{s_0} \mathbf{u}_{s_0} + R_\lambda \operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \qquad \mathbf{w}_{\lambda_0} = \mathbf{u}_{s_0}, \tag{2.52b}$$

where now $R_\lambda : \Lambda(\lambda_0, s_0) \longrightarrow \mathcal{D}^\perp$ is such that $\Lambda(\lambda, s_0) = \operatorname{graph}(R_\lambda)$. Similar to (2.46) we have

$$\langle -s_0^2 \mathbf{w}_\lambda, S\mathbf{w}_\lambda \rangle + \langle (N_{s_0} - s_0^2\lambda)\dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle = 0,$$

and using the identity (2.33) with $\mathbf{u} = \mathbf{w}_\lambda$ and $\mathbf{v} = \dot{\mathbf{w}}_\lambda$ yields

$$s_0\, \omega(\operatorname{Tr}_{s_0} \mathbf{w}_\lambda, \operatorname{Tr}_{s_0} \dot{\mathbf{w}}_\lambda) = \langle (N_{s_0} - s_0^2\lambda)\mathbf{w}_\lambda, S\dot{\mathbf{w}}_\lambda \rangle - \langle S\mathbf{w}_\lambda, (N_{s_0} - s_0^2\lambda)\dot{\mathbf{w}}_\lambda \rangle.$$

The previous two equations along with (2.52a) give

$$s_0\, \omega(\operatorname{Tr}_{s_0} \mathbf{w}_\lambda, \operatorname{Tr}_{s_0} \dot{\mathbf{w}}_\lambda) = -\langle s_0^2 \mathbf{w}_\lambda, S\mathbf{w}_\lambda \rangle. \tag{2.53}$$

Therefore the crossing form (2.24) is

$$\mathfrak{m}_{\lambda_0}(q) = \omega\left(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \operatorname{Tr}_{s_0} \dot{\mathbf{w}}_\lambda\big|_{\lambda=\lambda_0}\right) = -s_0\langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle = -2s_0 \left\langle u_{s_0}, v_{s_0}\right\rangle,$$

where we used (2.53) evaluated at $\lambda = \lambda_0$. □

Recalling (2.41), at a simple crossing $(0, s_0)$ one of u_{s_0} or v_{s_0} is always trivial. Degeneracy of the λ-crossing form immediately follows.

Corollary 2.24. *All conjugate points $(0, s_0)$ for which $\dim\ker(N_{s_0}) = 1$ are non-regular in the λ direction, i.e. at all such points $\mathfrak{m}_{\lambda_0} = 0$.*

For the case of higher dimensional crossings, we have the following corollary to Lemma 2.23.

Corollary 2.25. *Assume* $\dim\ker(N_{s_0}-s_0^2\lambda_0) = n$ *and let* $\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}, \dots, \mathbf{u}_{s_0}^{(n)}\}$ *be a basis for* $\ker(N_{s_0}-s_0^2\lambda_0)$. *The* $n\times n$ *symmetric matrix* $\mathfrak{M}_{\lambda_0}$ *induced from the* n*-dimensional quadratic form* (2.51) *is given by*

$$[\mathfrak{M}_{\lambda_0}]_{ij} = -s_0\left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle, \qquad i,j=1,\dots n. \tag{2.54}$$

Consequently, when $\lambda_0 = 0$ *and* $n=2$, $\mathfrak{m}_{\lambda_0}$ *is nondegenerate if and only if* $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\rangle \neq 0$.

Proof. The first statement is proved as in Corollary 2.22. When $\lambda_0 = 0$ and $n=2$, due to (2.42), (2.54) reduces to

$$\mathfrak{M}_{\lambda_0} = -s_0\begin{pmatrix} \langle \mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(1)}\rangle & \langle \mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)}\rangle \\ \langle \mathbf{u}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(1)}\rangle & \langle \mathbf{u}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(2)}\rangle \end{pmatrix} = -s_0\begin{pmatrix} 0 & \langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\rangle \\ \langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\rangle & 0 \end{pmatrix}, \tag{2.55}$$

from which nondegeneracy of $\mathfrak{m}_{\lambda_0}$ occurs if and only if the condition stated holds. □

It follows from Corollaries 2.24 and 2.25 that a calculation of the Maslov index at $\lambda = 0$ in the λ-direction is not possible using the first order crossing form (2.24) if $\dim\ker(N_{s_0}) = 1$, or if $\dim\ker(N_{s_0}) = 2$ and $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)}\rangle = 0$. In light of this, we define:

Definition 2.26. *The correction term* $\mathfrak{c}$ *is*

$$\mathfrak{c} := \operatorname{Mas}\left(\Lambda(s,\lambda),\mathcal{D}; s\in[1-\varepsilon,1]\right) + \operatorname{Mas}\left(\Lambda(\lambda,1),\mathcal{D};\lambda\in[0,\varepsilon]\right) \tag{2.56}$$

for $0<\varepsilon\ll 1$.

That is, $\mathfrak{c}$ denotes the contribution to the Maslov index from the top left corner of the Maslov box (consisting of the arrival along Γ_2 and the departure along Γ_3).

Remark 2.27. To see that this does not depend on the choice of $0<\varepsilon\ll 1$, we observe that $(0,1)$ is an isolated crossing for both Γ_2 and Γ_3. For Γ_2 this follows from the non-degeneracy of $\mathfrak{m}_{s_0}$ in Lemma 2.21 and Corollary 2.22. For Γ_3 we use the fact that the set $\{\lambda : \Lambda(\lambda,1)\cap\mathcal{D}\neq\{0\}\} = \operatorname{Spec}(N)\cap\mathbb{R}$ is discrete (because N has compact resolvent), so there exists $\hat\lambda > 0$ such that $\Lambda(\lambda,1)\cap\mathcal{D} = \{0\}$ for $0<\lambda<\hat\lambda$.

We circumvent the issue of the non-regular crossing in Section 2.3.4 via a homotopy argument. This will be possible after having analysed the local behaviour of the eigenvalue curves in Section 2.3. In the meantime, we compute the second order crossing form (2.28) from [DJ11, Proposition 3.10].

Lemma 2.28. *Assume the conditions of Lemma 2.23. If the first order quadratic form in* (2.51) *is identically zero, then the second order quadratic form* (2.28) *is given by*

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3\langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0}\rangle, \qquad q = \operatorname{Tr}_{s_0}\mathbf{u}_{s_0}, \tag{2.57}$$

where $\mathbf{u}_{s_0}\in\ker(N_{s_0}-s_0^2\lambda_0)$ *and* $\mathbf{v}_{s_0}\in\operatorname{dom}(N_{s_0})$ *solves* $(N_{s_0}-s_0^2\lambda_0)\mathbf{v}_{s_0} = \mathbf{u}_{s_0}$. *The* $n\times n$ *matrix* $\mathfrak{M}_{\lambda_0}^{(2)}$ *of the symmetric bilinear form associated with* $\mathfrak{m}_{\lambda_0}^{(2)}$ *has entries*

$$\left[\mathfrak{M}_{\lambda_0}^{(2)}\right]_{ij} = -2s_0^3\left\langle \mathbf{v}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)}\right\rangle, \tag{2.58}$$

where $\mathbf{v}_{s_0}^{(i)} \in \operatorname{dom}(N_{s_0})$ *solves* $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$. *In the case* $\lambda_0 = 0$ *and* $n = 1$, *we have*

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \begin{cases} -2s_0^3 \langle \widehat{v}_{s_0}, u_{s_0} \rangle & 0 \in \operatorname{Spec}(L_+^{s_0}) \setminus \operatorname{Spec}(L_-^{s_0}), \\ -2s_0^3 \langle \widehat{u}_{s_0}, v_{s_0} \rangle & 0 \in \operatorname{Spec}(L_-^{s_0}) \setminus \operatorname{Spec}(L_+^{s_0}), \end{cases} \tag{2.59}$$

where $\widehat{v}_{s_0} \in \operatorname{dom}(L_-^{s_0})$ *and* $\widehat{u}_{s_0} \in \operatorname{dom}(L_+^{s_0})$ *solve* $-L_-^{s_0}\widehat{v}_{s_0} = u_{s_0}$ *and* $L_+^{s_0}\widehat{u}_{s_0} = v_{s_0}$ *respectively. In the case* $\lambda_0 = 0$ *and* $n = 2$ *we have*

$$\mathfrak{M}_{\lambda_0}^{(2)} = -2s_0^3 \begin{pmatrix} \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle & 0 \\ 0 & \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle \end{pmatrix}, \tag{2.60}$$

where $\widehat{v}_{s_0}^{(1)} \in \operatorname{dom}(L_-^{s_0})$ *and* $\widehat{u}_{s_0}^{(2)} \in \operatorname{dom}(L_+^{s_0})$ *solve* $-L_-^{s_0}\widehat{v}_{s_0}^{(1)} = u_{s_0}^{(1)}$ *and* $L_+^{s_0}\widehat{u}_{s_0}^{(2)} = v_{s_0}^{(2)}$ *respectively.*

Remark 2.29. The equation $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$ is always solvable by virtue of the Fredholm Alternative, since $\mathfrak{m}_{s_0} = 0$ means $\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \rangle = 0$ for all i, j and hence implies $\mathbf{u}_{s_0}^{(i)}$ is orthogonal to $\ker(N_{s_0}^* - s_0^2\lambda_0)$. Such a solution is not unique; however, only the component of the solution in $\ker(N_{s_0} - s_0^2\lambda_0)^\perp$ (which is unique) contributes to (2.57). It therefore suffices to consider those $\mathbf{v}_{s_0}^{(i)}$ satisfying $\mathbf{v}_{s_0}^{(i)} \perp \mathbf{u}_{s_0}^{(j)}$ for all $j = 1, \ldots, n$. Notice that the $\mathbf{v}_{s_0}^{(i)}$ are generalised eigenfunctions: if $\mathfrak{m}_{\lambda_0} = 0$, the eigenvalue $s_0^2\lambda_0 \in \operatorname{Spec}(N_{s_0})$ has n Jordan chains of length (at least) two. We thus see that loss of regularity of the crossing coincides precisely with loss of semisimplicity of the eigenvalue, which agrees with the result of [Cor19, Theorem 6.1].

Proof of Lemma 2.28. Consider a C^2 family of vectors $\lambda \mapsto \mathbf{w}_\lambda$ satisfying (2.52). Then

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \omega\left(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \operatorname{Tr}_{s_0} \ddot{\mathbf{w}}_\lambda\right)\big|_{\lambda=\lambda_0}.$$

Differentiating (2.52a) twice with respect to λ, applying $\langle \cdot, S\mathbf{w}_\lambda \rangle$ and rearranging yields

$$\left\langle (N_{s_0} - s_0^2\lambda)\ddot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \right\rangle = 2s_0^2 \langle \dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle.$$

Now using (2.33) with $\mathbf{u} = \mathbf{w}_\lambda$ and $\mathbf{v} = \ddot{\mathbf{w}}_\lambda$, we have

$$s_0\,\omega(\operatorname{Tr}_{s_0} \mathbf{w}_\lambda, \operatorname{Tr}_{s_0} \ddot{\mathbf{w}}_\lambda) = \langle (N_{s_0} - s_0^2\lambda)\mathbf{w}_\lambda, S\ddot{\mathbf{w}}_\lambda \rangle - \langle S\mathbf{w}_\lambda, (N_{s_0} - s_0^2\lambda)\ddot{\mathbf{w}}_\lambda \rangle.$$

Combining (2.52a) with the previous two equations, we get

$$s_0\,\omega(\operatorname{Tr}_{s_0} \mathbf{w}_\lambda, \operatorname{Tr}_{s_0} \ddot{\mathbf{w}}_\lambda) = -2s_0^2 \langle \dot{\mathbf{w}}_\lambda, S\mathbf{w}_\lambda \rangle.$$

Evaluating this last equation at $\lambda = \lambda_0$ and dividing through by s_0, we see that

$$\mathfrak{m}_{\lambda_0}^{(2)}(q) = \omega(\operatorname{Tr}_{s_0} \mathbf{u}_{s_0}, \operatorname{Tr}_{s_0} \ddot{\mathbf{w}}_\lambda)\big|_{\lambda=\lambda_0} = -2s_0 \langle \dot{\mathbf{w}}_{\lambda_0}, S\mathbf{u}_{s_0} \rangle.$$

To compute $\dot{\mathbf{w}}_{\lambda_0}$, we see that differentiating (2.52a) with respect to λ, evaluating at $\lambda = \lambda_0$ and rearranging yields

$$\left(N_{s_0} - s_0^2\lambda_0\right) \dot{\mathbf{w}}_{\lambda_0} = s_0^2 \mathbf{u}_{s_0}. \tag{2.61}$$

Setting $s_0^2\, \mathbf{v}_{s_0} = \dot{\mathbf{w}}_{\lambda_0}$, equation (2.57) follows.

The same arguments as in the proof of Corollary 2.22 are used to prove (2.58). Equations (2.59) and (2.60) follow from the structure of the eigenvectors and generalised eigenvectors when $\lambda_0 = 0$. If $0 \in \operatorname{Spec}(L_-^{s_0}) \setminus \operatorname{Spec}(L_+^{s_0})$ and $\widehat{u}_{s_0}$ is as stated in the lemma, we have

$$\begin{pmatrix} 0 & -L_-^{s_0} \\ L_+^{s_0} & 0 \end{pmatrix} \begin{pmatrix} \widehat{u}_{s_0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_{s_0} \end{pmatrix} = \mathbf{u}_{s_0},$$

so $\mathbf{v}_{s_0} = (\widehat{u}_{s_0}, 0)^\top$ and hence $\langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0} \rangle = \langle \widehat{u}_{s_0}, v_{s_0} \rangle$. If $0 \in \operatorname{Spec}(L_+^{s_0}) \setminus \operatorname{Spec}(L_-^{s_0})$, we similarly find that $\mathbf{v}_{s_0} = (0, \widehat{v}_{s_0})^\top$ and hence $\langle \mathbf{v}_{s_0}, S\mathbf{u}_{s_0} \rangle = \langle \widehat{v}_{s_0}, u_{s_0} \rangle$. Finally, if $\dim\ker(N_{s_0}) = 2$, we have

$$\mathbf{v}_{s_0}^{(1)} = \begin{pmatrix} 0 \\ \widehat{v}_{s_0}^{(1)} \end{pmatrix}, \quad \mathbf{v}_{s_0}^{(2)} = \begin{pmatrix} \widehat{u}_{s_0}^{(2)} \\ 0 \end{pmatrix}, \tag{2.62}$$

with $\mathbf{u}_{s_0}^{(i)}$ given by (2.42). It follows that $\langle \mathbf{v}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)} \rangle = \langle \mathbf{v}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(1)} \rangle = 0$ and

$$\langle \mathbf{v}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(1)} \rangle = \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle, \qquad \langle \mathbf{v}_{s_0}^{(2)}, S\mathbf{u}_{s_0}^{(2)} \rangle = \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle, \tag{2.63}$$

which completes the proof. □

Remark 2.30. The Maslov index is in general not monotone in λ, in the sense that the form (2.51) is indefinite. Consequently, it does not necessarily give an exact count of the crossings along Γ_3 for $\lambda > 0$, which by Proposition 2.19 equals the number of real positive eigenvalues of N. Nonetheless, the Maslov index always provides a lower bound for this count, and this will be used in the proof of Theorem 2.2. In special cases it *is* possible to have monotonicity in λ; this will be used to obtain stability results in Theorem 2.11, cf. Lemma 2.52.

2.2.4 Bounding the real eigenvalue count

Before proving Theorem 2.2, we list some preliminary results. The first is a version of the Morse Index theorem (see [Mil63, §15],[Sma65]) for scalar-valued Schrödinger operators on bounded domains with Dirichlet boundary conditions. Recall that the Morse indices P and Q are the numbers of negative eigenvalues of the operators L_+ and L_-, respectively.

Lemma 2.31. *The Morse index of L_+ equals the number of conjugate points for L_+ in $(0,1)$,*

$$P = \#\{s_0 \in (0,1) : 0 \in \operatorname{Spec}(L_+^{s_0})\}, \tag{2.64}$$

and likewise for L_- and Q.

The following lemma will not be needed until the proof of Lemma 2.51, but we list it here since its proof uses the same ideas that are used to prove the previous lemma.

Lemma 2.32. *If $Q = 0$ (respectively, $P = 0$) then L_-^s (respectively, L_+^s) is a strictly positive operator for all $s \in (0,1)$, and is nonnegative for $s = 1$.*

Proof. This follows from monotonicity of the eigenvalues of the Schrödinger operators $L_\pm^s$ in the spatial parameter s, see [Sma65]. Indeed, the eigenvalues $\lambda_j^\pm(s) \in \operatorname{Spec}(L_\pm^s)$ are strictly decreasing functions of s, so $\lambda_j^\pm(1) \geq 0$ implies $\lambda_j^\pm(s) > 0$ for $s \in (0,1)$. □

The following selfadjoint formulation of the eigenvalue problem will be needed in Lemma 2.35. Some of the ideas used here, especially the use of the square root of a strictly positive operator to convert the eigenvalue problem to a selfadjoint one, can be found in [Pel11, §4].

Lemma 2.33. *Fix $s \in (0,1]$ and suppose $\lambda \in \mathbb{C}\backslash\{0\}$. If L_-^s is a nonnegative operator, the eigenvalue problem*

$$\begin{cases} \text{There exists } v_s \in \operatorname{dom}(L_-^s),\, u_s \in \operatorname{dom}(L_+^s) \text{ such that:} \\ \quad -L_-^s v_s = s^2\lambda u_s, \quad L_+^s u_s = s^2 \lambda v_s \end{cases} \tag{2.65}$$

is equivalent to

$$\begin{cases} \text{There exists } w_s \in \operatorname{dom}\left(L_-^s|_{X_c}\right)^{1/2} \text{ with } \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \operatorname{dom}(L_+^s) \\ \text{and } L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s \in \operatorname{dom}(L_-^s) \text{, such that:} \\ \quad \left(L_-^s|_{X_c}\right)^{1/2} \Pi L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2} w_s = -s^4\lambda^2 w_s, \end{cases} \tag{2.66}$$

where the domains $\operatorname{dom}(L_\pm^s)$ are given by (2.9), $X_c := \ker(L_-^s)^\perp \subseteq L^2(0,\ell)$ and Π is the orthogonal projection $\Pi : L^2(0,\ell) \to X_c$. If L_+^s is nonnegative, then (2.65) is equivalent to

$$\begin{cases} \text{There exists } w_s \in \operatorname{dom}\left(L_+^s|_{X_c}\right)^{1/2} \text{ with } \Pi\left(L_+^s|_{X_c}\right)^{1/2} w_s \in \operatorname{dom}(L_-^s) \\ \text{and } L_-^s \Pi \left(L_+^s|_{X_c}\right)^{1/2} w_s \in \operatorname{dom}(L_+^s) \text{, such that:} \\ \quad \left(L_+^s|_{X_c}\right)^{1/2} \Pi L_-^s \Pi \left(L_+^s|_{X_c}\right)^{1/2} w_s = -s^4\lambda^2 w_s, \end{cases} \tag{2.67}$$

where now $X_c := \ker(L_+^s)^\perp \subseteq L^2(0,\ell)$.

Proof. We begin with the case $L_-^s \geq 0$. We prove the equivalence of (2.65) and (2.66) via their equivalence with:

$$\begin{cases} \text{There exists } u_s \in \operatorname{dom}(L_+^s) \cap X_c \text{ with } L_+^s u_s \in \operatorname{dom}(L_-^s) \text{, such that:} \\ \quad L_-^s L_+^s u_s = -s^4\lambda^2 u_s. \end{cases} \tag{2.68}$$

Defining the restricted operator $L_-^s|_{X_c}$ acting in X_c by

$$L_-^s|_{X_c} v := L_-^s v, \qquad v \in \operatorname{dom}(L_-^s|_{X_c}) := \operatorname{dom}(L_-^s) \cap X_c,$$

note that $L_-^s|_{X_c} > 0$ and $\left(L_-^s|_{X_c}\right)^{1/2}$ is a well-defined and invertible operator acting in X_c.

(2.65) $\implies$ (2.68): Clearly $L_+^s u_s = s^2\lambda v_s \in \operatorname{dom}(L_-^s)$, and $u_s = -\frac{1}{s^2\lambda} L_-^s v_s \in \operatorname{Ran} L_-^s = X_c$ because L_-^s is selfadjoint and Fredholm. Applying L_-^s to the second equation in (2.65) yields the equation in (2.68).

(2.68) $\implies$ (2.66): Set $w_s := \left(L_-^s|_{X_c}\right)^{-1/2} u_s$. Then $w_s \in \operatorname{dom}\left(L_-^s|_{X_c}\right)^{1/2}$, and since $u_s \in X_c$ we have $\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = \Pi u_s = u_s \in \operatorname{dom}(L_+^s)$, and $L_+^s \Pi u_s = L_+^s u_s \in \operatorname{dom}(L_-^s)$. Now $L_+^s u_s = \Pi L_+^s u_s + (I - \Pi) L_+^s u_s$, where the projection $(I-\Pi) : L^2(0,\ell) \to \ker(L_-^s) \subset \operatorname{dom}(L_-^s)$. Then $\Pi L_+^s u_s \in \operatorname{dom}(L_-^s) \cap X_c = \operatorname{dom}(L_-^s|_{X_c})$. Thus $L_-^s L_+^s u_s = L_-^s \Pi L_+^s \Pi u_s = L_-^s|_{X_c} \Pi L_+^s \Pi u_s = \left(L_-^s|_{X_c}\right)^{1/2} \left(L_-^s|_{X_c}\right)^{1/2} \Pi L_+^s \Pi u_s$. Substituting this into the equation in (2.68) and multiplying by $\left(L_-^s|_{X_c}\right)^{-1/2}$ gives the equation in (2.66).

(2.66) $\implies$ (2.65): Set $u_s := \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_+^s)$ and $v_s := \frac{1}{s^2\lambda} L_+^s \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s \in \mathrm{dom}(L_-^s)$. Then $L_+^s u_s = L_+^s \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2\lambda v_s$, and since Π projects onto $\mathrm{Ran}(L_-^s)$, $-L_-^s v_s = -\Pi L_-^s v_s = \frac{-1}{s^2\lambda}\Pi L_-^s L_+^s \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = \frac{-1}{s^2\lambda}\Pi L_-^s(\Pi + (I-\Pi)L_+^s\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s$ $= \frac{-1}{s^2\lambda}\Pi L_-^s \Pi L_+^s \Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2\lambda\Pi\left(L_-^s|_{X_c}\right)^{1/2} w_s = s^2\lambda u_s$.

The case $L_+^s \geq 0$ uses similar arguments, except now (2.65) and (2.67) are equivalent via:

$$\begin{cases} \text{There exists } v_s \in \mathrm{dom}(L_-^s) \cap X_c \text{ with } L_-^s v_s \in \mathrm{dom}(L_+^s)\text{, such that:} \\ L_+^s L_-^s v_s = -s^4\lambda^2 v_s. \end{cases}$$

We omit the details. □

We are now ready to compute the Maslov index of Γ_2^ε, the restriction of Γ_2 to $[\tau, 1-\varepsilon]$.

Lemma 2.34. *The Maslov index of the Lagrangian path $s \mapsto \Lambda(0,s) \subset \mathbb{R}^8$, $s \in [\tau, 1-\varepsilon]$ is*

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) = Q - P. \tag{2.69}$$

Proof. Consider the crossing form

$$\mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2}\left[-\left(u_{s_0}'(\ell)\right)^2 + \left(v_{s_0}'(\ell)\right)^2\right]$$

from (2.44) and recall (2.41). If $(0, s_0)$ is a simple crossing, we obtain $\mathfrak{m}_{s_0} < 0$ if $0 \in \mathrm{Spec}(L_+^{s_0})$ and $\mathfrak{m}_{s_0} > 0$ if $0 \in \mathrm{Spec}(L_-^{s_0})$. On the other hand, if $0 \in \mathrm{Spec}(L_+^{s_0}) \cap \mathrm{Spec}(L_-^{s_0})$, the 2×2 matrix $\mathfrak{M}_{s_0}$ in (2.50) has eigenvalues of opposite sign, so we conclude that

$$\mathrm{sign}(\mathfrak{m}_{s_0}) = \begin{cases} -1 & 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0}), \\ +1 & 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0}), \\ 0 & 0 \in \mathrm{Spec}(L_+^{s_0}) \cap \mathrm{Spec}(L_-^{s_0}). \end{cases} \tag{2.70}$$

From the definition (2.25) we then have

$$\begin{aligned} \mathrm{Mas}(\Lambda(0,s), \mathcal{D}; s \in [\tau, 1-\varepsilon]) &= -\#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})\} \\ &\quad + \#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})\} \\ &= -\#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_+^{s_0})\} \\ &\quad + \#\{s_0 \in [\tau, 1-\varepsilon] : 0 \in \mathrm{Spec}(L_-^{s_0})\}, \end{aligned}$$

and the result follows using Lemma 2.31. □

Next, we prove that there are no crossings along Γ_1 and Γ_4; we refer to Fig. 2.2.

Lemma 2.35. $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_1) = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0$ *provided $\tau > 0$ is sufficiently small and $\lambda_\infty > 0$ is sufficiently large.*

Proof. For the case of no crossings along Γ_1, we prove that N_s has no real eigenvalues for $s = \tau$ small enough. Seeking a contradiction, assume there exists $\tau^2\lambda \in \operatorname{Spec}(N_\tau) \cap \mathbb{R}$ with eigenfunction $\mathbf{u}_\tau = (u_\tau, v_\tau)^\top$.

First, note that the operators $L_\pm^\tau$ with domains given by (2.9) are strictly positive: by the Poincaré and Cauchy-Schwarz inequalities,

$$\langle L_+^\tau v, v\rangle = \|v'\|^2 - \langle \tau^2 g(\tau x)v, v\rangle \geq C\|v\|^2 - \tau^2\|g\|_\infty \|v\|^2$$

for some $C > 0$ and all $v \in \operatorname{dom}(L_+^\tau)$, so we choose τ small enough that $C > \tau^2\|g\|_\infty$. Owing to the decoupling of the eigenvalue equations for N_τ when $\lambda = 0$, it follows that $0 \notin \operatorname{Spec}(N_\tau)$.

Next, for $\lambda \in \mathbb{R}\backslash\{0\}$, we note that by Lemma 2.33 the eigenvalue equations for N_τ are equivalent to

$$\left(L_-^\tau\right)^{1/2} L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau = -\tau^4\lambda^2 w_\tau, \tag{2.71}$$

since the positivity of L_-^τ implies that $X_c = \ker(L_-^\tau)^\perp$ is all of $L^2(0,\ell)$ and hence the resulting projection Π is the identity. Applying $\langle\, \cdot, w_\tau\rangle$ to (2.71), we immediately see that the right hand side is negative, while for the left hand side we obtain

$$\begin{aligned}\langle \left(L_-^\tau\right)^{1/2} L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau, w_\tau\rangle &= \langle L_+^\tau \left(L_-^\tau\right)^{1/2} w_\tau, \left(L_-^\tau\right)^{1/2} w_\tau\rangle \\ &\geq C_+ \langle \left(L_-^\tau\right)^{1/2} w_\tau, \left(L_-^\tau\right)^{1/2} w_\tau\rangle \\ &= C_+\langle L_-^\tau w_\tau, w_\tau\rangle \\ &\geq C_+ C_- \|w_\tau\|^2 > 0,\end{aligned}$$

for some positive constants $C_\pm$ (using the positivity of $L_\pm^\tau$ and selfadjointness of $\left(L_-^\tau\right)^{1/2}$), a contradiction. We conclude that no such real $\tau^2\lambda \in \operatorname{Spec}(N_\tau)$ exists, and there are no crossings along Γ_1.

Moving to Γ_4, we show that the spectrum of N_s lies in a vertical strip around the imaginary axis in the complex plane for all $s \in (0,1]$). For this, it suffices to show that $\operatorname{Spec}(iN_s)$ lies in a horizontal strip around the real axis, since $\operatorname{Spec}(N_s) = -i \operatorname{Spec}(iN_s)$ by the spectral mapping theorem. Fixing $s \in (0,1]$ we have

$$iN_s = iD + iB_s(x) \tag{2.72}$$

where iD is selfadjoint and $iB_s(x)$ is bounded. It then follows from [Kat80, Remark 3.2, p.208] and [Kat80, eq.(3.16), p.272] that

$$\zeta \in \operatorname{Spec}(iD + iB_s(x)) \implies |\operatorname{Im}\zeta| \leq \|iB_s(x)\|, \tag{2.73}$$

as required. Choosing $\lambda_\infty > \sup_{s\in(0,1]} \|B_s(x)\|$ ensures there are no crossings along Γ_4. □

We are our ready to prove our first main result.

Proof of Theorem 2.2. As already observed in (2.39), the homotopy invariance and additivity of the Maslov index yield

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_1) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0, \tag{2.74}$$

hence

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) = 0 \tag{2.75}$$

by Lemma 2.35. Again using additivity and recalling the definition of $\mathfrak{c}$ in Definition 2.26, we rewrite this as

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) + \mathfrak{c} + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = 0, \tag{2.76}$$

where Γ_2^ε was defined in Lemma 2.34 and Γ_3^ε is the restriction of Γ_3 to $[\varepsilon, \lambda_\infty]$. Using Lemma 2.34 we thus obtain

$$\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = P - Q - \mathfrak{c}. \tag{2.77}$$

As discussed in Remark 2.30, the lack of monotonicity in λ means that $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$ does not necessarily count the number of real, positive eigenvalues of N. Nonetheless, we still have that

$$n_+(N) \geq |\,\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)|, \tag{2.78}$$

and (2.11) follows. □

2.3 The eigenvalue curves

In this section we analyse the real eigenvalue curves of N_s in the λs-plane. We consider the general case of a crossing (λ_0, s_0) corresponding to an eigenvalue $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ with $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$, paying special attention to the cases $\lambda_0 = 0$ and $n = 1, 2$. We use the results obtained to compute the correction term $\mathfrak{c}$ from Theorem 2.2, and relate a component of it to the signature of the second order crossing form (2.57) in Proposition 2.50.

2.3.1 Numerical description

We begin with a brief description of a tool that is useful for numerically computing the eigenvalue curves. The idea is to globally characterise the set of points (λ, s) such that $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ as the zero set of a function called the *characteristic determinant*.

Converting the restricted problem (2.10) with $y \in [0, s\ell]$ to a first order system yields

$$\frac{d}{dy}\begin{pmatrix} u \\ v \\ r \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -g(y) & -\lambda & 0 & 0 \\ -\lambda & h(y) & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ r \\ z \end{pmatrix}. \tag{2.79}$$

Notice that we use the substitution $\partial_y v = -z$ in order to preserve the Hamiltonian structure. Rescaling as in Section 2.2.2, we define $u_s(x) := u(sx)$ for $x \in [0, \ell]$, and similarly for v_s, r_s and

z_s. Then, the equivalent system on $[0, \ell]$ is

$$\frac{d}{dx}\begin{pmatrix} u_s \\ v_s \\ r_s \\ z_s \end{pmatrix} = \begin{pmatrix} 0 & 0 & s & 0 \\ 0 & 0 & 0 & -s \\ -sg(sx) & -s\lambda & 0 & 0 \\ -s\lambda & sh(sx) & 0 & 0 \end{pmatrix}\begin{pmatrix} u_s \\ v_s \\ r_s \\ z_s \end{pmatrix}. \tag{2.80}$$

Consider a fundamental matrix solution $\Phi(x; \lambda, s) \in \mathbb{R}^{4\times 4}$ to (2.80) with $\Phi(0; \lambda, s) = I_4$. For convenience, we write Φ as the block matrix

$$\Phi(x; \lambda, s) = \begin{pmatrix} U(x; \lambda, s) & X(x; \lambda, s) \\ V(x; \lambda, s) & Y(x; \lambda, s) \end{pmatrix}, \qquad U, V, X, Y \in \mathbb{R}^{2\times 2},$$

where

$$U(0; \lambda, s) = Y(0; \lambda, s) = I_2, \qquad V(0; \lambda, s) = X(0; \lambda, s) = 0_2. \tag{2.81}$$

Because Φ is a matrix solution for (2.80), we have

$$\frac{d}{dx}\begin{pmatrix} U & X \\ V & Y \end{pmatrix} = \begin{pmatrix} 0 & s\sigma_3 \\ s\left(SB(sx) - \lambda S\right) & 0 \end{pmatrix}\begin{pmatrix} U & X \\ V & Y \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.82}$$

Proposition 2.36. *For all $(\lambda, s) \in \mathbb{R} \times (0, 1]$, the following are equivalent:*

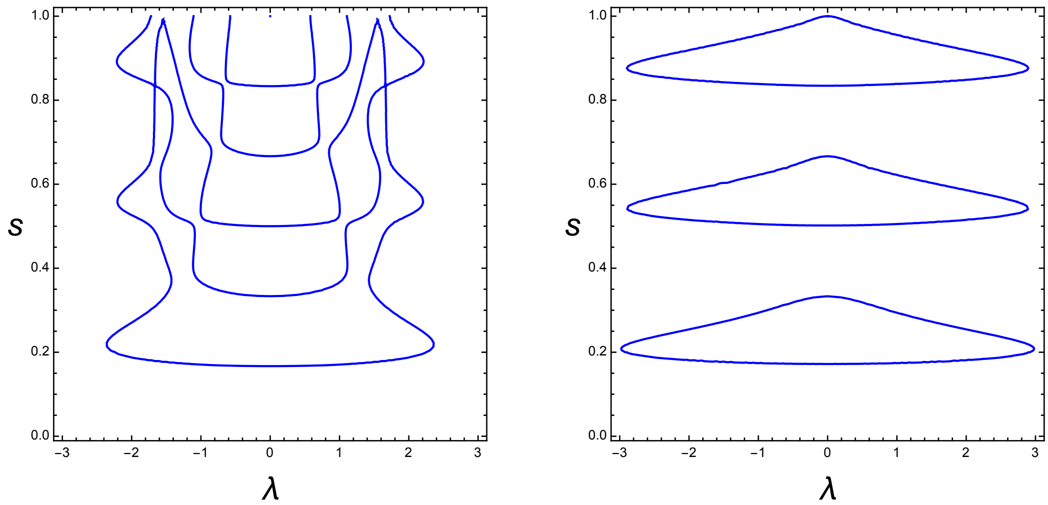
1. $\lambda \in \operatorname{Spec}(N|_{[0,s\ell]}) \cap \mathbb{R}$,
2. $s^2\lambda \in \operatorname{Spec}(N_s) \cap \mathbb{R}$,
3. $\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\}$,
4. $\det X(\ell; \lambda, s) = 0$.

We thus call $\det X(\ell; \lambda, s)$ the *characteristic determinant*: the real eigenvalue curves in the λs-plane are given by the zero set $\{(\lambda, s) : \det X(\ell; \lambda, s) = 0\}$. Figure 2.3 illustrates some examples of these curves under Hypothesis 2.5.

Proof. The discussion following (2.32) gives the equivalence of (1) and (2), while the equivalence of (2) and (3) was given in Proposition 2.19. We show the equivalence of (3) and (4). Fix $s \in (0, 1]$ and $\lambda \in \mathbb{R}$ and consider the 8×4 matrix

$$\mathcal{Z}(\lambda, s) := \begin{pmatrix} U(0; \lambda, s) & X(0; \lambda, s) \\ U(\ell; \lambda, s) & X(\ell; \lambda, s) \\ -V(0; \lambda, s) & -Y(0; \lambda, s) \\ V(\ell; \lambda, s) & Y(\ell; \lambda, s) \end{pmatrix} = \begin{pmatrix} I_2 & 0_2 \\ U(\ell; \lambda, s) & X(\ell; \lambda, s) \\ 0_2 & -I_2 \\ V(\ell; \lambda, s) & Y(\ell; \lambda, s) \end{pmatrix}.$$

Notice that the columns of $\mathcal{Z}(\lambda, s)$ are precisely the rescaled trace (cf. (2.31)) of four linearly independent functions in $\mathcal{K}_{\lambda,s}$ (recall that the entries of $Y(\cdot\,; \lambda, s)$ and $V(\cdot\,; \lambda, s)$ satisfy $r_s = s^{-1}\partial_x u_s$ and $z_s = -s^{-1}\partial_x v_s$), and thus are a basis for our Lagrangian subspace $\Lambda(\lambda, s)$.



(a) (b)

Figure 2.3: Real eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ under Hypothesis 2.5(i) associated with a T-periodic stationary state ϕ_0 with nonlinearity $f(\phi^2) = \phi^2$ and $\beta = -2$. In (a) ϕ_0 is a positive Jacobi dnoidal function (i.e. an orbit located inside the homoclinic orbit in the right half plane in Fig. 2.1a) satisfying $\phi_0'(0) = \phi_0'(\ell) = 0$ with $\ell = 3T = 9.9398$. In (b) ϕ_0 is a Jacobi cnoidal function (i.e. an orbit located outside the homoclinic orbit in Fig. 2.1a) satisfying $\phi_0(0) = \phi_0(\ell) = 0$ with $\ell = 3T/2 = 10.0391$.

A nontrivial intersection of the four-dimensional linear subspaces $\Lambda(\lambda, s)$ and $\mathcal{D}$ of $\mathbb{R}^8$ occurs if and only if the 8×8 matrix formed by their bases has zero determinant. Therefore,

$$\Lambda(\lambda, s) \cap \mathcal{D} \neq \{0\} \iff \det \begin{pmatrix} I & 0 & 0 & 0 \\ U(\ell;\lambda,s) & X(\ell;\lambda,s) & 0 & 0 \\ -0 & -I & I & 0 \\ V(\ell;\lambda,s) & Y(\ell;\lambda,s) & 0 & I \end{pmatrix} = 0 \iff \det X(\ell;\lambda,s) = 0,$$

as required. □

2.3.2 Analytic description

We will generalise Theorem 2.9 to Theorem 2.40, which is a consequence of the following general result. We remind the reader that $n \le 2$ in the current paper; see Remark 2.20. Below, dot denotes $d/d\lambda$.

Proposition 2.37. *Assume* $\dim\ker(N_{s_0} - s_0^2\lambda_0) = n$ *with basis* $\{\mathbf{u}_{s_0}^{(1)}, \dots, \mathbf{u}_{s_0}^{(n)}\}$. *There exists an* $n \times n$ *matrix* $M(\lambda, s)$, *defined near* (λ_0, s_0), *such that* $s^2\lambda \in \mathrm{Spec}(N_s)$ *if and only if* $\det M(\lambda, s) = 0$. *This matrix satisfies* $M(\lambda_0, s_0) = 0$ *and*

$$\frac{\partial M_{ij}}{\partial \lambda}(\lambda_0, s_0) = -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \quad \frac{\partial M_{ij}}{\partial s}(\lambda_0, s_0) = \left\langle (\partial_s B_{s_0} - 2s_0\lambda_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle. \tag{2.83}$$

Moreover, if $\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \rangle = 0$ *for all* $i, j = 1, \dots, n$, *then*

$$\frac{\partial^2 M_{ij}}{\partial \lambda^2}(\lambda_0, s_0) = -2s_0^4 \left\langle \mathbf{v}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \tag{2.84}$$

where $\mathbf{v}_{s_0}^{(i)} \in \operatorname{dom}(N_{s_0})$ *solves the inhomogeneous equation* $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$.

Remark 2.38. Just as in Remark 2.29, for (2.84) it suffices to consider those solutions to the inhomogeneous equation that satisfy $\mathbf{v}_{s_0}^{(i)} \perp \mathbf{u}_{s_0}^{(j)}$ for $i, j = 1, \ldots, n$.

The definition of M, which requires some preparation, is given in (2.94).

Proof. We construct $M(\lambda, s)$ using Lyapunov–Schmidt reduction. The first step is to split the eigenvalue equation $(N_s - s^2\lambda)\mathbf{u} = 0$ into two parts, one of which can always be solved uniquely. Let P denote the L^2-orthogonal projection onto $\ker(N_{s_0}^* - s_0^2\lambda_0)$, so that $I - P$ is the projection onto $\ker(N_{s_0}^* - s_0^2\lambda_0)^\perp = \operatorname{Ran}(N_{s_0} - s_0^2\lambda_0)$. It follows that $s^2\lambda$ is an eigenvalue of N_s if and only if there exists a nonzero $\mathbf{u} \in \operatorname{dom}(N_s)$ such that both

$$P(N_s - s^2\lambda)\mathbf{u} = 0 \tag{2.85}$$

and

$$(I - P)(N_s - s^2\lambda)\mathbf{u} = 0 \tag{2.86}$$

hold.

We first consider (2.86). Defining $X_0 = \ker(N_{s_0} - s_0^2\lambda_0)^\perp \cap H^2(0,\ell) \cap H_0^1(0,\ell)$, we have that any $\mathbf{u} \in H^2(0,\ell) \cap H_0^1(0,\ell)$ can be written uniquely as

$$\mathbf{u} = \sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)} + \tilde{\mathbf{u}},$$

where $t_i \in \mathbb{R}$ and $\tilde{\mathbf{u}} \in X_0$. This means (2.86) holds if and only if there exists a vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and a function $\tilde{\mathbf{u}} \in X_0$ such that

$$(I - P)(N_s - s^2\lambda)\left(\sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)} + \tilde{\mathbf{u}}\right) = 0. \tag{2.87}$$

We claim that for each $(\mathbf{t}, \lambda, s)$ there exists a unique $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) \in X_0$ satisfying (2.87). Writing this equation out explicitly, it is

$$(I - P)(N_s - s^2\lambda)\tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) = -(I - P)(N_s - s^2\lambda)\sum_{i=1}^n t_i \mathbf{u}_{s_0}^{(i)}.$$

We define

$$T(\lambda, s)\colon X_0 \to \operatorname{Ran}(N_{s_0} - s_0^2\lambda_0), \qquad T(\lambda, s) = (I - P)\big(N_s - s^2\lambda\big)\Big|_{X_0}, \tag{2.88}$$

and observe that $T(\lambda_0, s_0)$ is invertible, hence $T(\lambda, s)$ is also invertible for nearby (λ, s). Defining

$$A(\lambda, s) : X_0^\perp \to X_0, \qquad A(\lambda, s) = -T^{-1}(\lambda, s)(I - P)\big(N_s - s^2\lambda\big)\Big|_{X_0^\perp}, \tag{2.89}$$

where $X_0^\perp = \ker(N_{s_0} - s_0^2\lambda_0)$, the unique solution to (2.87) is thus

$$\tilde{\mathbf{u}}(\mathbf{t}, \lambda, s) = A(\lambda, s) \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)}. \tag{2.90}$$

So far we have shown that the equation $(I - P)(N_s - s^2\lambda)\mathbf{u} = 0$ is satisfied if and only if $\mathbf{u}$ has the form

$$\mathbf{u} = \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} + A(\lambda, s) \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} = \big(I + A(\lambda, s)\big) \sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} \tag{2.91}$$

for some $\mathbf{t} \in \mathbb{R}^n$. We conclude that there exists $\mathbf{u}$ for which $(N_s - s^2\lambda)\mathbf{u} = 0$ holds if and only if

$$P(N_s - s^2\lambda)\big(I + A(\lambda, s)\big) \left(\sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} \right) = 0 \tag{2.92}$$

for some $\mathbf{t} \in \mathbb{R}^n$. Moreover, $\mathbf{u}$ is nonzero if and only if $\mathbf{t}$ is nonzero. Finally, we observe that $\ker(N_{s_0}^* - s_0^2\lambda_0)$ is spanned by $\{S\mathbf{u}_{s_0}^{(1)}, S\mathbf{u}_{s_0}^{(2)}, \ldots, S\mathbf{u}_{s_0}^{(n)}\}$, and so (2.92) is equivalent to

$$\left\langle (N_s - s^2\lambda)\big(I + A(\lambda, s)\big) \left(\sum_{i=1}^{n} t_i \mathbf{u}_{s_0}^{(i)} \right), S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0, \qquad j = 1, \ldots, n. \tag{2.93}$$

Defining the $n \times n$ matrix $M(\lambda, s)$ by

$$M_{ji}(\lambda, s) = \left\langle (N_s - s^2\lambda)\big(I + A(\lambda, s)\big)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \quad i, j = 1, \ldots, n, \tag{2.94}$$

the system of n equations (2.93) may be written as $M(\lambda, s)\mathbf{t} = 0$, which is satisfied for a nonzero vector $\mathbf{t}$ if and only if $\det M(\lambda, s) = 0$. This completes the first part of the proof.

It follows that $M(\lambda_0, s_0) = 0$. We then compute

$$\frac{\partial M_{ij}}{\partial \lambda}(\lambda_0, s_0) = \left\langle -s_0^2\big(I + A(\lambda_0, s_0)\big)\mathbf{u}_{s_0}^{(i)} + (N_{s_0} - s_0^2\lambda_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle \tag{2.95}$$

$$= -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \tag{2.96}$$

where in the second line we have used the fact that $A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = 0$ and

$$\left\langle (N_{s_0} - s_0^2\lambda_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle = \left\langle \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, (N_{s_0}^* - s_0^2\lambda_0)S\mathbf{u}_{s_0}^{(j)} \right\rangle = 0,$$

because $S\mathbf{u}_{s_0}^{(j)} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$. The s derivative is computed similarly.

Finally, if $\partial_\lambda M(\lambda_0, s_0) = 0$, we have

$$\frac{\partial^2 M_{ij}}{\partial \lambda^2}(\lambda_0, s_0) = -2s_0^2 \left\langle \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \tag{2.97}$$

where $\langle (N_{s_0} - s_0^2\lambda_0)\partial_{\lambda\lambda} A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \rangle = 0$ again using $S\mathbf{u}_{s_0}^{(j)} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$. To compute $\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}$, we use the definition of $A(\lambda, s)$ to write

$$T(\lambda, s)A(\lambda, s)\mathbf{u}_{s_0}^{(i)} = -(I - P)\big(N_s - s^2\lambda\big)\mathbf{u}_{s_0}^{(i)}.$$

Differentiating in λ and again using the fact that $A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = 0$, we get

$$T(\lambda_0, s_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)} = s_0^2(I - P)\mathbf{u}_{s_0}^{(i)}. \tag{2.98}$$

The fact that $\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \rangle = 0$ for all i, j implies $(I-P)\mathbf{u}_{s_0}^{(i)} = \mathbf{u}_{s_0}^{(i)}$. Setting $s_0^2\mathbf{v}_{s_0}^{(i)} = \partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}^{(i)}$, we see from the definition of T that

$$T(\lambda_0, s_0)(s_0^2\mathbf{v}_{s_0}^{(i)}) = s_0^2(I - P)(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)} = s_0^2(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(i)}$$

and the result follows. □

Comparison with the symmetric matrices (2.48), (2.54) and (2.58) associated with the first and second order crossing forms reveals that the partial derivatives of the matrix M satisfy

$$\frac{\partial M}{\partial s}(\lambda_0, s_0) = s_0\, \mathfrak{M}_{s_0}, \quad \frac{\partial M}{\partial \lambda}(\lambda_0, s_0) = s_0\, \mathfrak{M}_{\lambda_0}, \quad \frac{\partial^2 M}{\partial \lambda^2}(\lambda_0, s_0) = s_0\, \mathfrak{M}_{\lambda_0}^{(2)}, \tag{2.99}$$

where the last formula holds when $\partial_\lambda M(\lambda_0, s_0) = 0$. In particular, in the case $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$ (so that M is a scalar), we have

$$\frac{\partial M}{\partial s}(\lambda_0, s_0) = s_0\, \mathfrak{m}_{s_0}(q), \quad \frac{\partial M}{\partial \lambda}(\lambda_0, s_0) = s_0\, \mathfrak{m}_{\lambda_0}(q), \quad \frac{\partial^2 M}{\partial \lambda^2}(\lambda_0, s_0) = s_0 \mathfrak{m}_{\lambda_0}^{(2)}(q), \tag{2.100}$$

where again the last formula holds when $\partial_\lambda M(\lambda_0, s_0) = 0$. Combining (2.100) with the implicit function theorem immediately yields the following Hadamard-type formulas for the derivatives of the real eigenvalue curves in terms of the crossing forms.

Corollary 2.39. *Under the assumption that* $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$*, the following hold:*

1. *If* $\mathfrak{m}_{\lambda_0} \neq 0$*, then there exists a* C^2 *curve* $\lambda(s)$ *near* s_0 *such that*

$$\lambda'(s_0) = -\frac{\mathfrak{m}_{s_0}(q)}{\mathfrak{m}_{\lambda_0}(q)}. \tag{2.101}$$

2. *If* $\mathfrak{m}_{s_0} \neq 0$*, then there exists a* C^2 *curve* $s(\lambda)$ *near* λ_0 *such that*

$$\dot{s}(\lambda_0) = -\frac{\mathfrak{m}_{\lambda_0}(q)}{\mathfrak{m}_{s_0}(q)}. \tag{2.102}$$

   Moreover, $\dot{s}(\lambda_0) = 0$ *if and only if* $\mathfrak{m}_{\lambda_0}(q) = 0$*, and in this case*

$$\ddot{s}(\lambda_0) = -\frac{\mathfrak{m}_{\lambda_0}^{(2)}(q)}{\mathfrak{m}_{s_0}(q)}. \tag{2.103}$$

Using this, we can construct a curve $s(\lambda)$ through any simple conjugate point and determine its concavity by an explicit formula.

Theorem 2.40. *If* $\dim\ker N_{s_0} = 1$*, then for* $|\lambda| \ll 1$ *there exists a* C^2 *curve* $s(\lambda)$ *such that* $s(\lambda)^2\lambda \in \operatorname{Spec}(N_{s(\lambda)})$*, and a continuous curve* $\mathbf{u}_{s(\lambda)}$ *of eigenfunctions such that* $\mathbf{u}_{s(\lambda)} \to \mathbf{u}_{s_0}$ *as* $\lambda \to 0$*. Moreover,* $s(0) = s_0$*,* $\dot{s}(0) = 0$*, and the concavity of* $s(\lambda)$ *can be determined as follows:*

1. *If* $0 \in \operatorname{Spec}(L_-^{s_0}) \setminus \operatorname{Spec}(L_+^{s_0})$ *with eigenfunction* $v_{s_0} \in \ker L_-^{s_0}$*, then*

$$\ddot{s}(0) = \frac{2s_0^5}{\ell} \frac{\langle \widehat{u}_{s_0}, v_{s_0} \rangle}{\left(v'_{s_0}(\ell)\right)^2} \tag{2.104}$$

where $\widehat{u}_{s_0} \in H^2(0,\ell) \cap H_0^1(0,\ell)$ *is the unique solution to* $L_+^{s_0} \widehat{u}_{s_0} = v_{s_0}$.

2. *If* $0 \in \operatorname{Spec}(L_+^{s_0}) \setminus \operatorname{Spec}(L_-^{s_0})$ *with eigenfunction* $u_{s_0} \in \ker L_+^{s_0}$*, then*

$$\ddot{s}(0) = -\frac{2s_0^5}{\ell} \frac{\langle \widehat{v}_{s_0}, u_{s_0} \rangle}{\left(u'_{s_0}(\ell)\right)^2} \tag{2.105}$$

where $\widehat{v}_{s_0} \in H^2(0,\ell) \cap H_0^1(0,\ell)$ *is the unique solution to* $-L_-^{s_0} \widehat{v}_{s_0} = u_{s_0}$.

Proof. Lemma 2.21 implies $\mathfrak{m}_{s_0} \neq 0$, so the existence of $s(\lambda)$ follows from Corollary 2.39. Corollary 2.24 then gives $\dot{s}(0) = 0$. From (2.91) we see that $\mathbf{u}_{s(\lambda)} = \big(I + A(\lambda, s(\lambda))\big)\mathbf{u}_{s_0}$ is an eigenfunction of $N_{s(\lambda)}$ for the eigenvalue $s^2(\lambda)\lambda$. Since $A(\lambda, s(\lambda))$ is continuous in λ and $A(0, s_0)\mathbf{u}_{s_0} = 0$, the convergence of $\mathbf{u}_{s(\lambda)}$ to $\mathbf{u}_{s_0}$ follows.

It thus remains to prove (2.104) and (2.105). If $0 \in \operatorname{Spec}(L_-^{s_0}) \setminus \operatorname{Spec}(L_+^{s_0})$ then u_{s_0} is trivial, so equations (2.44) and (2.59) give

$$\mathfrak{m}_{s_0}(q) = \frac{\ell}{s_0^2}\left(v'_{s_0}(\ell)\right)^2, \qquad \mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3 \langle \widehat{u}_{s_0}, v_{s_0} \rangle. \tag{2.106}$$

Substituting these into (2.103) immediately gives (2.104). The case $0 \in \operatorname{Spec}(L_+^{s_0}) \setminus \operatorname{Spec}(L_-^{s_0})$ is almost identical. Here we have

$$\mathfrak{m}_{s_0}(q) = -\frac{\ell}{s_0^2}\left(u'_{s_0}(\ell)\right)^2, \qquad \mathfrak{m}_{\lambda_0}^{(2)}(q) = -2s_0^3 \langle \widehat{v}_{s_0}, u_{s_0} \rangle,$$

and (2.105) follows. □

2.3.3 When $\lambda_0 = 0$ has geometric multiplicity two

In this section we focus on the case of a geometrically double eigenvalue at zero. Since $0 \in \operatorname{Spec}(L_+^{s_0}) \cap \operatorname{Spec}(L_-^{s_0})$, we have $\ker(N_{s_0}) = \operatorname{span}\{\mathbf{u}_{s_0}^{(1)}, \mathbf{u}_{s_0}^{(2)}\}$ where the $\mathbf{u}_{s_0}^{(i)}$ are given in (2.42). Applying Proposition 2.37 with $\lambda_0 = 0$ and $n = 2$, we will show the following. Again, dot denotes $d/d\lambda$.

Theorem 2.41. *Suppose* $\dim\ker N_{s_0} = 2$*, and denote the corresponding eigenfunctions of* $L_+^{s_0}$ *and* $L_-^{s_0}$ *by* $u_{s_0}^{(1)}$ *and* $v_{s_0}^{(2)}$*, respectively.*

1. *If* $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$*, then* $s^2\lambda \notin \operatorname{Spec}(N_s)$ *for* (λ, s) *in a punctured neighbourhood of* $(0, s_0)$.

2. *If* $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle = 0$ *and*

$$\frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \neq 0, \tag{2.107}$$

where $\widehat{u}^{(2)}_{s_0} \in \operatorname{dom}(L^{s_0}_+)$ *and* $\widehat{v}^{(1)}_{s_0} \in \operatorname{dom}(L^{s_0}_-)$ *denote solutions to*

$$L^{s_0}_+ \widehat{u}^{(2)}_{s_0} = v^{(2)}_{s_0}, \qquad -L^{s_0}_- \widehat{v}^{(1)}_{s_0} = u^{(1)}_{s_0}, \tag{2.108}$$

then for $|\lambda| \ll 1$ *there exist* C^2 *curves* $s_1(\lambda)$ *and* $s_2(\lambda)$ *such that*

(*i*) $s^2_{1,2}(\lambda)\lambda \in \operatorname{Spec}\left(N_{s_{1,2}(\lambda)}\right)$,

(*ii*) $s_{1,2}(0) = s_0$,

(*iii*) $\dot{s}_{1,2}(0) = 0$,

and the concavities satisfy

$$\ddot{s}_1(0) = -\frac{2s_0^5}{\ell} \frac{\langle \widehat{v}^{(1)}_{s_0}, u^{(1)}_{s_0} \rangle}{\left(\partial_x u^{(1)}_{s_0}(\ell)\right)^2}, \qquad \ddot{s}_2(0) = \frac{2s_0^5}{\ell} \frac{\langle \widehat{u}^{(2)}_{s_0}, v^{(2)}_{s_0} \rangle}{\left(\partial_x v^{(2)}_{s_0}(\ell)\right)^2}. \tag{2.109}$$

Moreover, there exist continuous curves $\mathbf{u}_{s_1(\lambda)}$ *and* $\mathbf{u}_{s_2(\lambda)}$ *of eigenfunctions such that*

$$\mathbf{u}_{s_1(\lambda)} \to \mathbf{u}^{(1)}_{s_0} = \begin{pmatrix} u^{(1)}_{s_0} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_{s_2(\lambda)} \to \mathbf{u}^{(2)}_{s_0} = \begin{pmatrix} 0 \\ v^{(2)}_{s_0} \end{pmatrix} \tag{2.110}$$

as $\lambda \to 0$.

The condition (2.107) will be discussed in Remark 2.45 below.

Remark 2.42. As in Remark 2.29 the solutions $\widehat{u}^{(2)}_{s_0}$ and $\widehat{v}^{(1)}_{s_0}$ in (2.108) are not unique, but the expressions in (2.107) and (2.109) do not depend on the choice of solution.

We prove the theorem by studying the zero set of $m(\lambda, s) := \det M(\lambda, s)$, where M is given in (**??**). We thus start with some elementary calculations for the higher order derivatives of m. These will be used to prove the existence of the eigenvalue curves $s_{1,2}(\lambda)$ and also to evaluate their first and second derivatives.

Lemma 2.43. *Under the assumptions of Theorem 2.41, we have*

$$m(0, s_0) = \frac{\partial m}{\partial s}(0, s_0) = \frac{\partial m}{\partial \lambda}(0, s_0) = \frac{\partial^2 m}{\partial s \partial \lambda}(0, s_0) = 0 \tag{2.111}$$

and

$$\frac{\partial^2 m}{\partial s^2}(0, s_0) = -\frac{2\ell^2}{s_0^2} \left(\partial_x u^{(1)}_{s_0}(\ell)\right)^2 \left(\partial_x v^{(2)}_{s_0}(\ell)\right)^2, \qquad \frac{\partial^2 m}{\partial \lambda^2}(0, s_0) = -2s_0^4 \left\langle u^{(1)}_{s_0}, v^{(2)}_{s_0} \right\rangle^2. \tag{2.112}$$

Moreover, if $\langle u^{(1)}_{s_0}, v^{(2)}_{s_0} \rangle = 0$, *then*

$$\frac{\partial^3 m}{\partial s \partial \lambda^2}(0, s_0) = 2\ell s_0^3 \left(\partial_x u^{(1)}_{s_0}(\ell)\right)^2 \left\langle \widehat{u}^{(2)}_{s_0}, v^{(2)}_{s_0} \right\rangle - 2\ell s_0^3 \left(\partial_x v^{(2)}_{s_0}(\ell)\right)^2 \left\langle \widehat{v}^{(1)}_{s_0}, u^{(1)}_{s_0} \right\rangle \tag{2.113}$$

$$\frac{\partial^3 m}{\partial \lambda^3}(0, s_0) = 0, \qquad \frac{\partial^4 m}{\partial \lambda^4}(0, s_0) = 24 s_0^8 \left\langle \widehat{u}^{(2)}_{s_0}, v^{(2)}_{s_0} \right\rangle \left\langle \widehat{v}^{(1)}_{s_0}, u^{(1)}_{s_0} \right\rangle \tag{2.114}$$

with $\widehat{u}^{(2)}_{s_0}$ *and* $\widehat{v}^{(1)}_{s_0}$ *as in* (2.108).

Proof. Writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $m = ad - bc$, we compute

$$\partial_s m = (\partial_s a)\, d + a\, (\partial_s d) - (\partial_s b)\, c - b\, (\partial_s c),$$
$$\partial_s^2 m = (\partial_s^2 a)\, d + 2\, (\partial_s a)\, (\partial_s d) + a\, (\partial_s^2 d) - (\partial_s^2 b)\, c - 2\, (\partial_s b)\, (\partial_s c) - b\, (\partial_s^2 c)$$

and so at $(0, s_0)$ we have

$$\partial_s m = 0, \qquad \partial_s^2 m = 2\, (\partial_s a)\, (\partial_s d) - 2\, (\partial_s b)\, (\partial_s c) \tag{2.115a}$$

because $a = b = c = d = 0$ there (recall that $M(\lambda_0, s_0) = 0$). Similarly, we find that

$$\partial_\lambda m = 0, \tag{2.115b}$$
$$\partial_\lambda^2 m = 2(\partial_\lambda a)(\partial_\lambda d) - 2(\partial_\lambda b)(\partial_\lambda c), \tag{2.115c}$$
$$\partial_{s\lambda} m = (\partial_s a)(\partial_\lambda d) + (\partial_\lambda a)(\partial_s d) - (\partial_s b)(\partial_\lambda c) - (\partial_\lambda b)(\partial_s c). \tag{2.115d}$$

at $(0, s_0)$. To evaluate the second derivatives, it remains to differentiate the components of M. By Proposition 2.37, for $i, j = 1, 2$ we have

$$\frac{\partial M_{ij}}{\partial \lambda}(0, s_0) = -s_0^2 \left\langle \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle, \qquad \frac{\partial M_{ij}}{\partial s}(0, s_0) = \left\langle \partial_s B_{s_0} \mathbf{u}_{s_0}^{(i)}, S\mathbf{u}_{s_0}^{(j)} \right\rangle. \tag{2.116}$$

It follows from (2.99) and (2.55) that

$$\frac{\partial M}{\partial \lambda}(0, s_0) = -s_0^2 \begin{pmatrix} 0 & \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \\ \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle & 0 \end{pmatrix},$$

so that at $(0, s_0)$, we have $\partial_\lambda a = \partial_\lambda d = 0$ and $\partial_\lambda b = \partial_\lambda c = -s_0^2 \langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle$. Similarly, it follows from (2.99) and (2.50) that

$$\frac{\partial M}{\partial s}(0, s_0) = \frac{\ell}{s_0} \begin{pmatrix} -\big(\partial_x u_{s_0}^{(1)}(\ell)\big)^2 & 0 \\ 0 & \big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2 \end{pmatrix}, \tag{2.117}$$

hence at $(0, s_0)$ we have $\partial_s a = -s_0^{-1} \ell \big(\partial_x u_{s_0}^{(1)}(\ell)\big)^2$, $\partial_s d = s_0^{-1} \ell \big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2$ and $\partial_s b = \partial_s c = 0$. The claimed formulas for $\partial_s^2 m$, $\partial_{s\lambda} m$ and $\partial_\lambda^2 m$ now follow from (2.115).

If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle = 0$, then $\partial_\lambda b = \partial_\lambda c = 0$ at $(0, s_0)$. This implies that $\partial_\lambda^3 m = 0$ and

$$\partial_\lambda^4 m = 6 \left((\partial_\lambda^2 a)\, (\partial_\lambda^2 d) - (\partial_\lambda^2 b)\, (\partial_\lambda^2 c) \right), \quad \partial_{s\lambda\lambda} m = (\partial_s a)\, (\partial_\lambda^2 d) + (\partial_\lambda^2 a)\, (\partial_s d) \tag{2.118}$$

at $(0, s_0)$. Using (2.99) and (2.60) we obtain

$$\frac{\partial^2 M}{\partial \lambda^2}(0, s_0) = -2s_0^4 \begin{pmatrix} \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle & 0 \\ 0 & \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle \end{pmatrix}, \tag{2.119}$$

hence $\partial_\lambda^2 b = \partial_\lambda^2 c = 0$ and it follows that

$$\partial_\lambda^4 m = 6(\partial_\lambda^2 a)(\partial_\lambda^2 d) = 24 s_0^8 \langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle.$$

The claimed formula for $\partial_{s\lambda\lambda} m$ follows directly from (2.118). $\square$

The next elementary lemma will be used to prove differentiability of the eigenvalue curves in the second part of Theorem 2.41. In what follows, dot denotes $d/d\lambda$.

Lemma 2.44. *If Δ is a smooth function with $\Delta(\lambda) = \alpha\lambda^4 + O(\lambda^5)$ as $|\lambda| \to 0$ for some $\alpha > 0$, then $\delta(\lambda) := \sqrt{\Delta(\lambda)}$ is C^2 near $\lambda = 0$, with $\dot{\delta}(0) = 0$ and $\ddot{\delta}(0) = 2\sqrt{\alpha}$.*

Proof. It is clear that δ is smooth except possibly at $\lambda = 0$. For the first derivative we note that $\delta(\lambda)/\lambda \to 0$ as $\lambda \to 0$, so $\dot{\delta}(0) = 0$. For $\lambda \neq 0$ we compute

$$\dot{\delta}(\lambda) = \frac{1}{2}\Delta(\lambda)^{-1/2}\dot{\Delta}(\lambda).$$

Using $\Delta(\lambda) = \alpha\lambda^4 + O(\lambda^5)$ and $\dot{\Delta}(\lambda) = 4\alpha\lambda^3 + O(\lambda^4)$, we see that $\dot{\delta}(\lambda) \to 0$ as $\lambda \to 0$ and conclude that δ is C^1. Next, we observe that

$$\frac{\dot{\delta}(\lambda) - \dot{\delta}(0)}{\lambda} = \frac{1}{2}\frac{\lambda^2}{\sqrt{\Delta(\lambda)}}\frac{\dot{\Delta}(\lambda)}{\lambda^3} \to 2\sqrt{\alpha},$$

and hence $\ddot{\delta}(0)$ exists. A similar argument gives

$$\ddot{\delta}(\lambda) = -\frac{1}{4}\frac{\dot{\Delta}(\lambda)^2}{\Delta(\lambda)^{3/2}} + \frac{1}{2}\frac{\ddot{\Delta}(\lambda)}{\sqrt{\Delta(\lambda)}} \to 2\sqrt{\alpha}$$

as $\lambda \to 0$, so δ is C^2. $\square$

Proof of Theorem 2.41. By assumption we have $m(0, s_0) = 0$. If $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$, Lemma 2.43 implies m has a strict local maximum at $(0, s_0)$, so m is negative (and in particular nonzero) in a punctured neighbourhood of $(0, s_0)$. This proves the first case.

For the second case we use the Malgrange preparation theorem (see [GG73, §IV.2]). We know from Lemma 2.43 that $m(0, s_0) = \partial_s m(0, s_0) = 0$ and $\partial_s^2 m(0, s_0) < 0$, so we can write

$$m(\lambda, s) = Q(\lambda, s)P(\lambda, s) \tag{2.120}$$

in a neighbourhood of $(0, s_0)$, where

$$P(\lambda, s) = (s - s_0)^2 + B(\lambda)(s - s_0) + C(\lambda), \tag{2.121}$$

Q, B and C are smooth, real-valued functions, and Q does not vanish in a neighbourhood of $(0, s_0)$. This means m locally has the same zero set as P.

We claim that the discriminant $\Delta(\lambda) = B^2(\lambda) - 4C(\lambda)$ satisfies

$$\Delta(\lambda) = \alpha\lambda^4 + O(\lambda^5) \text{ as } |\lambda| \to 0, \qquad \alpha = \frac{\ddot{B}(0)^2}{4} - \frac{C^{(4)}(0)}{6} > 0. \tag{2.122}$$

To see this, we compute the Taylor expansion of $\Delta(\lambda) = B(\lambda)^2 - 4C(\lambda)$ about $\lambda = 0$ and show that $\Delta(0) = \dot{\Delta}(0) = \ddot{\Delta}(0) = \dddot{\Delta}(0) = 0$. For this it suffices to show that $B(0) = \dot{B}(0) = C(0) = \dot{C}(0) = \ddot{C}(0) = \dddot{C}(0) = 0$. That $\Delta^{(4)}(0) = 4!\alpha$ follows from the definition of $\Delta(\lambda)$.

Using Lemma 2.43 we obtain

$$m(0, s_0) = Q(0, s_0)C(0) = 0.$$

Since $Q(0, s_0) \neq 0$, this implies $C(0) = 0$. Similarly, we find that

$$\begin{aligned}
\partial_\lambda m(0, s_0) &= Q(0, s_0)\dot{C}(0) = 0 \\
\partial_\lambda^2 m(0, s_0) &= Q(0, s_0)\ddot{C}(0) = 0 \\
\partial_\lambda^3 m(0, s_0) &= Q(0, s_0)\dddot{C}(0) = 0 \\
\partial_\lambda^4 m(0, s_0) &= Q(0, s_0)C^{(4)}(0)
\end{aligned}$$

and

$$\begin{aligned}
\partial_s m(0, s_0) &= Q(0, s_0)B(0) = 0 \\
\partial_{s\lambda} m(0, s_0) &= Q(0, s_0)\dot{B}(0) = 0 \\
\partial_{s\lambda\lambda} m(0, s_0) &= Q(0, s_0)\ddot{B}(0),
\end{aligned}$$

which gives

$$B(0) = \dot{B}(0) = C(0) = \dot{C}(0) = \ddot{C}(0) = \dddot{C}(0) = 0.$$

We now observe that

$$\partial_s^2 m(0, s_0) = Q(0, s_0)\, \partial_s^2 P(0, s_0) = 2Q(0, s_0).$$

Using the first formula from (2.112), this implies that

$$Q(0, s_0) = -\frac{\ell^2}{s_0^2} \left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2. \tag{2.123}$$

Therefore, using (2.114),

$$C^{(4)}(0) = \frac{\partial_\lambda^4 m(0, s_0)}{Q(0, s_0)} = -24 \frac{s_0^{10}}{\ell^2} \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle \langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2 \left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2}. \tag{2.124}$$

We similarly use (2.113) to compute

$$\ddot{B}(0) = \frac{\partial_{s\lambda\lambda} m(0, s_0)}{Q(0, s_0)} = \frac{2s_0^5}{\ell} \left\{ \frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} - \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \right\}. \tag{2.125}$$

Therefore

$$\alpha = \frac{\ddot{B}(0)^2}{4} - \frac{C^{(4)}(0)}{6} = \frac{s_0^{10}}{\ell^2} \left(\frac{\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} + \frac{\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \right)^2 > 0 \tag{2.126}$$

on account of (2.107), thus proving the claim.

Given (2.122), we have $\Delta(\lambda) > 0$ for small nonzero λ, and so the equation $P(\lambda, s) = 0$ has two solutions in s,

$$s_\pm(\lambda) := \frac{-B(\lambda) \pm \sqrt{\Delta(\lambda)}}{2} + s_0. \tag{2.127}$$

It then follows from Lemma 2.44 that both $s_\pm(\lambda)$ are C^2 in a neighbourhood of $\lambda = 0$, with $\dot{s}_\pm(0) = -\dot{B}(0)/2 = 0$ and

$$\ddot{s}_\pm(0) = \frac{-\ddot{B}(0) \pm 2\sqrt{\alpha}}{2}, \tag{2.128}$$

so the curves $s_\pm(\lambda)$ satisfy properties (i)–(iii) in the theorem. Substituting (2.125) and (2.126) into (2.128), we obtain

$$\ddot{s}_\pm(0) = \frac{s_0^5}{\ell}\left\{ \frac{\left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} - \frac{\left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} \pm \left| \frac{\left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} + \frac{\left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \right| \right\}. \tag{2.129}$$

If the quantity inside the absolute value (which is nonzero by (2.107)) is positive, we get

$$\ddot{s}_+(0) = \frac{2s_0^5}{\ell} \frac{\left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2}, \qquad \ddot{s}_-(0) = -\frac{2s_0^5}{\ell} \frac{\left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2}, \tag{2.130}$$

in which case we define $s_1 := s_-$ and $s_2 := s_+$. If it is negative we get

$$\ddot{s}_-(0) = \frac{2s_0^5}{\ell} \frac{\left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2}, \qquad \ddot{s}_+(0) = -\frac{2s_0^5}{\ell} \frac{\left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2}, \tag{2.131}$$

and we define $s_1 := s_+$ and $s_2 := s_-$.

To prove the existence of a continuous family of eigenfunctions, we define $M_1(\lambda) = M(\lambda, s_1(\lambda))$. If $\big(t_1(\lambda), t_2(\lambda)\big)^\top \in \ker M_1(\lambda)$ is nonzero, we know from (2.91) that

$$\mathbf{u}_{s_1(\lambda)} = \big(I + A(\lambda, s_1(\lambda))\big)\left(t_1(\lambda)\mathbf{u}_{s_0}^{(1)} + t_2(\lambda)\mathbf{u}_{s_0}^{(2)}\right)$$

is an eigenfunction of $N_{s_1(\lambda)}$ for the eigenvalue $s_1^2(\lambda)\lambda$. We therefore need to understand the kernel of $M_1(\lambda)$.

By construction we have $M_1(0) = 0$. Since $(\partial_\lambda M)(0, s_0) = 0$ and $\dot{s}_1(0) = 0$, we find that $\dot{M}_1(0) = 0$ and $\ddot{M}_1(0) = (\partial_\lambda^2 M)(0, s_0) + (\partial_s M)(0, s_0)\ddot{s}_1(0)$. Using (2.109), (2.117) and (2.119), we get

$$\ddot{M}_1(0) = -2s_0^4\big(\partial_x v_{s_0}^{(2)}(\ell)\big)^2 \left(\frac{\left\langle \widehat{v}_{s_0}^{(1)}, u_{s_0}^{(1)} \right\rangle}{\left(\partial_x u_{s_0}^{(1)}(\ell)\right)^2} + \frac{\left\langle \widehat{u}_{s_0}^{(2)}, v_{s_0}^{(2)} \right\rangle}{\left(\partial_x v_{s_0}^{(2)}(\ell)\right)^2} \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.132}$$

which is nonzero by (2.107). Writing $M_1(\lambda) = \left(\begin{smallmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{smallmatrix}\right)$, it follows that $d(\lambda) \neq 0$ for small, nonzero values of λ, and so we can choose

$$\begin{pmatrix} t_1(\lambda) \\ t_2(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ -c(\lambda)/d(\lambda) \end{pmatrix} \in \ker M_1(\lambda)$$

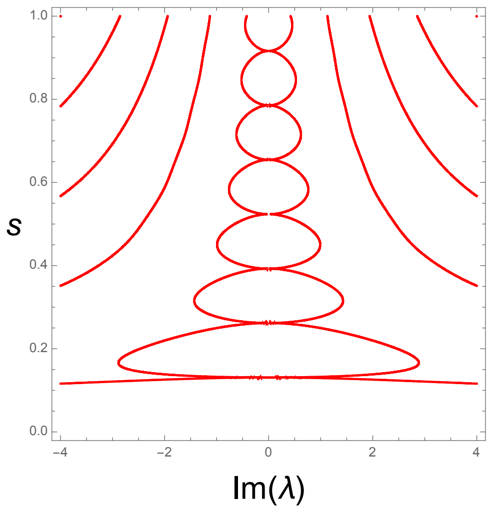


Figure 2.4: Imaginary eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap i\mathbb{R}$, where $L_-^s = L_+^s = -\partial_{xx} - 4s^2$ and $\ell = 12$. Viewed from the ηs-plane where $\eta = \mathrm{Re}(\lambda)$, a series of isolated crossings appear at $\eta = 0$ as s increases from 0 to 1.

for $\lambda \neq 0$. Since $c(0) = \dot{c}(0) = \ddot{c}(0) = d(0) = \dot{d}(0) = 0$ but $\ddot{d}(0) \neq 0$, we get $c(\lambda)/d(\lambda) \to 0$ as $\lambda \to 0$, and so

$$\lim_{\lambda \to 0} \left(I + A(\lambda, s_1(\lambda))\right) \left(t_1(\lambda)\mathbf{u}_{s_0}^{(1)} + t_2(\lambda)\mathbf{u}_{s_0}^{(2)}\right) = \mathbf{u}_{s_0}^{(1)}$$

as claimed. The result for $\mathbf{u}_{s_2(\lambda)}$ is proved in the same way. □

Remark 2.45. The condition (2.107) implies $\Delta(\lambda) > 0$ for small nonzero λ, and hence guarantees the existence of $s_\pm(\lambda)$. It also guarantees that $\ddot{s}_+(0) \neq \ddot{s}_-(0)$, as can be seen from (2.129). If (2.107) fails then $\alpha = 0$ and we cannot use the result of Lemma 2.44. In this (nongeneric) case one may compute higher derivatives of m in order to determine higher order coefficients in the Taylor expansion of $\Delta(\lambda)$, but we do not pursue this here.

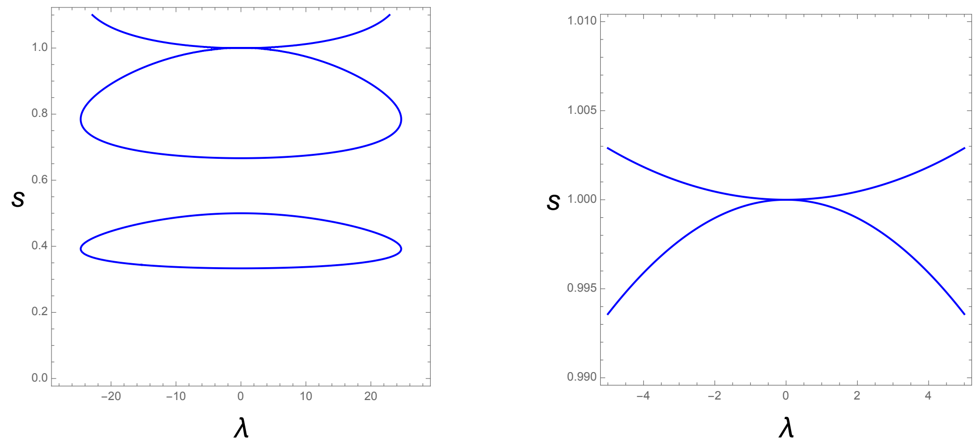
The following examples illustrate the two scenarios detailed in Theorem 2.41.

Example 2.46. The conditions in case (1) of Theorem 2.41 are satisfied if we take $L_+^s = L_-^s$, in which case $u_{s_0}^{(1)} = v_{s_0}^{(2)}$ at any crossing $(0, s_0)$, so that $\langle u_{s_0}^{(1)}, v_{s_0}^{(2)} \rangle \neq 0$. Each isolated crossing $(\lambda, s) = (0, s_0)$ is a consequence of a pair of purely imaginary eigenvalues passing through the origin as s increases. For clarity, in Fig. 2.4 we have plotted the *imaginary* eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap i\mathbb{R}$ for the case when $L_-^s = L_+^s = -\partial_{xx} - 4s^2$ and $\ell = 12$ (here $\lambda \in \mathbb{C}$).

Example 2.47. Let $L = -\partial_{xx} + V(x)$ with domain (2.9), and define $L_\pm = L - \lambda_\pm$, where $\lambda_\pm \in \mathrm{Spec}(L)$ are distinct eigenvalues with eigenfunctions u_1 and v_2, so that $L_+ u_1 = L_- v_2 = 0$. Since $L_\pm$ is selfadjoint and $\lambda_+ \neq \lambda_-$, we have $\langle u_1, v_2 \rangle = 0$, and the conditions of case (2) in Theorem 2.41 are satisfied. (Recall the notation of (2.40) when $s_0 = 1$.)

The equations $L_+ \widehat{u}_2 = v_2$ and $-L_- \widehat{v}_1 = u_1$ are solved by $\widehat{u}_2 = \frac{1}{\lambda_- - \lambda_+} v_2$ and $\widehat{v}_1 = \frac{1}{\lambda_- - \lambda_+} u_1$, and it follows that

$$\int_0^\ell \widehat{u}_2\, v_2\, dx = \frac{1}{\lambda_- - \lambda_+} \int_0^\ell v_2^2\, dx \quad \text{and} \quad \int_0^\ell \widehat{v}_1\, u_1 dx = \frac{1}{\lambda_- - \lambda_+} \int_0^\ell u_1^2\, dx$$



(a) **(b)**

Figure 2.5: (a) Real eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ where $L^s_- = -\partial_{xx} - 4\pi^2 s^2$, $L^s_+ = -\partial_{xx} - 9\pi^2 s^2$ and $\ell = 1$, and (b) a blow-up of the conjugate point $(\lambda, s) = (0, 1)$.

are nonzero and have the same sign. According to (2.109) this means the curves $s_{1,2}(\lambda)$ passing through $(0, 1)$ will have opposite concavity. This is illustrated in Fig. 2.5, where we have plotted the real eigenvalue curves for a domain of length $\ell = 1$, choosing $L = -\partial_{xx}$, $\lambda_+ = 9\pi^2$ and $\lambda_- = 4\pi^2$.

2.3.4 The Maslov index at the non-regular corner

We are now in a position to calculate the corner term $\mathfrak{c}$ appearing in Theorem 2.2 (and defined in Definition 2.26) using the tools developed in Sections 2.3.2 and 2.3.3.

Since a non-regular crossing occurs at the initial point of Γ_3, we cannot use (2.25) to compute the Maslov index. We therefore take advantage of homotopy invariance, deforming the corner of the Maslov box to a path that only has simple regular crossings.

The index can then be deduced from the local behaviour of the eigenvalue curves through $(0, 1)$ (see Theorems 2.9 and 2.41), which we quantify as follows. Given the curve $s(\lambda)$ from Theorem 2.9, there is an interval $(0, \hat{\lambda})$ on which either $s(\lambda) > 1$ or $s(\lambda) < 1$, since the set $\{\lambda : s(\lambda) = 1\}$ is discrete; cf. Remark 2.27. Therefore, the quantity

$$s^\sharp(0) := \lim_{\lambda \to 0^+} \mathrm{sign}\,\big(s(\lambda) - 1\big) \in \{\pm 1\} \tag{2.133}$$

is well-defined. In the case that $s = s(\lambda)$ is analytic, $s^\sharp(0)$ is the sign of the first nonzero Taylor coefficient at $\lambda = 0$.

Remark 2.48. Recall from Theorem 2.9 that $\dot{s}(0) = 0$. Therefore, in the generic case where $\ddot{s}(0) \neq 0$, we simply have

$$s^\sharp(0) = \mathrm{sign}\,\ddot{s}(0). \tag{2.134}$$

That is, the VK-type integrals in Theorem 2.9 determine $s^\sharp(0)$ (and hence the index $\mathfrak{c}$) provided the integrals are nonzero. However, it is important to note that the dichotomy $s^\sharp(0) = \pm 1$ holds even if $\ddot{s}(0) = 0$.

The same considerations apply to the curves $s_{1,2}(\lambda)$ from Theorem 2.41 (for which $\dot{s}_{1,2}(0) = 0$), so we define $s^\sharp_{1,2}(0)$ analogously, and emphasize that in the generic case $\ddot{s}_{1,2}(0) \neq 0$ we have

$$s^\sharp_{1,2}(0) = \operatorname{sign} \ddot{s}_{1,2}(0). \tag{2.135}$$

With this notation in place, we are ready to calculate $\mathfrak{c}$.

Theorem 2.49. *The corner term $\mathfrak{c}$ from Definition 2.26 is calculated as follows:*

(1) *Suppose* $\dim\ker(N) = 1$*, and let* $s = s(\lambda)$ *be the eigenvalue curve through* $(0, 1)$*.*

  (i) *If* $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ *then*

$$\mathfrak{c} = \frac{1}{2}(s^\sharp(0) - 1).$$

  That is, $\mathfrak{c} = 0$ *if* $s^\sharp(0) = +1$ *and* $\mathfrak{c} = -1$ *if* $s^\sharp(0) = -1$*.*

  (ii) *If* $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ *then*

$$\mathfrak{c} = \frac{1}{2}(1 - s^\sharp(0)).$$

  That is, $\mathfrak{c} = 0$ *if* $s^\sharp(0) = +1$ *and* $\mathfrak{c} = +1$ *if* $s^\sharp(0) = -1$*.*

(2) *Suppose* $\dim\ker(N) = 2$*, with* $\ker(L_+) = \operatorname{span}\{u_1\}$ *and* $\ker(L_-) = \operatorname{span}\{v_2\}$*. If* $\langle u_1, v_2\rangle \neq 0$*, then* $\mathfrak{c} = 0$*. If* $\langle u_1, v_2\rangle = 0$ *and the condition* (2.107) *holds, we denote by* $s_{1,2}(\lambda)$ *the eigenvalue curves passing through* $(0, 1)$*, as in Theorem 2.41. Then*

$$\mathfrak{c} = \frac{1}{2}(s^\sharp_1(0) - s^\sharp_2(0)). \tag{2.136}$$

We remark that formula (2.136) is simply the sum of the formulas for $\mathfrak{c}$ in cases (i) and (ii) of the simple case, identifying s with s_1 if $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ and s with s_2 if $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$. It is perhaps interesting to note that in (2.136) we have $\mathfrak{c} \in \{-1, 0, 1\}$, so that $\mathfrak{c}$ can never be $+2$ or -2, despite it being the contribution to the Maslov index from a two dimensional crossing in this case.

Proof. We use a homotopy argument, deforming the top left corner of the Maslov box as shown in Fig. 2.6.

We first consider the case $\dim\ker(N) = 1$. If $s^\sharp(0) > 0$ then the deformed path does not intersect $\mathcal{D}$, so we have $\mathfrak{c} = 0$. On the other hand, if $s^\sharp(0) < 0$, there will be a crossing at some point $(\lambda_*, s_*) = (\lambda_*, s(\lambda_*))$ with $0 < \lambda_* \ll 1$. This segment of the deformed path is parameterized by increasing s, so the relevant crossing form is

$$\mathfrak{m}_{s_*}(q) = \frac{1}{s_*}\big\langle\big(\partial_s B_{s_*} - 2s_*\lambda_*\big)\mathbf{u}_{s_*}, S\mathbf{u}_{s_*}\big\rangle, \tag{2.137}$$

where $q = \mathrm{Tr}_{s_*} \mathbf{u}_{s_*}$. From Theorem 2.40 we obtain a continuous family of eigenfunctions with $\mathbf{u}_{s(\lambda)} \to \mathbf{u}$ as $\lambda \to 0$, so we can use Lemma 2.21 to compute

$$\lim_{\lambda\to 0} \frac{1}{s(\lambda)} \big\langle \big(\partial_s B_{s(\lambda)} - 2s(\lambda)\lambda\big)\mathbf{u}_{s(\lambda)}, S\mathbf{u}_{s(\lambda)} \big\rangle = \big\langle \partial_s B_1 \mathbf{u}_1, S\mathbf{u}_1 \big\rangle = \ell \left[-\big(u_1'(\ell)\big)^2 + \big(v_1'(\ell)\big)^2 \right].$$

By continuity this has the same sign as the crossing form (2.137) at (λ_*, s_*), so we conclude that $\mathfrak{c} = -1$ if $0 \in \mathrm{Spec}(L_+)$ and $\mathfrak{c} = 1$ if $0 \in \mathrm{Spec}(L_-)$.

The argument for the case $\dim\ker(N) = 2$ is similar. Depending on the values of $s_1^\sharp(0)$ and $s_2^\sharp(0)$, there will be zero, one or two crossings that contribute to the index $\mathfrak{c}$. These are necessarily simple crossings, since $s_1(\lambda) \neq s_2(\lambda)$ for $\lambda \neq 0$ (see Remark 2.45). Moreover, if either $s_1^\sharp(0)$ or $s_2^\sharp(0)$ is positive, it does not contribute to the index.

Suppose $s_1^\sharp(0) < 0$, so there is a crossing at some point $(\lambda_*, s_*) = (\lambda_*, s_1(\lambda_*))$. As in the first case, we need to compute the crossing form

$$\mathfrak{m}_{s_*}(q) = \frac{1}{s_*} \big\langle \big(\partial_s B_{s_*} - 2s_*\lambda_*\big)\mathbf{u}_{s_*}, S\mathbf{u}_{s_*} \big\rangle.$$

We use Theorem 2.41 to get

$$\lim_{\lambda\to 0} \frac{1}{s_1(\lambda)} \big\langle \big(\partial_s B_{s_1(\lambda)} - 2s_1(\lambda)\lambda\big)\mathbf{u}_{s_1(\lambda)}, S\mathbf{u}_{s_1(\lambda)} \big\rangle = \big\langle \partial_s B_1 \mathbf{u}_1^{(1)}, S\mathbf{u}_1^{(1)} \big\rangle = -\ell \left(\partial_x u_1^{(1)}(\ell)\right)^2 < 0,$$

and hence conclude that the crossing form at (λ_*, s_*) is negative. Similarly, if $s_2^\sharp(0) < 0$, there is a crossing at some point $(\lambda_*, s_2(\lambda_*))$ whose crossing form is positive, because

$$\lim_{\lambda\to 0} \frac{1}{s_2(\lambda)} \big\langle \big(\partial_s B_{s_2(\lambda)} - 2s_2(\lambda)\lambda\big)\mathbf{u}_{s_2(\lambda)}, S\mathbf{u}_{s_2(\lambda)} \big\rangle = \big\langle \partial_s B_1 \mathbf{u}_1^{(2)}, S\mathbf{u}_1^{(2)} \big\rangle = \ell \left(\partial_x v_1^{(2)}(\ell)\right)^2 > 0.$$

In summary, the curve s_1 contributes 0 to $\mathfrak{c}$ if $s_1^\sharp(0) > 0$ and -1 if $s_1^\sharp(0) < 0$, whereas s_2 contributes 0 if $s_2^\sharp(0) > 0$ and 1 if $s_2^\sharp(0) < 0$. Adding these contributions completes the proof. □

We conclude this section by relating the concavity of the eigenvalue curves to the second order Maslov crossing form.

Proposition 2.50. *Assume the first order crossing form $\mathfrak{m}_{\lambda_0}$ is identically zero at the crossing $(\lambda_0, s_0) = (0, 1)$. If the second order crossing form $\mathfrak{m}_{\lambda_0}^{(2)}$ given in Lemma 2.28 is nondegenerate, then*

$$\mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon]) = -n_-(\mathfrak{m}_{\lambda_0}^{(2)}). \tag{2.138}$$

Proof. We will prove this statement in the cases relevant to the current paper, that is, when $\dim\ker(N) = 1, 2$. Recall that nondegeneracy of $\mathfrak{m}_{\lambda_0}^{(2)}$ implies that $\ddot{s}(0) \neq 0$ if $\dim\ker(N) = 1$ and $\ddot{s}_{1,2}(0) \neq 0$ if $\dim\ker(N) = 2$. Therefore, (2.134) and (2.135) hold.

For the right hand side of (2.138), if $\dim\ker(N) = 1$, recall using (2.44) that $\mathfrak{m}_{s_0} > 0$ if $0 \in \mathrm{Spec}(L_-^{s_0}) \backslash \mathrm{Spec}(L_+^{s_0})$, and $\mathfrak{m}_{s_0} < 0$ if $0 \in \mathrm{Spec}(L_+^{s_0}) \backslash \mathrm{Spec}(L_-^{s_0})$. Then, using the Hadamard formula (2.103), we find that

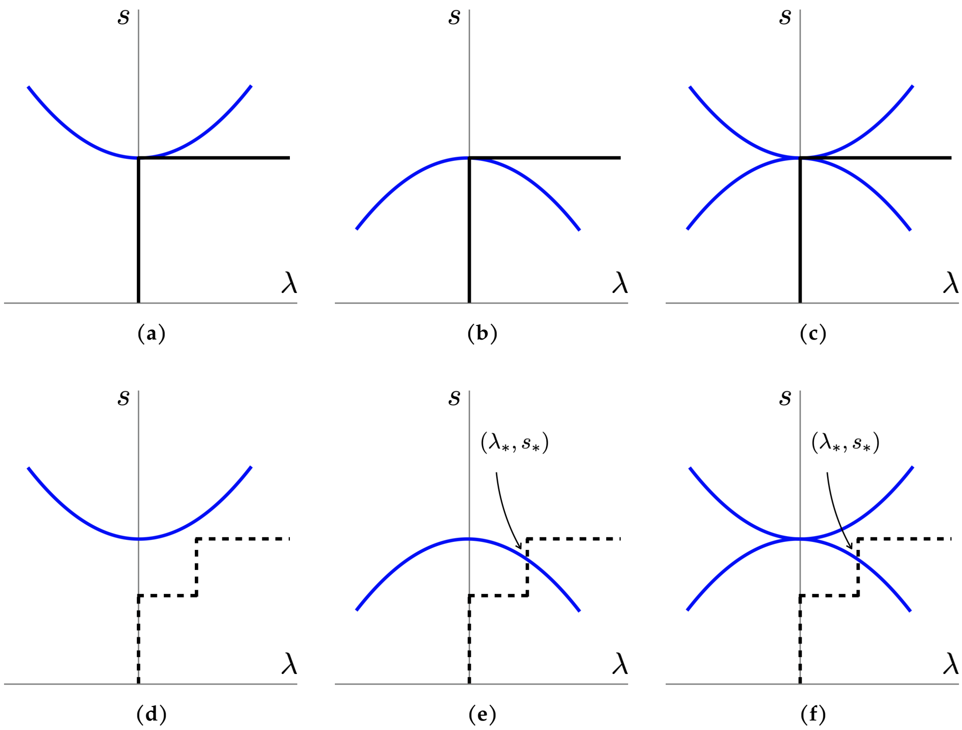


Figure 2.6: Neighbourhood of the crossing $(\lambda_0, s_0) = (0, 1)$ featuring the eigenvalue curves (parabolas in blue) and the portion of the Maslov box passing through the corner $(0, 1)$ (in black) when (a) $\dim\ker(N) = 1$ and $s^\sharp(0) > 0$, (b) $\dim\ker(N) = 1$ and $s^\sharp(0) < 0$, and (c) $\dim\ker(N) = 2$ and $s_1^\sharp(0)s_2^\sharp(0) < 0$. The path (dashed) to which we homotope the top left corner of the Maslov box in (a), (b) and (c) is given in (d), (e) and (f) respectively.

(i) If $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ then $n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 0 & \ddot{s}(0) > 0, \\ 1 & \ddot{s}(0) < 0. \end{cases}$

(ii) If $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ then $n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 1 & \ddot{s}(0) > 0, \\ 0 & \ddot{s}(0) < 0. \end{cases}$

If $\dim\ker(N) = 2$, consider the matrix $\mathfrak{M}^{(2)}_{\lambda_0}$ of the second order form $\mathfrak{m}^{(2)}_{\lambda_0}$, which is given in (2.60). Using (2.109), we see that:

(iii) If $0 \in \mathrm{Spec}(L_+) \cap \mathrm{Spec}(L_-)$ then $n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 0 & \ddot{s}_1(0) > 0,\ \ddot{s}_2(0) < 0, \\ 1 & \ddot{s}_1(0)\ddot{s}_2(0) > 0, \\ 2 & \ddot{s}_1(0) < 0,\ \ddot{s}_2(0) > 0. \end{cases}$

For the left hand side of (2.138), let us define $\mathfrak{a} := \mathrm{Mas}(\Lambda(s, 0), \mathcal{D}; s \in [1 - \varepsilon, 1])$ and $\mathfrak{b} := \mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon])$, and notice from (2.56) that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$. From the proof of Lemma 2.34 we know that the crossing form at $(0, 1)$ has $n_+(\mathfrak{m}_{s_0}) = \dim\ker(L_-)$, so Definition 2.13 gives $\mathfrak{a} = \dim\ker(L_-)$. Therefore

$$\mathfrak{b} = \mathfrak{c} - \dim\ker(L_-). \tag{2.139}$$

Using the values of $\mathfrak{c}$ computed in Theorem 2.49, we confirm that $\mathfrak{b} = -n_-(\mathfrak{m}^{(2)}_{\lambda_0})$ in cases (i), (ii) and (iii) described above, as claimed. □

2.4 Applications

In this section we give some applications of the theory of Sections 2.2 and 2.3. We begin with the proof of Corollaries 2.7 and 2.8 and Theorem 2.11, which are consequences of Theorem 2.2 and Theorem 2.49. We then give formulas for the concavity of the NLS spectral curves, and recover the classical VK criterion for a particular one-parameter family of stationary states. Finally, we relate our results to the Krein index theory.

2.4.1 The Jones–Grillakis instability theorem

We first prove the compact interval analogue of the Jones–Grillakis instability theorem, Corollary 2.7, and its consequence Corollary 2.8.

Proof of Corollary 2.7. From Theorem 2.2 we have $n_+(N) \geq 1$ provided $P - Q \neq \mathfrak{c}$. The result now follows from Theorem 2.49, which guarantees $\mathfrak{c} \in \{-1, 0\}$ when $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$, and $\mathfrak{c} \in \{0, 1\}$ when $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$. □

Proof of Corollary 2.8. We claim that $Q = 0$, $P \geq 1$ and $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ under the assumptions of the Corollary. Once this has been shown, the result follows immediately from Corollary 2.7.

Since ϕ is nonconstant and satisfies Neumann boundary conditions, we have $0 \in \mathrm{Spec}(L_+)$, with eigenfunction ϕ'. Moreover, each stationary point of ϕ in the interior of its domain corresponds to a conjugate point for L_+: If $\phi'(x_0) = 0$ for some $x_0 \in (0, \ell)$, then $0 \in \mathrm{Spec}(L_+^{s_0})$ for $s_0 = x_0/\ell$, with eigenfunction $\phi(s_0 x)$. It then follows from Lemma 2.31 that $P \geq 1$.

We next consider L_-^s for $s \in (0, 1]$. Under Hypothesis 2.5, the general solution to the differential equation $L_-^s w = 0$ is

$$w(x) = c_1 \phi(sx) + c_2 \phi(sx) \int_0^x \frac{1}{\phi(st)^2}\, dt, \tag{2.140}$$

where the second fundamental solution was obtained via the method of reduction of order, and is well defined since $\phi(x) \neq 0$ for all $x \in [0, \ell]$ implies $1/\phi^2$ is integrable. It follows that

$$\phi(sx) \int_0^x \frac{1}{\phi(st)^2}\, dt \geq 0 \tag{2.141}$$

for all $x \in [0, \ell]$, with equality when $x = 0$. Dirichlet boundary conditions on w then dictate that $c_1 = c_2 = 0$, and we conclude that $0 \notin \mathrm{Spec}(L_-^s)$ for all $s \in (0, 1]$. In particular, $0 \notin \mathrm{Spec}(L_-)$, and Lemma 2.31 implies $Q = 0$. □

2.4.2 VK-type (in)stability criteria

For the proof Theorem 2.11 we will need two preliminary results. The first of these mimics [Gri88, Corollary 1.1], and follows from the equivalent selfadjoint formulation of the eigenvalue problem (2.65); see Lemma 2.33.

Lemma 2.51. *If* $Q = 0$ *or* $P = 0$ *then* $\operatorname{Spec}(N_s) \subset \mathbb{R} \cup i\mathbb{R}$ *for all* $s \in (0, 1]$.

Proof. Fix $s \in (0, 1]$. If $Q = 0$ then L_-^s is nonnegative by Lemma 2.32. By Lemma 2.33 the eigenvalue problem (2.65) is equivalent to (2.66). The operator $\left(L_-^s|_{X_c}\right)^{1/2} \Pi L_+^s \Pi \left(L_-^s|_{X_c}\right)^{1/2}$ acting in X_c is selfadjoint, and therefore $s^4\lambda^2 \in \mathbb{R}$. Then $s \in \mathbb{R}$ implies $\lambda \in \mathbb{R} \cup i\mathbb{R}$. The case $P = 0$ follows similarly. □

We next prove that the Maslov index is monotone in λ if either $Q = 0$ or $P = 0$.

Lemma 2.52. *If* $Q = 0$ *then the crossing form* $\mathfrak{m}_{\lambda_0}$ *is strictly positive for any crossing with* $\lambda_0 > 0$ *and* $s_0 = 1$, *while if* $P = 0$ *then* $\mathfrak{m}_{\lambda_0}$ *is strictly negative at all such crossings. Consequently,*

$$n_+(N) = \begin{cases} \operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) & \text{if } Q = 0, \\ -\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) & \text{if } P = 0. \end{cases} \tag{2.142}$$

(*Recall that* $\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = \operatorname{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [\varepsilon, \lambda_\infty])$.)

Proof. Assume $\lambda_0 > 0$ with eigenfunction $\mathbf{u}_1 = (u_1, v_1)^\top$, so that (2.65) holds with $\lambda = \lambda_0$ and $s = 1$. Note that both u_1 and v_1 are necessarily nontrivial due to the coupling of the eigenvalue equations for $\lambda \neq 0$. If $Q = 0$, we apply $\langle \cdot, v_1 \rangle$ to the first equation of (2.65) to obtain

$$\langle L_- v_1, v_1 \rangle = -\lambda_0 \langle u_1, v_1 \rangle = \frac{\lambda_0}{2} \mathfrak{m}_{\lambda_0}(q), \quad q = \operatorname{Tr} \mathbf{u}_1, \tag{2.143}$$

using formula (2.51). Now $0 \neq u_1 \in \operatorname{Ran}(L_-)$ implies v_1 has a component lying in $\ker(L_-)^\perp$. Since $Q = 0$, it follows that $\langle L_- v_1, v_1 \rangle > 0$. Thus $\mathfrak{m}_{\lambda_0}(q) > 0$ at all crossings along Γ_3^ε if $Q = 0$. If $P = 0$, one applies $\langle \cdot, u_1 \rangle$ to the second equation of (2.65) at $(\lambda_0, 1)$, and a similar argument yields that $\langle L_+ u_1, u_1 \rangle = -\frac{\lambda_0}{2} \mathfrak{m}_{\lambda_0}(q) > 0$. Thus $\mathfrak{m}_{\lambda_0}(q) < 0$ at all crossings on Γ_3^ε if $P = 0$. □

Proof of Theorem 2.11. Consider the eigenvalue curve $s = s(\lambda)$ through the point $(\lambda, s) = (0, 1)$, for which $\dot{s}(0) = 0$ as stated in Theorem 2.9.

We start with the case $P = 1, Q = 0$ and $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$. If $\ddot{s}(0) > 0$, then by Theorem 2.49 we have $\mathfrak{c} = 0$. Since $Q = 0$, by Lemma 2.52 and (2.77) we have $n_+(N) = P - \mathfrak{c} = 1$. On the other hand, if $\ddot{s}(0) < 0$, then by Theorem 2.49 we have $\mathfrak{c} = 1$, and by the same argument $n_+(N) = P - \mathfrak{c} = 0$. It then follows from Lemma 2.51 that $\operatorname{Spec}(N) \subset i\mathbb{R}$.

The case where $Q = 1, P = 0$ and $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ is similar. If $\ddot{s}(0) > 0$, then $\mathfrak{c} = 0$ by Theorem 2.49, and Lemma 2.52 and (2.77) imply $n_+(N) = Q + \mathfrak{c} = 1$. If $\ddot{s}(0) < 0$, then $\mathfrak{c} = -1$ by Theorem 2.49, hence $n_+(N) = 0$. By Lemma 2.51 we deduce that $\operatorname{Spec}(N) \subset i\mathbb{R}$. □

2.4.3 Concavity computations for NLS

Working under Hypothesis 2.5, in this subsection we compute the sign of $\ddot{s}(0)$ via the VK-type integrals given in Theorem 2.9. In what follows, $s(\lambda)$ is the eigenvalue curve through $(\lambda_0, s_0) = (0, 1)$.

2.4.3.1 The L_+ integral

We first consider the case when L_+ has a nontrivial kernel. The following result allows us to compute $\ddot{s}(0)$ when ϕ satisfies Neumann boundary conditions.

Proposition 2.53. *Assume Hypothesis 2.5 and that $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ with eigenfunction ϕ'. If $\{p, q\}$ is a fundamental set of solutions to the differential equation $L_- v = 0$ initialised at the identity, then $q(\ell) \neq 0$ and*

$$\operatorname{sign} \ddot{s}(0) = \operatorname{sign} \left[\left(\int_0^\ell p^2 dx \right) - \frac{p(\ell)}{q(\ell)} \ell^2 \right]. \tag{2.144}$$

Proof. First, note that $\ker(N) = \operatorname{span}\{(\phi', 0)^\top\}$. Now by case (2) of Theorem 2.9 we have

$$\operatorname{sign} \ddot{s}(0) = \operatorname{sign} \int_0^\ell \widehat{v}\, \phi' \, dx$$

where $\widehat{v}$ is the unique solution to the inhomogeneous boundary value problem

$$L_- \widehat{v} = \phi', \qquad \widehat{v}(0) = \widehat{v}(\ell) = 0. \tag{2.145}$$

Let $\{p, q\}$ be a fundamental set of solutions to the homogeneous equation $L_- \widehat{v} = 0$ such that

$$\begin{pmatrix} p(0) & q(0) \\ p'(0) & q'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.146}$$

Since $\phi(0) \neq 0$, the first solution is given by $p(x) = \phi(x)/\phi(0)$. We have $p'(\ell) = 0$, $p(\ell) \neq 0$, while $q(\ell) \neq 0$ since $q(0) = 0$ and $0 \notin \operatorname{Spec}(L_-)$. By Abel's identity,

$$p(x) q'(x) - q(x) p'(x) = 1 \quad \forall\, x \in [0, \ell]. \tag{2.147}$$

The general solution to the differential equation $L_- \widehat{v} = \phi'$ is thus

$$\widehat{v}(x) = A p(x) + B q(x) - \frac{x \phi(x)}{2}, \tag{2.148}$$

where it is easily verified that $-x\phi(x)/2$ is a particular solution. Imposing the boundary conditions on $\widehat{v}$ to determine the constants A and B, we find that the unique solution to (2.145) is

$$\widehat{v}(x) = \frac{1}{2} \left(\frac{\ell \phi(\ell)}{q(\ell)} q(x) - x \phi(x) \right).$$

It remains to compute sign $\int_0^\ell \widehat{v}\phi' dx$. Since $\phi(x) = p(x)\phi(0)$, we have

$$\begin{aligned}\int_0^\ell \widehat{v}(x)\phi'(x)dx &= \int_0^\ell \frac{1}{2}\left(\frac{\ell\phi(\ell)}{q(\ell)}q(x) - x\phi(x)\right)p'(x)\phi(0)\,dx \\ &= \frac{\phi(0)^2\ell p(\ell)}{2q(\ell)}\int_0^\ell q(x)p'(x)dx - \frac{\phi(0)^2}{2}\int_0^\ell xp(x)p'(x)dx.\end{aligned}$$

For the second integral we obtain

$$\int_0^\ell xp(x)p'(x)dx = \frac{1}{2}\left(\ell p(\ell)^2 - \int_0^\ell p(x)^2dx\right),$$

while for the first we integrate by parts and appeal to (2.147) to arrive at

$$\int_0^\ell q(x)p'(x)dx = \frac{1}{2}\left(q(\ell)p(\ell) - \ell\right).$$

Therefore

$$\begin{aligned}\int_0^\ell \widehat{v}(x)\phi'(x)dx &= \frac{\phi(0)^2\ell p(\ell)}{4q(\ell)}\left(q(\ell)p(\ell) - \ell\right) - \frac{\phi(0)^2}{4}\left(\ell p(\ell)^2 - \int_0^\ell p(x)^2dx\right) \\ &= \frac{\phi(0)^2}{4}\left(\int_0^\ell p(x)^2dx - \frac{p(\ell)}{q(\ell)}\ell^2\right)\end{aligned}$$

and (2.144) follows. □

Remark 2.54. If ϕ is nonvanishing, the second solution q can be determined using reduction of order; see (2.149) and also the proof of Corollary 2.8. When ϕ has zeros the second solution is given by the Rofe–Beketov formula [Sch00, Lemma 2]; however, the resulting expression is significantly more complicated and does not appear to be useful for our analysis.

The following result serves as an application of Proposition 2.53 in the case when the stationary state is either strictly positive or strictly negative over its domain.

Corollary 2.55. *Under the assumptions of Proposition 2.53, for nonconstant solutions to (2.13) satisfying $\phi(x) \neq 0$ for all $x \in [0, \ell]$, we have $\ddot{s}(0) > 0$.*

Proof. In the case when ϕ has no zeros on the interval $[0, \ell]$, the method of reduction of order allows us to write

$$q(x) = p(x)\int_0^x \frac{1}{p(t)^2}dt, \tag{2.149}$$

where the nonvanishing of p ensures $1/p^2$ is integrable. This gives

$$\int_0^\ell p(x)^2dx - \frac{p(\ell)}{q(\ell)}\ell^2 = \frac{\left(\int_0^\ell \frac{1}{p^2}dx\right)\left(\int_0^\ell p^2dx\right) - \ell^2}{\left(\int_0^\ell \frac{1}{p^2}dx\right)},$$

and so

$$\operatorname{sign}\ddot{s}(0) = \operatorname{sign}\left[\left(\int_0^\ell \frac{1}{p^2}dx\right)\left(\int_0^\ell p^2dx\right) - \ell^2\right]. \tag{2.150}$$

By virtue of the Cauchy Schwarz inequality,

$$\ell = \int_0^\ell p(x)\frac{1}{p(x)}\,dx \leq \sqrt{\int_0^\ell p^2(x)dx}\sqrt{\int_0^\ell \frac{1}{p(x)^2}dx}$$

where we have equality only when p and $1/p$ are linearly dependent, that is, when ϕ is constant. Since we have assumed a nonconstant solution, the inequality is strict, and we conclude that (2.150) is positive. □

Remark 2.56. The statement of Corollary 2.55 may also be proven using Remark 2.10, since $L_- > 0$ for stationary states that are nonvanishing over $[0, \ell]$ (as was shown in the proof of Corollary 2.8). However, the proof given above is a nice illustration of Proposition 2.53, a more general result that holds for any nonconstant ϕ.

2.4.3.2 The L_- integral: Recovering classical VK

We now consider the case when L_- has a nontrivial kernel (spanned by ϕ). We show that the associated VK-type integral in equation (2.18) of Theorem 2.9 recovers a compact interval analogue of the classical VK integral expression

$$\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty}\phi^2\,dx \tag{2.151}$$

associated with a stationary state $\phi \in L^2(\mathbb{R})$ solving (2.13) (see [Pel11, Theorem 4.4, p.215]). The key observation is that $\partial_\beta\phi(\cdot;\beta)$ solves the differential equation $L_+\widehat{u} = \phi$ associated with case (1) of Theorem 2.9, and this naturally leads to the expressions (2.153) and (2.154), which clearly resemble (2.151). This is not true for the equation $L_-\widehat{v} = \phi'$ associated with case (2) of Theorem 2.9, for which a recovery of a compact interval analogue of (2.151) is thus not possible. In what follows, $\phi'(x;\beta)$ refers to $\frac{d\phi}{dx}(x;\beta)$, while the β derivative will be denoted by ∂_β.

Proposition 2.57. *Assume Hypothesis 2.5 and let ϕ_0 be a solution to (2.13) with parameter β_0 that satisfies $\phi_0(0) = \phi_0(\ell) = 0$. There exists a unique one-parameter family of solutions $\beta \mapsto \hat{\phi}(\cdot;\beta)$ to (2.13), defined in a neighbourhood of β_0, such that*

$$\hat{\phi}(0;\beta) = \hat{\phi}(\ell;\beta) = 0 \tag{2.152}$$

for all β near β_0 and $\hat{\phi}(\cdot;\beta_0) = \phi_0$. In terms of this family, the VK-type integral in (2.18) is

$$\int_0^\ell \widehat{u}\,v\,dx = \frac{1}{2}\frac{\partial}{\partial\beta}\bigg|_{\beta=\beta_0}\int_0^\ell \hat{\phi}(x;\beta)^2\,dx. \tag{2.153}$$

More generally, if $\beta \mapsto \phi(\cdot;\beta)$ is any C^1 family of solutions to (2.13) satisfying $\phi(\cdot;\beta_0) = \phi_0$, then the integral in (2.18) can be written

$$\begin{aligned}\int_0^\ell \widehat{u}\,v\,dx &= \frac{1}{2}\frac{\partial}{\partial\beta}\bigg|_{\beta=\beta_0}\int_0^\ell \phi(x;\beta)^2\,dx \\ &+ \left((-1)^Q\partial_\beta\phi(0;\beta_0) + \partial_\beta\phi(\ell;\beta_0)\right)\left(\frac{\partial_\beta\phi(0;\beta_0) + (-1)^Q\partial_\beta\phi(\ell;\beta_0)}{q(\ell)} + \partial_\beta\phi'(\ell;\beta_0)\right).\end{aligned} \tag{2.154}$$

Furthermore, if $P = 1$, $Q = 0$ and (2.153) or (2.154) is positive (resp. negative), then the standing wave $\widehat{\psi}(x,t) = e^{i\beta_0 t}\phi_0(x)$ is spectrally unstable (resp. spectrally stable).

Proof. The existence of ϕ_0 implies that the associated operators

$$L_- = -\partial_{xx} - f(\phi_0^2) - \beta_0,$$
$$L_+ = -\partial_{xx} - 2f'(\phi_0^2)\phi_0^2 - f(\phi_0^2) - \beta_0$$

have $\phi_0 \in \ker(L_-)$ and hence $0 \in \operatorname{Spec}(L_-)\setminus\operatorname{Spec}(L_+)$. Consider the function

$$F\colon \big(H^2(0,\ell)\cap H_0^1(0,\ell)\big)\times\mathbb{R} \longrightarrow L^2(0,\ell), \qquad F(\phi,\beta) = \phi'' + f(\phi^2)\phi + \beta\phi, \tag{2.155}$$

in terms of which (2.13) and (2.152) become $F(\phi,\beta)=0$. It can be shown that F is continuously Fréchet differentiable (see [Col12, §2.2]), with

$$DF(\phi_0,\beta_0)(u,\gamma) = \gamma\phi_0 - L_+u. \tag{2.156}$$

Since $0 \notin \operatorname{Spec}(L_+)$, this implies $DF(\phi_0,\beta_0)(\cdot,0) = -L_+$ is invertible, so the implicit function theorem guarantees the existence of a C^1 function

$$(\beta_0-\epsilon,\beta_0+\epsilon) \to H^2(0,\ell)\cap H_0^1(0,\ell), \quad \beta \mapsto \hat{\phi}(\cdot;\beta), \tag{2.157}$$

such that $F(\hat{\phi}(\cdot;\beta),\beta) = 0$ for all $|\beta-\beta_0| < \epsilon$.

Turning to the integral in (2.18), where now $v = \phi_0$, we need to solve

$$L_+\widehat{u} = \phi_0, \qquad \widehat{u}(0) = \widehat{u}(\ell) = 0. \tag{2.158}$$

Using the family constructed above, which is C^1 in β, we differentiate (2.13) with respect to β and evaluate at β_0 to obtain

$$L_+\partial_\beta\hat{\phi}(x;\beta_0) = \phi_0(x). \tag{2.159}$$

Now differentiating (2.152) (which holds for *all* β near β_0) with respect to β and evaluating at β_0 yields

$$\partial_\beta\hat{\phi}(0;\beta_0) = \partial_\beta\hat{\phi}(\ell;\beta_0) = 0. \tag{2.160}$$

Therefore, $\widehat{u}(x) = \partial_\beta\hat{\phi}(x;\beta_0)$ is the *unique* solution to (2.158), and substituting this into the VK-type integral in (2.18) with $v = \phi_0$ yields (2.153).

Now let $\beta \mapsto \phi(\cdot;\beta)$ be an arbitrary family of solutions to (2.13) (again for β close to β_0) such that $\phi(x;\beta_0) = \phi_0(x)$. To solve (2.158), note that (2.159) still holds for the family $\phi(\cdot;\beta_0)$, and thus the general solution to $L_+\widehat{u} = \phi_0$ is

$$\widehat{u}(x) = Ap(x) + Bq(x) + \partial_\beta\phi(x;\beta_0), \tag{2.161}$$

where $\{p,q\}$ is now a fundamental set of solutions to the homogeneous equation $L_+\widehat{u} = 0$ satisfying (2.146). Since $\phi'(0;\beta_0) \neq 0$, we may set $p(x) = \phi'(x;\beta_0)/\phi'(0;\beta_0)$. A brief look at the Hamiltonian for (2.13) indicates that intersections of any fixed orbit with $\phi = 0$ are symmetric about $\phi' = 0$; from this, along with Sturm-Liouville theory applied to $\phi(\cdot;\beta_0) = \phi_0 \in \ker(L_-)$, we deduce that we necessarily have $\phi'(\ell;\beta_0) = (-1)^{Q+1}\phi'(0;\beta_0)$, and therefore that $p(\ell) = (-1)^{Q+1}$. Evaluating (2.13) at $x=\ell$ we also find that $\phi''(\ell;\beta_0)=0$, hence $p'(\ell)=0$. Thus

$$\begin{pmatrix} p(\ell) & q(\ell) \\ p'(\ell) & q'(\ell) \end{pmatrix} = \begin{pmatrix} (-1)^{Q+1} & * \\ 0 & (-1)^{Q+1} \end{pmatrix} \tag{2.162}$$

where $q'(\ell) = (-1)^{Q+1}$ because (2.162) must have unit determinant by virtue of Abel's identity (see (2.147)). In addition, $q(\ell) \neq 0$ since $0 \notin \mathrm{Spec}(L_+)$ and $q(0)=0$.

Imposing the boundary conditions $\widehat{u}(0) = \widehat{u}(\ell) = 0$ and using (2.162) allows us to determine the constants A and B. We find that the unique solution to (2.158) is

$$\widehat{u}(x) = -\partial_\beta\phi(0;\beta_0)\,p(x) + \frac{(-1)^{Q+1}\partial_\beta\phi(0;\beta_0) - \partial_\beta\phi(\ell;\beta_0)}{q(\ell)}\,q(x) + \partial_\beta\phi(x;\beta_0). \tag{2.163}$$

Multiplying (2.163) by ϕ_0 and integrating the first two terms by parts yields (2.154). The statement regarding spectral stability follows immediately from Theorem 2.11. □

Remark 2.58. The one-parameter family constructed abstractly in (2.157) via the implicit function theorem leads to the simplest expression for the VK-type integral on a compact interval. However, this is only useful in practice if one can determine this family explicitly, which may not be possible. For this reason, we have included formula (2.154), which holds for *any* one-parameter family of solutions to the standing wave equation that starts at ϕ_0.

Remark 2.59. When the spatial domain is the entire real line, it is known that for power-law nonlinearities of the form $f(\phi^2) = \phi^{2p}$, $p>0$, strictly positive localised stationary states (for which $\beta<0$, $P=1$ and $Q=0$) are spectrally stable[1] for $p \le 2$ and spectrally unstable for $p>2$ (see [Pel11, Corollary 4.3, p.216]). The result follows from a change in sign of the VK integral (2.151) (see [Pel11, Theorem 4.4, p.215]). Moving to the compact interval, we investigated whether an analogous phenomenon holds for stationary states ϕ_0 that likewise satisfy $\beta<0$, $P=1$ and $Q=0$. We found that our numerical experiments were in line with the result on the real line when $p=1,2$, for which we found no spectrally unstable waves. Interestingly, however, for $p\in(2,p_0), p_0\approx 5$, we observed the existence of a β-dependent threshold value of the interval length $\ell=\ell^*$ separating spectral stability ($\ell<\ell^*$) and spectral instability ($\ell>\ell^*$). This agrees with the instability result on the real line (for these values of p), in the sense that we recover it (numerically) upon taking $\ell\to+\infty$. Theorem 2.11 indicates that this change in stability at $\ell=\ell^*$ should be reflected in a change in concavity of the eigenvalue curve passing through $(\lambda,s)=(0,1)$, and indeed we observe this numerically. Figure 2.7 displays the real eigenvalue curves for three T-periodic stationary states ϕ_0 satisfying the Dirichlet boundary conditions $\phi_0(0)=\phi_0(\ell)=0$, $\ell=T/2$, for differing ℓ. The sign of $\ddot{s}(0)$ at $(\lambda,s)=(0,1)$ switches from negative to positive as ℓ increases through $\ell=\ell^*$. By Theorem 2.11 the underlying standing wave becomes unstable, which is confirmed by the emergence of a positive real eigenvalue in Fig. 2.7c.

Remark 2.60. In the previous example, note that at the critical value $\ell=\ell^*$ we have $\dim\ker(N)=1$ and $\ddot{s}(0)=0$. This corresponds to the non-generic case in Remark 2.48 where $s^\sharp(0)\neq \mathrm{sign}\,\ddot{s}(0)$

[1] The critical case $p=2$ is spectrally stable but nonlinearly unstable due to algebraically growing solutions of the linearised system; see [Pel11, Remark 4.3, p.217].

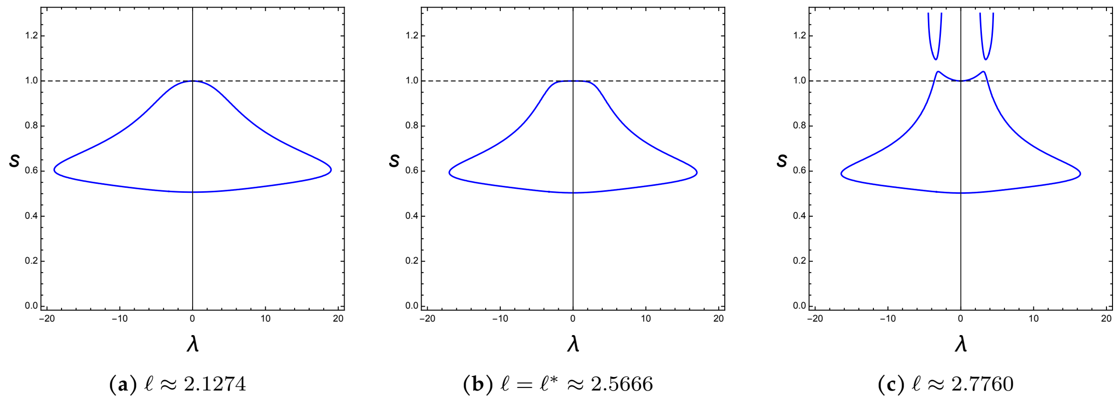


Figure 2.7: Eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{R}$ under Hypothesis 2.5(i) for T-periodic stationary states ϕ_0 satisfying $\phi_0(0) = \phi_0(\ell) = 0$, with nonlinearity $f(\phi^2) = \phi^6$, $\beta = -2$, and domain length $\ell = T/2$ indicated. These ϕ_0 correspond to orbits located outside the homoclinic orbit and in the right half plane of Fig. 2.1a. (Note the phase plane for (2.13) with $f(\phi^2) = \phi^6$ is qualitatively similar to Fig. 2.1a.) Eigenvalues of N are given by intersections with the dashed line at $s = 1$. At $\ell = \ell^*$, we computed $\ddot{s}(0) \approx 0$ to four decimal places.

and the second order crossing form $\mathfrak{m}^{(2)}_{\lambda_0}$ in Lemma 2.28 is degenerate. A brief calculation using the Fredholm Alternative indicates that the algebraic multiplicity of $\lambda = 0 \in \mathrm{Spec}(N)$ is at least four.

2.4.4 Connections with existing eigenvalue counts

We now give a comparison of our lower bound (2.11) with the one given in [KKS04, Eq.(3.9)] (see (2.172) below); see also [KP13, Theorem 7.1.16]. We will show that the contribution to the Maslov index from the non-regular crossing (see Definition 2.26) is equal to the difference in negative indices of matrices arising in constrained eigenvalue counts for $L_\pm$. We refer the reader to [CM19] for an alternate approach to the constrained eigenvalue problem using the Maslov index. Throughout this section, $\{\mathbf{u}_1, \dots \mathbf{u}_n\}$ is a basis for $\ker(N)$ with $n \leq 2$. We assume the crossing $(\lambda_0, s_0) = (0, 1)$ is non-regular in the λ direction, with first order crossing form $\mathfrak{m}_{\lambda_0}$ in (2.51) that is identically zero. We further assume that the second-order crossing form $\mathfrak{m}^{(2)}_{\lambda_0}$ in (2.57) is nondegenerate. The notation $n_-(A)$ refers to the number of negative eigenvalues of the selfadjoint operator or symmetric matrix A. Recall then that $P = n_-(L_+)$ and $Q = n_-(L_-)$.

Define the diagonal, selfadjoint operator

$$L := \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \qquad \mathrm{dom}(L) := \mathrm{dom}(N), \tag{2.164}$$

so that $N = JL$. The eigenvalue problem (2.5) may then be written as

$$JL\mathbf{u} = \lambda \mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}(\ell) = 0. \tag{2.165}$$

We denote the generalised eigenvectors of $N = JL$ by $\widehat{\mathbf{v}}_i$, i.e.

$$JL\widehat{\mathbf{v}}_i = \mathbf{u}_i, \quad JL\mathbf{u}_i = 0, \qquad i = 1, \dots, n. \tag{2.166}$$

As in Remark 2.29, the Fredholm Alternative and the fact that $\mathfrak{m}_{\lambda_0} = 0$ guarantee the existence of solutions to the first n equations in (2.166), so the algebraic multiplicity of $\lambda = 0$ is at least $2n$. Nondegeneracy of $\mathfrak{M}^{(2)}_{\lambda_0}$ then implies the algebraic multiplicity is exactly $2n$.

The matrix D in [KKS04, eq.(3.1)] is the $n \times n$ matrix with entries

$$D_{ij} = \langle \widehat{\mathbf{v}}_i, L\widehat{\mathbf{v}}_j \rangle = -\langle \widehat{\mathbf{v}}_i, J\mathbf{u}_j \rangle, \tag{2.167}$$

where the second equality follows since $JL\widehat{\mathbf{v}}_i = \mathbf{u}_i$ implies $L\widehat{\mathbf{v}}_i = J^{-1}\mathbf{u}_i = -J\mathbf{u}_i$. It is used to determine the number of negative eigenvalues of L restricted to $\operatorname{Ran} JL = [\ker(JL)^*]^\perp$ (see [KKS04, Theorem 3.1]). Denoting $\dim\ker L_\pm = z_\pm \in \{0,1\}$ so that $z_+ + z_- = n$, notice that the off-diagonal structure of JL implies that its eigenvectors and generalised eigenvectors may be written as

$$\mathbf{u}_i = \begin{cases} (u_i, 0)^\top, \\ (0, v_i)^\top, \end{cases} \qquad \widehat{\mathbf{v}}_i = \begin{cases} (0, \widehat{v}_i)^\top, & i = 1, \ldots, z_+, \\ (\widehat{u}_i, 0)^\top, & i = z_+ + 1, \ldots, n, \end{cases} \tag{2.168}$$

where, by (2.166), the functions $u_i, v_i, \widehat{u}_i, \widehat{v}_i$ satisfy

$$\begin{aligned} -L_-\widehat{v}_i &= u_i, \quad & L_+ u_i &= 0, \qquad & i &= 1, \ldots, z_+, \\ L_+\widehat{u}_i &= v_i, \quad & L_- v_i &= 0, \qquad & i &= z_+ + 1, \ldots, n. \end{aligned}$$

The matrix D thus has the block form (as in [KKS04, §3.3])

$$D = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix},$$

where

$$\begin{aligned} [D_-]_{ij} &= \langle \widehat{v}_i, L_-\widehat{v}_j \rangle = -\langle \widehat{v}_i, u_j \rangle, & i,j &= 1, \ldots, z_+, \\ [D_+]_{ij} &= \langle \widehat{u}_{z_++i}, L_+\widehat{u}_{z_++j} \rangle = \langle \widehat{u}_{z_++i}, v_{z_++j} \rangle, & i,j &= 1, \ldots, z_-. \end{aligned} \tag{2.169}$$

The matrices D_+ and D_- are themselves used in constrained eigenvalue counts. Namely, if D_+ and D_- are nondegenerate, then

$$n_-(\Pi L_+ \Pi) = P - n_-(D_+), \qquad n_-(\Pi L_- \Pi) = Q - n_-(D_-), \tag{2.170}$$

where Π is the orthogonal projection onto $[\ker(L_-) \oplus \ker(L_+)]^\perp$ (see [KKS04, Lemma 3.1]).

Now noticing that the entries of $\mathfrak{M}^{(2)}_{\lambda_0}$ are given by

$$\left[\mathfrak{M}^{(2)}_{\lambda_0}\right]_{ij} = -2\langle \widehat{\mathbf{v}}_i, S\mathbf{u}_j \rangle = \begin{cases} -2\langle \widehat{v}_i, u_j \rangle, & i,j = 1, \ldots, z_+ \\ -2\langle \widehat{u}_i, v_j \rangle, & i,j = z_+ + 1, \ldots, n, \\ 0 & \text{elsewhere}, \end{cases}$$

on account of (2.58) and (2.168), we are lead to the observation that

$$\mathfrak{M}^{(2)}_{\lambda_0} = 2\begin{pmatrix} D_- & 0 \\ 0 & -D_+ \end{pmatrix}. \tag{2.171}$$

Clearly $\mathfrak{M}^{(2)}_{\lambda_0}$ is nonsingular if and only if D_+ and D_- are nonsingular. Under this condition, in the notation of the current paper equation (3.9) from [KKS04] reads

$$n_+(N) \geq |n_-(\Pi L_+ \Pi) - n_-(\Pi L_- \Pi)| = |P - Q - n_-(D_+) + n_-(D_-)|. \tag{2.172}$$

Comparing (2.172) with (2.11), we might naïvely expect that $\mathfrak{c} = n_-(D_+) - n_-(D_-)$. We confirm this in the following proposition.

Proposition 2.61. *If $n \leq 2$ and $\mathfrak{M}^{(2)}_{\lambda_0}$ is nondegenerate, then*

$$\mathfrak{c} = n_-(D_+) - n_-(D_-). \tag{2.173}$$

That is, the contribution to the Maslov index from the crossing $(\lambda, s) = (0, 1)$ is precisely the difference of the "correction factors" counting the mismatch in negative dimensions between $L_\pm$ and their constrained counterparts (see (2.170)).

Proof. Recall the definition of $\mathfrak{b}$ given in the proof of Proposition 2.50. By the same Proposition, if $n \leq 2$ we have

$$\mathfrak{b} = -n_-(\mathfrak{M}^{(2)}_{\lambda_0}) = -\big(n_-(D_-) + n_-(-D_+)\big), \tag{2.174}$$

where the last equality follows from (2.171). Notice that D_+ is a $z_- \times z_-$ matrix. Since D_+ is nondegenerate, it follows that

$$n_-(-D_+) = z_- - n_-(D_+). \tag{2.175}$$

Thus, by (2.174),

$$\mathfrak{b} = -n_-(D_-) - (\dim \ker L_- - n_-(D_+)), \tag{2.176}$$

and using (2.139) and rearranging gives (2.173). □

A direct relationship between the matrices $D_\pm$ and the concavities of the eigenvalue curves follows from Theorem 2.9, Lemma 2.28, Theorem 2.41 and equation (2.171). In particular, it is straightforward to show that:

(i) If $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ then $z_+ = 0$ and

$$\operatorname{sign} \mathfrak{m}^{(2)}_{\lambda_0}(q) = -\operatorname{sign} D_+ = -\operatorname{sign} \ddot{s}(0). \tag{2.177a}$$

(ii) If $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ then $z_- = 0$ and

$$\operatorname{sign} \mathfrak{m}^{(2)}_{\lambda_0}(q) = \operatorname{sign} D_- = \operatorname{sign} \ddot{s}(0). \tag{2.177b}$$

(iii) If $0 \in \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+)$ then $z_- = z_+ = 1$ and

$$\operatorname{sign} \ddot{s}_1(0) = \operatorname{sign} D_-, \qquad \operatorname{sign} \ddot{s}_2(0) = \operatorname{sign} D_+ \tag{2.177c}$$

(provided (2.107) holds so that $\operatorname{sign} \ddot{s}_1(0) = -\operatorname{sign}\langle \widehat{v}_1, u_1 \rangle$ and $\operatorname{sign} \ddot{s}_2(0) = \operatorname{sign}\langle \widehat{u}_2, v_2 \rangle$).

We finish the present work with an application of our results to a formula relating the number of eigenvalues of JL that are either unstable or susceptible to instability-inducing bifurcations, to the negative index of the constrained operator $L|_{X_c}$, $X_c := \mathrm{Ran}(JL)$, known as the *Hamiltonian–Krein index theorem* (see [KP13, Theorem 7.1.5] or [LZ22, Theorem 2.3]). For the eigenvalue problem (2.5) – (2.7), because L is diagonal and the symplectic matrix J is invertible, this formula reduces to that in [KKS04, Theorem 3.3], which in the notation of the current paper reads

$$k_r + 2k_c + 2k_i^- = P + Q - n_-(D_-) - n_-(D_+). \tag{2.178}$$

Here, $k_r := n_+(N)$, k_c is the number eigenvalues lying in the open first quadrant, and k_i^- is the number of eigenvalues on the positive imaginary axis with negative Krein signature (see [KKS04]). Note that (2.178) holds provided D_+ and D_- are nonsingular (and since P, Q and n are finite, where $\dim\ker(JL) = \frac{1}{2}\dim \mathrm{gker}(JL) = n$; see [KP13, §7.1.3] or [KKS04] for details). In light of our earlier results, this leads to the following.

Proposition 2.62. *Equation* (2.178) *may be written in one of the following equivalent forms:*

$$\begin{aligned} k_r + 2k_c + 2k_i^- &= -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) + 2P - 2n_-(D_+), &(2.179)\\ &= \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) + 2Q - 2n_-(D_-). &(2.180)\end{aligned}$$

Proof. Using Proposition 2.61 and Lemma 2.34 we can rearrange (2.178) to read

$$k_r + 2k_c + 2k_i^- = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) + \mathfrak{c} + 2P - 2n_-(D_+). \tag{2.181}$$

Then (2.179) follows from (2.181) using (2.76). A similar manipulation yields

$$k_r + 2k_c + 2k_i^- = -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) - \mathfrak{c} + 2Q - 2n_-(D_-), \tag{2.182}$$

in which case (2.180) follows from (2.182) via (2.76). □

Corollary 2.63. *If $P = 0$ or $Q = 0$, then $k_c = k_i^- = 0$.*

Proof. If $P = 0$, then by Lemma 2.52, we have $k_r = n_+(N) = -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$. Furthermore, if $P = 0$ then L_+ is a nonnegative operator in $L^2(0, \ell)$, and in particular $n_-(D_+) = 0$. Cancelling terms on both sides of (2.179), we get

$$2k_c + 2k_i^- = 0, \tag{2.183}$$

as required. Note we could have argued that $k_c = 0$ using Lemma 2.51. The case $Q = 0$ is similar: $k_r = n_+(N) = \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$ by Lemma 2.52, and we have $L_- \geq 0$ in $L^2(0, \ell)$. Thus $n_-(D_-) = 0$, and (2.180) yields the result. □

In the case that $L_\pm$ are invertible, the previous result agrees with that given in [HK08, Corollary 2.26], where the dimension of intersecting cones is zero because $P = 0$ or $Q = 0$. The result for $Q = 0$ is a special case of the formula in [KKS04, Remark 3.1, Eq.(3.10)].

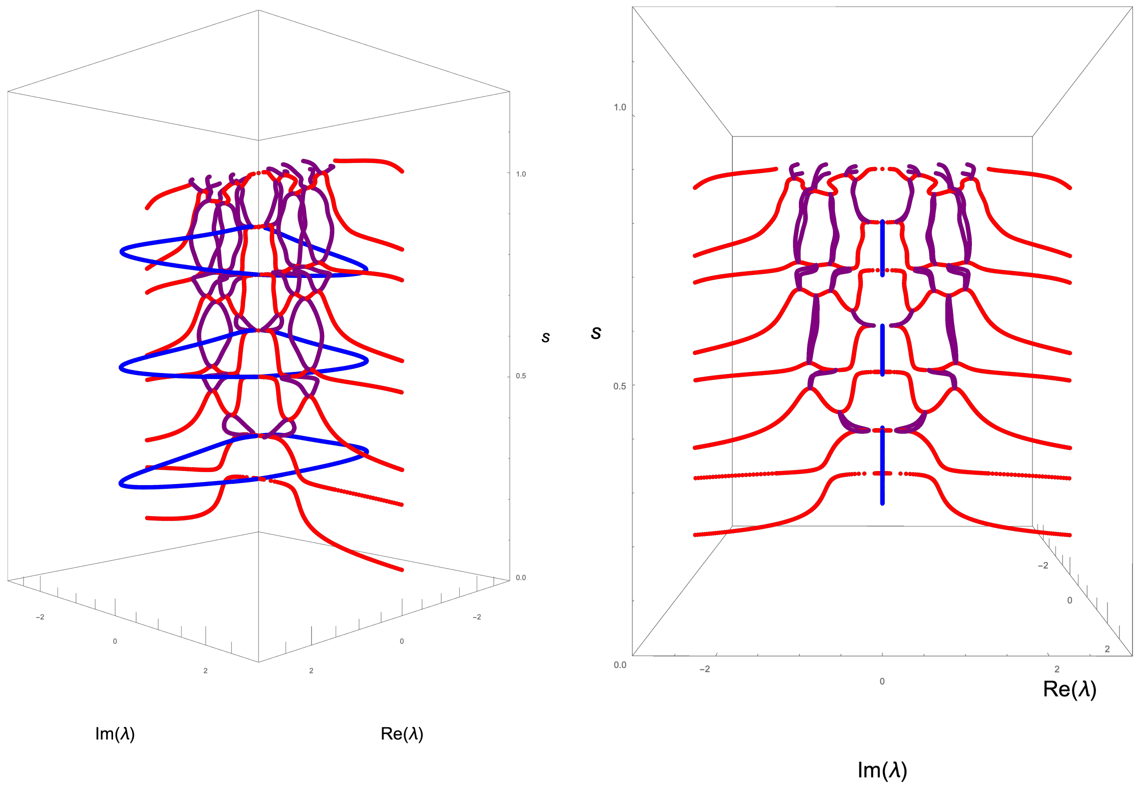
Corollary 2.64. *If either $k_r = 0$ or the Maslov index of the path $\lambda \to \Lambda(\lambda, 1), \lambda \in [\varepsilon, \lambda_\infty]$, $0 < \varepsilon \ll 1$ is monotone in λ, then $k_c + k_i^- = Q - n_-(D_-) = P - n_-(D_+)$.*

Proof. If $k_r = 0$, the statement follows from (2.179) and (2.180) upon noticing that $k_r = n_+(N) = 0$ implies $\operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = 0$ by (2.78).

Monotonicity of the Lagrangian path stated means that the crossing form (2.51) has the same sign at all crossings along Γ_3. In this case, $k_r = n_+(N) = \pm \operatorname{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)$ and the statement follows from (2.179) or (2.180). □

Remark 2.65. Monotonicity in λ is guaranteed if $P = 0$ or $Q = 0$. However, the Maslov index is in general not monotone when $P, Q \geq 1$, and attempts to compute the terms k_c and k_i^- in these cases using the formulas above have so far been limited.

We finish with a numerical example to illustrate the scenario in Corollary 2.64. In Fig. 2.8 we have plotted the *complex* eigenvalue curves for $s \in (0, 1]$ under Hypothesis 2.5(i), associated with a Jacobi cnoidal function ϕ_0 (see Fig. 2.1a) satisfying $\phi_0'(0) = \phi_0'(\ell) = 0$. Precisely, the blue curves represent real eigenvalues, the red curves represent imaginary eigenvalues, and the purple curves represent eigenvalues lying off the real and imaginary axes. It was computed that the minimum point of each blue connected component (for which $\lambda = 0$) corresponds to a point of nontrivial kernel for L_+^s, while the maximum point of each such component corresponds to a point of nontrivial kernel for L_-^s. Note that by a simple rescaling we can apply the formulas of the current section to the rescaled operators $N_s, L_\pm^s$ for *any* $s \in (0, 1]$. Consider then a horizontal plane at $s = s_* \approx 0.85$ in Fig. 2.8, which coincides with the maximum point of the top blue connected component. By the above considerations and Lemma 2.31 applied to the interval $(0, s^*)$ instead of $(0, 1)$, we have $P = n_-(L_+^{s_*}) = 3$ and $Q = n_-(L_-^{s_*}) = 2$. Since $0 \in \operatorname{Spec}(L_-^{s^*}) \setminus \operatorname{Spec}(L_+^{s^*})$, D_- is null (see (2.169)) and hence $n_-(D_-) = 0$. Figure 2.8 clearly shows $k_r = 0$ for $s = s^*$, and by Corollary 2.64 we deduce that $n_-(D_+) = 1$ and $k_c + k_i^- = 2$. (It was confirmed numerically that $k_c = 2$.) A similar analysis can be done for any of the minima or maxima of the blue connected components in Fig. 2.8, or indeed for any horizontal plane which does not intersect the blue curves (for which $k_r = 0$).



(a) (b)

Figure 2.8: Real (blue), imaginary (red) and complex (purple) eigenvalue curves $s^2\lambda \in \mathrm{Spec}(N_s) \cap \mathbb{C}$, $\lambda \in [-3, 3] \times [-3i, 3i] \subset \mathbb{C}$, $s \in (0, 1]$, under Hypothesis 2.5(i) for a T-periodic stationary state ϕ_0 with $f(\phi^2) = \phi^2$ satisfying $\phi_0'(0) = \phi_0'(\ell) = 0$, where $\ell = 2T = 13.3854$. Here, ϕ_0 is a Jacobi cnoidal function corresponding to an orbit located outside the homoclinic orbit in Fig. 2.1a. Figures (a) and (b) give two different viewpoints of the same curves. The eigenvalues were computed using Mathematica's `NDEigenvalues` command.

Chapter 3

A fourth-order Hamiltonian system on the line

3.1 Introduction

The fourth-order cubic nonlinear Schrödinger (NLS) equation

$$i\Psi_t = -\frac{\beta_4}{24}\Psi_{xxxx} + \frac{\beta_2}{2}\Psi_{xx} - \gamma|\Psi|^2\Psi. \tag{3.1}$$

models the propagation of pulses in media with Kerr nonlinearity that are subject to both quartic and quadratic dispersion [KH94, ABK94, BGBK21, TABRdS19]. Here Ψ is the slowly varying complex envelope of the pulse, and β_2, β_4, and γ are real coefficients.

Solutions to (3.1) of the form $\Psi(x,t) = e^{i\beta t}\phi(x)$, $\beta \in \mathbb{R}$, are called *standing wave* solutions. Following the convention of [BGBK21], when the wave profile ϕ is a homoclinic orbit of the associated standing wave equation (given in (3.4)), we will call Ψ a *soliton solution* of (3.1). Karlsson and Höök [KH94] discovered an exact analytic family of soliton solutions to (3.1) with a squared hyperbolic secant profile. Akhmediev, Buryak and Karlsson [ABK94] observed oscillatory behaviour in the tails of solitons for certain values of β. Akhmediev and Buryak [BA95] showed the existence of bound states of two-solitons (i.e. double-hump pulses ϕ) in the same parameter regime, and derived a stability criterion by analysing the dependence of the associated Hamiltonian on the energy. Karpman and Shagalov [Kar96, KS97, KS00] considered the extension of (3.1) to higher-order nonlinearities and multiple space dimensions. All of these works considered the case $\beta_4 < 0$ and $\beta_2 < 0$.

More recently, (3.1) has been the focus of a number of studies following the experimental discovery of *pure quartic solitons* (PQSs) in silicon photonic crystal waveguides [BRdSS+16]. These solitons exist through a balance of negative quartic dispersion and the nonlinear Kerr effect, for which $\beta_2 = 0$ and $\beta_4 < 0$. They have attracted much attention for their potential applicability to ultrafast lasers due to their favourable energy-width scaling [BRdSHE17, TABRdS19]. Following the discovery of PQSs, Tam *et al.* [TABRdS19] numerically investigated their existence and spectral stability. They also showed [TABRdS18, TABRdS20] that PQSs and solitons of the classical second-order NLS equation, for which $\beta_4 = 0$, are in fact part of a broader continuous family of soliton solutions to (3.1) for nonpositive dispersion coefficients β_4 and β_2.

Extending the work of Tam *et al.*, Bandara *et al.* [BGBK21] used a dynamical systems approach to find infinite families of multi-hump soliton solutions to (3.1) for $\beta_4 \neq 0$ and $\beta_2 \neq 0$. To do so, they identified solitons of (3.1) as orbits of the stationary state equation satisfied by the wave profile that are homoclinic to the origin. As a consequence of the stationary state equation being Hamiltonian, fourth-order and having two reversible symmetries, they explain that infinitely many homoclinic solutions are created when the origin transitions from a real saddle (having only purely real eigenvalues) to a saddle focus (having complex conjugate eigenvalues) as a parameter is changed. This holds provided there exists a symmetric homoclinic orbit at the point of transition (see also [CT93]). In parameter regimes where this spectral behaviour occurs, they use continuation techniques to numerically compute these homoclinic orbits, which are characterised as heteroclinic cycles between the origin and periodic orbits in the zero energy level (zero set of the Hamiltonian). Depending on the symmetry properties of the periodic orbits and the types of connections from the origin to them, the orbits are organised into infinite families accordingly. They then use numerical simulations to investigate the stability of the waves computed. They found that while many of the multi-pulse solutions were unstable, some were only weakly unstable, and therefore possibly observable in experiments over a number of dispersion lengths.

A more rigorous stability analysis was undertaken by Natali and Pastor [NP15]. They proved the orbital stability of an exact solution to the nondimensionalised equivalent of (3.1) (see (3.2)). This solution represents the family of exact solutions to (3.1) found by Karlsson and Höök in [KH94]. As observed in [NP15] (also [TABRdS20, §II]), this solution exists only for a fixed value of the frequency parameter, and is *not* part of a continuous family of solutions in that parameter. The failure of the existence of such a family renders the classical results of Grillakis, Shatah and Strauss [GSS87, GSS90] inadmissible since [GSS87, Assumption 2] does not hold in this instance.

Under certain assumptions, Parker and Aceves [PA21] proved the existence and orbital stability of a single-hump solitary wave (not the exact analytical solution of Karlsson and Höök). For any such solitary wave, they determined the existence of an associated family of multi-hump solitons, which they proved to be unstable by showing the associated linearised operator has a positive real eigenvalue. The main results of [PA21] are formulated under a number of hypotheses which will not be required in our analysis.

In this paper, we further develop the spectral stability theory for *arbitrary* single and multi-hump soliton solutions to (3.1). Our results may be applied to the infinite families of multipulse solitons numerically computed in [BGBK21]. Our goal is to determine the existence of positive real eigenvalues for the linearised operator associated with any soliton solution to (3.1). We do not require Hypothesis 2, the first part of Hypothesis 3 or Hypothesis 4 of [PA21]. The main tool of our analysis is a topological invariant from symplectic geometry known as the *Maslov index*. It is a signed count of the intersections of a path in the manifold of Lagrangian subspaces of a symplectic vector space with a certain codimension-one set, the *train* of a fixed reference plane.

Our main results are as follows. In Theorem 3.2, we provide a lower bound for the number of positive real eigenvalues associated with soliton solutions to the nondimensionalised equivalent of (3.1). The bound is given in terms of the *Morse indices* (here, the number of positive eigenvalues) of two related selfadjoint operators, as well as a certain correction term which represents a contribution to the Maslov index from a *non-regular* crossing. This includes, as a corollary, the Jones-Grillakis instability theorem, which gives sufficient conditions on the aforementioned terms for the existence of a positive, real eigenvalue. We also provide a complete proof of the *Vakhitov-Kolokolov* (VK) stability criterion (see Theorem 3.5), where spectral (in)stability is determined by the sign of a certain integral. This includes, as a special case, the stability result of [PA21]. An advantage of our analysis is in the interpretation of P and Q, afforded by the Maslov index, as the number of *conjugate points* for each of the operators L_+ and L_-. All of the required data is therefore encoded at $\lambda = 0$. As highlighted in [BJ22], numerically this is a desirable feature that a calculation with the Evans function [AGJ90] does not possess. In light of this, an alternate form of (3.17), which may be more useful for numerical computations, is given in Remark 3.32. Our results are formulated under two genericity conditions (Hypotheses 3.16 and 3.17) the removal of which will be the subject of future work.

The key feature of the eigenvalue problem herein that allows us to make use of the Maslov index is the infinitesimally symplectic structure of the eigenvalue equations which preserve Lagrangian planes. The stable and unstable subspaces of the asymptotic system give rise to two-parameter families of Lagrangian planes, the stable and unstable bundles. Their nontrivial intersection at a common $x \in \mathbb{R}$ encodes (real) eigenvalues. By exploiting homotopy invariance of the Maslov index, we will detect positive real eigenvalues by instead analysing the intersections of the unstable bundle at $\lambda = 0$ with the train of the stable subspace of the asymptotic system.

The Maslov index has been used to study the spectrum of homoclinic orbits in a number of works [Jon88, BJ95, Cor19, CH14, HLS18, BCJ$^+$18, How23, How21]. In these cases, the Maslov index is used to detect purely real unstable eigenvalues. If monotonicity in the spectral parameter holds, as is often the case in selfadjoint problems [HLS18, BCJ$^+$18, How23], then it is possible to give an exact count of these eigenvalues in terms of a related Lagrangian path for which the spectral parameter is zero. Howard, Latushkin and Sukhtayev [HLS18] proved the equality of the Morse and Maslov indices for Schrödinger operators on the line, where the symmetric matrix-valued potential approaches constant end-states. They apply their results to analyse the stability of nonlinear waves in various reaction-diffusion systems. Jones [Jon88] and Bose and Jones [BJ95] used the Maslov index to study the stability of homoclinic orbits in the NLS equation and a gradient reaction-diffusion system respectively. Chen and Hu [CH14] proved a stability result for standing pulses in a doubly-diffusive FitzHugh-Nagumo equation. Beck *et al.* [BCJ$^+$18] proved the instability of pulses in gradient reaction-diffusion systems, generalising the instability result for pulses in scalar reaction-diffusion equations (see [KP13, §2.3.3]). Cornwell and Jones [Cor19, CJ18] used the Maslov index to analyse the stability of travelling waves in skew-gradient systems. Despite the eigenvalue equations not having a Hamiltonian structure, for a nonstandard symplectic form they preserve Lagrangian planes. They proved the stability of a particular travelling pulse in a doubly diffusive FitzHugh-Nagumo system by showing the Maslov index to be zero in the travelling wave co-ordinate z at $\lambda = 0$, despite lacking monotonicity in z.

A notable feature of the current problem is the occurrence of non-regular crossings, i.e. non-transversal intersections of the Lagrangian path with the train. We find instances where the *crossing form* is either identically zero, or degenerate with nonzero rank. In particular, the crossing form associated with the zero eigenvalue of the linearised operator (i.e. the conjugate point at the top left corner of the *Maslov box*) is identically zero crossing in the λ direction. This is a feature of eigenvalue problems of the form (3.12); see, for example, [CCLM23]. In addition, crossings in the x direction (when $\lambda = 0$) have a degenerate crossing form which is not identically zero. This phenomenon appears to be the result of the eigenvalue equations being fourth order, and has been encountered in [How21,How23]. In those papers, Howard and co-authors use a formulation of the Maslov index based on the spectral flow of a family of unitary matrices. Nonetheless, a degenerate crossing form can be still be observed in the spatial variable (see, for example, [How23, §6] and [How21, §5.2]). The issue is circumvented due to the crossing form being semidefinite in a neighbourhood of the crossing. By contrast, this semidefiniteness property does not hold in our case. In addition, it is unclear how to apply *Hörmander's index* (see [How21]), as was done for the fourth order problem on the line in [How23]. The complication is the requirement of a basis of vectors for the unstable bundle (along $\lambda = 0$) at $x = +\infty$. At this point, the bundle intersects the stable subspace in a one-dimensional subspace because $\lambda = 0$ is a simple eigenvalue. It is unclear how to determine this subspace. In this paper, we use the approach of [GPP04b,GPP04a] to locally compute the Maslov index via the *partial signatures* of an associated family of symmetric bilinear forms. This allows us to handle non-regular crossings without perturbative arguments, as in [RS93]. This will involve the use of *higher-order crossing forms*, which generalise the (first-order) crossing form defined in [RS93].

Recently in [CCLM23], a similar lower bound to that in Theorem 3.2 was derived for an eigenvalue problem of the form of (3.12) on a compact interval, where $L_\pm$ are Schrödinger operators. There, the "correction term" $\mathfrak{c}$ was computed via an analysis of the *eigenvalue curves*, offering a geometric interpretation of the corresponding term in the lower bound of [KP13, Theorem 7.1.16]. The fact that the spatial domain is the entire real line renders a similar calculation in the present setting intractable.

3.1.1 Statement of main results

We will work with the following nondimensionalised version of (3.1) corresponding to the case of nonzero quartic dispersion ($\beta_4 \neq 0$) and positive Kerr nonlinearity ($\gamma > 0$):

$$i\psi_t = -\sigma_4\psi_{xxxx} + \sigma_2\psi_{xx} - |\psi|^2\psi, \tag{3.2}$$

where $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $\sigma_4 = \operatorname{sign}\beta_4$ and $\sigma_2 = \operatorname{sign}\beta_2$. (For the transformations used to obtain (3.2) from (3.1) for $\beta_4 \neq 0, \beta_2 \neq 0$, we refer the reader to [BGBK21, Table 1].) We will treat the case when the quartic dispersion coefficient is negative, i.e. $\sigma_4 = -1$, and we assume that $\beta_2 \neq 0$, hence $\sigma_2 \in \{\pm 1\}$.

Our focus will be to determine the spectral stability of standing wave solutions

$$\psi(x,t) = e^{i\beta t}\phi(x), \qquad \phi(x) \in \mathbb{R}, \tag{3.3}$$

to (3.2), subject to perturbations in $L^2(\mathbb{R};\mathbb{C})$. Note that the wave profile ϕ satisfies the *standing wave equation*

$$\phi'''' + \sigma_2\phi'' + \beta\phi - \phi^3 = 0, \tag{3.4}$$

as seen upon substituting (3.3) into (3.2). Using the change of variables

$$\phi_1 = \phi'' + \sigma_2\phi, \qquad \phi_2 = \phi, \qquad \phi_3 = \phi', \qquad \phi_4 = \phi''', \tag{3.5}$$

we may write (3.4) as the first order Hamiltonian system

$$\begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \\ \phi_4' \end{pmatrix} = \begin{pmatrix} \phi_4 + \sigma_2\phi_3 \\ \phi_3 \\ \phi_1 - \sigma_2\phi_2 \\ -\sigma_2\phi_1 + \phi_2 - \beta\phi_2 + \phi_2^3 \end{pmatrix}. \tag{3.6}$$

Motivated by the families of homoclinic orbits discovered in [BGBK21], we consider orbits of (3.6) that are homoclinic to the origin, which correspond to soliton solutions of (3.2). We will assume that the origin in (3.6) is hyperbolic. Noting that the eigenvalues of the linearisation about the origin are given by

$$\mu^2 = \frac{1}{2}\left(-\sigma_2 \pm \sqrt{1-4\beta}\right) \tag{3.7}$$

(where we used that $\sigma_2^2 = 1$), hyperbolicity holds provided

$$\begin{cases} \beta > 0 & \text{if} \quad \sigma_2 = -1 \\ \beta > \frac{1}{4} & \text{if} \quad \sigma_2 = 1. \end{cases} \tag{3.8}$$

In the first part of (3.8), we additionally require

$$\beta \neq \frac{1}{4} \quad \text{if} \quad \sigma_2 = -1. \tag{3.9}$$

Linearising (3.2) by substituting the complex-valued perturbation

$$\psi(x,t) = \left[\phi(x) + \varepsilon\left(u(x) + iv(x)\right)e^{\lambda t}\right]e^{i\beta x} \tag{3.10}$$

for $u, v \in L^2(\mathbb{R};\mathbb{R})$ into (3.2), collecting $O(\varepsilon)$ terms and separating into real and imaginary parts leads to the following linearised dynamics in u and v:

$$\begin{aligned} -u'''' - \sigma_2 u'' - \beta u + 3\phi^2 u &= \lambda v \\ -v'''' - \sigma_2 v'' - \beta v + \phi^2 v &= -\lambda u. \end{aligned} \tag{3.11}$$

We can write (3.11) as the spectral problem

$$N\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.12}$$

where N is the linear operator

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \qquad \begin{cases} L_- = -\partial_x^4 - \sigma_2 \partial_x^2 - \beta + \phi^2, \\ L_+ = -\partial_x^4 - \sigma_2 \partial_x^2 - \beta + 3\phi^2, \end{cases} \tag{3.13}$$

with

$$\mathrm{dom}(N) = H^4(\mathbb{R}) \times H^4(\mathbb{R}), \qquad \mathrm{dom}(L_\pm) = H^4(\mathbb{R}). \tag{3.14}$$

Our goal is to determine whether the spectrum of the unbounded and densely defined linear operator N intersects the open right half plane. Because N is Hamiltonian, its spectrum has four-fold symmetry in $\mathbb{C}$, and instability follows from any part of the spectrum lying off the imaginary axis. We will show in Section 3.2 that the essential spectrum of N is confined to the imaginary axis. Regarding the point spectrum, it is a requirement of the Maslov index that the eigenvalue parameter be real (the detection of complex eigenvalues via the Maslov index remains an open problem). Our task therefore is to detect *positive real eigenvalues* $\lambda \in \mathrm{Spec}(N) \cap \mathbb{R}^+$. We will give a lower bound for the count of these eigenvalues in terms of the Morse indices of the operators $L_\pm$, which are selfadjoint with the domain in (3.14) (see, for example, [Wei87]). The Morse indices of $L_\pm$ are only well-defined if their essential spectra are confined to the negative half line, and we show in Section 3.2 that this is indeed the case under the assumptions (3.8)–(3.9).

We point out here that the equation $L_-\phi = 0$ is just (3.4), and, differentiating (3.4) with respect to x, we have $L_+\phi_x = 0$. Thus

$$0 \in \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+), \tag{3.15}$$

where $\phi \in \ker(L_-)$ and $\phi_x \in \ker(L_+)$. We will assume these are the only functions lying in the kernel.

Hypothesis 3.1. $\dim\ker(L_+) = \dim\ker(L_-) = 1$, where $\ker(L_-) = \mathrm{span}\{\phi\}$ and $\ker(L_+) = \mathrm{span}\{\phi_x\}$.

Notice that when $\lambda = 0$, the eigenvalue equations (3.12) decouple into the two independent equations $L_-v = 0$ and $L_+u = 0$, so that $\ker(N) = \ker(L_+) \oplus \ker(L_-)$. Hypothesis 3.1 therefore implies that $\ker(N) = \mathrm{span}\{(\phi_x, 0)^\top, (0, \phi)^\top\}$.

Let us denote

$$\begin{aligned} P &:= \#\{\text{positive eigenvalues of } L_+\}, \\ Q &:= \#\{\text{positive eigenvalues of } L_-\}, \\ n_+(N) &:= \#\{\text{positive real eigenvalues of } N\}, \end{aligned}$$

and define the quantities

$$\mathcal{I}_1 := \int_{-\infty}^{\infty} \phi_x \, \widehat{v} \, dx, \qquad \mathcal{I}_2 := \int_{-\infty}^{\infty} \phi \, \widehat{u} \, dx, \tag{3.16}$$

where $\widehat{v}$ is any solution in $H^4(\mathbb{R})$ to $-L_-v = \phi_x$ and $\widehat{u}$ is any solution in $H^4(\mathbb{R})$ to $L_+u = \phi$. Under Hypothesis 3.1 and the conditions (3.8)–(3.9), as well as two genericity conditions Hypotheses 3.16 and 3.17 which will be given in Section 3.4, our main result is the following:

Theorem 3.2. *Suppose $\mathcal{I}_1, \mathcal{I}_2 \neq 0$. The number of positive, real eigenvalues of the operator N satisfies*

$$n_+(N) \geq |P - Q - \mathfrak{c}|, \tag{3.17}$$

where $\mathfrak{c}$ is computed via

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0,\ \mathcal{I}_2 < 0, \\ 0 & \mathcal{I}_1\mathcal{I}_2 > 0, \\ -1 & \mathcal{I}_1 < 0,\ \mathcal{I}_2 > 0. \end{cases} \tag{3.18}$$

Remark 3.3. The equations $-L_-v = \phi'$ and $L_+u = \phi$ each satisfy a solvability condition that guarantees the existence of solutions $\widehat{u}$ and $\widehat{v}$. In the case that either $\mathcal{I}_1$ or $\mathcal{I}_2$ vanishes, an extra calculation is needed to compute the correction term $\mathfrak{c}$ (the definition of which is given in (3.91)); for details, see Remark 3.34). Finally, our theorem will also hold in the case of any integer power-law nonlinearity in (3.2), i.e. in the case of standing wave solutions to

$$i\psi_t = -\sigma_4\psi_{xxxx} + \sigma_2\psi_{xx} - f(|\psi|^2)\psi, \qquad f(|\psi|^2) = |\psi|^{2p},\ \ p \in \mathbb{Z}^+. \tag{3.19}$$

(See Remark 3.23.) However, with the standing wave solutions of [BGBK21] in mind, we have stated our results for the cubic case.

The following *Jones-Grillakis* instability theorem [Jon88, Gri88, KP13] is an immediate consequence of Theorem 3.2.

Corollary 3.4. *Standing waves for which $P - Q \neq -1, 0, 1$ are unstable.*

In this work we do not require existence of standing waves; rather, we prove that if a standing wave exists with the spectral properties of L_+ and L_- stated, then its linearised operator N satisfies Theorem 3.2.

We also have the following *Vakhitov-Kolokolov* type criterion [VK73, Pel11].

Theorem 3.5. *Suppose $P = 1$ and $Q = 0$. The standing wave $\widehat{\psi}$ is spectrally unstable if $\mathcal{I}_2 > 0$ and is spectrally stable if $\mathcal{I}_2 < 0$.*

Remark 3.6. If there exists a C^1 family of solutions $\beta \to \partial_\beta\phi(x;\beta) \in H^4(\mathbb{R})$ to the standing wave equation, then $\widehat{u} = \partial_\beta\phi(x;\beta)$ and the integral $\mathcal{I}_2$ is precisely that appearing in the classical Vakhitov-Kolokolov criterion for standing waves in the usual (second-order) NLS equation (see [Pel11, §4.2]), i.e.

$$\mathcal{I}_2 = \frac{1}{2}\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty}\phi^2 dx. \tag{3.20}$$

The paper is organised as follows. In Section 3.2 we write down the first order system associated with (3.12), compute the essential spectra of the operators $L_, L_+$ and N, and define the stable and unstable bundles, the main objects of our analysis. In Section 3.3 we provide some background material on the Maslov index before setting up the homotopy argument that will lead to the proof of the lower bound in Theorem 3.2. In Section 3.4 we use the Maslov index to prove that the Morse index of each of the operators L_- and L_+ is equal to the associated number of conjugate points. In Section 3.5 we prove Theorems 3.2 and 3.5.

3.2 Set-up

We first compute the essential spectra of the operators $L_\pm, N$. Using the change of variables

$$\begin{aligned} u_1 &= u'' + \sigma_2 u, & u_2 &= u, & u_3 &= u', & u_4 &= u''', \\ v_1 &= v'' + \sigma_2 v, & v_2 &= -v, & v_3 &= -v', & v_4 &= v''', \end{aligned} \tag{3.21}$$

we convert (3.11) to the (infinitesimally symplectic) first order system

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}' = \left(\begin{array}{cccc|cccc} & & & & \sigma_2 & 0 & 1 & 0 \\ & 0 & & & 0 & -\sigma_2 & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -\sigma_2 & 0 & & & & \\ 0 & -1 & 0 & -\sigma_2 & & 0 & & \\ -\sigma_2 & 0 & \alpha(x) & \lambda & & & & \\ 0 & -\sigma_2 & \lambda & \eta(x) & & & & \end{array}\right) \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}, \tag{3.22}$$

where

$$\alpha(x) := 3\phi(x)^2 - \beta + 1, \qquad \eta(x) := -\phi(x)^2 + \beta - 1.$$

Setting

$$B = \begin{pmatrix} \sigma_2 & 0 & 1 & 0 \\ 0 & -\sigma_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C(x;\lambda) = \begin{pmatrix} 1 & 0 & -\sigma_2 & 0 \\ 0 & -1 & 0 & -\sigma_2 \\ -\sigma_2 & 0 & \alpha(x) & \lambda \\ 0 & -\sigma_2 & \lambda & \eta(x) \end{pmatrix},$$

we can write (3.22) as

$$\mathbf{w}_x = A(x;\lambda)\mathbf{w}, \tag{3.23}$$

where

$$\mathbf{w} = (u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)^\top, \qquad A(x;\lambda) = \begin{pmatrix} 0 & B \\ C(x;\lambda) & 0 \end{pmatrix}. \tag{3.24}$$

The asymptotic system for (3.22) is given by

$$\mathbf{w}_x = A_\infty(\lambda)\mathbf{w}, \tag{3.25}$$

where

$$A_\infty(\lambda) := \lim_{x\to\pm\infty} A(x,\lambda).$$

(The endstates as $x \to \pm\infty$ are the same because ϕ is homoclinic to the origin.) It now follows from [KP13, Theorem 3.1.11] that the essential spectrum of N is given by the set of $\lambda \in \mathbb{C}$ for which the matrix $A_\infty(\lambda)$ has a purely imaginary eigenvalue. A short calculation shows that

$$\mathrm{Spec}_{\mathrm{ess}}(N) = \{\lambda \in \mathbb{C} : \lambda^2 = -\left(-k^4 + \sigma_2 k^2 - \beta\right)^2 \ \text{ for some } \ k \in \mathbb{R}\} \subseteq i\mathbb{R}. \tag{3.26}$$

Notice we require $\beta \neq 0$ in order to have $0 \notin \mathrm{Spec}_{\mathrm{ess}}(N)$.

The essential spectra of the operators $L_\pm$ is computed similarly. The first order systems associated with the eigenvalue equations for each of the operators L_+ and L_- will be given in Section 3.4 (see (3.95) and (3.98)). It follows from a similar calculation on the asymptotic matrices associated with those systems that

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \{\lambda \in \mathbb{R} : \lambda = -k^4 + \sigma_2 k^2 - \beta \ \text{ for some } \ k \in \mathbb{R}\}. \tag{3.27}$$

Given its biquadratic structure, if the equation in (3.27) has no real roots for k then the essential spectra of L_+ and L_- will be confined to the negative half line. The equation in (3.27) has no real roots for k if and only if the associated discriminant is positive, i.e.

$$16\beta^3 - 8\beta^2 + \beta = \beta(4\beta - 1)^2 > 0, \tag{3.28}$$

and, in addition, we have either

$$-8\sigma_2 > 0 \qquad \text{or} \qquad 4\beta - 1 > 0. \tag{3.29}$$

(See [Ree22], and note we have used that $\sigma_2^2 = 1$). Both (3.28) and (3.29) are satisfied for the values of β given in (3.8), (3.9). For these values of β we therefore have

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \begin{cases} (-\infty, -\beta) & \sigma_2 = -1, \\ (-\infty, -\beta - \frac{1}{4}] & \sigma_2 = 1, \end{cases} \tag{3.30}$$

so that $\mathrm{Spec}_{\mathrm{ess}}(L_\pm) \subset \mathbb{R}^-$. In addition to hyperbolicity of the asymptotic matrices for the L_+ and L_- eigenvalue problems, the values of β given in (3.8), (3.9) will actually guarantee that those asymptotic matrices have an equal number of eigenvalues with positive and negative real part.

Note that the assumptions (3.8) actually guarantee that the matrix $A_\infty(\lambda)$ is hyperbolic, with an equal number of eigenvalues with positive and negative real part. Precisely, the eight eigenvalues are

$$\pm\frac{\sqrt{-\sigma_2 \pm \sqrt{1 - 4\beta \pm 4\lambda i}}}{\sqrt{2}}. \tag{3.31}$$

We denote the corresponding stable and unstable subspaces (i.e. the eigenspaces associated with eigenvalues with negative and positive real part) by $\mathbb{S}(\lambda)$ and $\mathbb{U}(\lambda)$ respectively.

Next, since $\mathrm{Spec}_{\mathrm{ess}}(N) \subset i\mathbb{R}\backslash\{0\}$, the operator $N - \lambda I$ of (3.12)–(3.14) is Fredholm for $\lambda \in \mathbb{R}$, and it follows from [San02, §3.3] that the densely-defined closed linear operator

$$T(\lambda) : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad T(\lambda)u := \frac{du}{dx} - A(\cdot;\lambda)u,$$

associated with (3.23) is also Fredholm. By [San02, Theorem 3.2, Remark 3.3], (3.22) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. That is, for each fixed $\lambda \in \mathbb{R}$, on each of the intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ the set of solutions to (3.22) is the direct sum of two subspaces, where one subspace consists solely of solutions that decay (exponentially) backwards in x, and the other of solutions that decay forwards in x. By flowing these subspaces under (3.22), each of these families can be extended to all of $\mathbb{R}$. This leads us to consider the spaces

$$\begin{aligned} \mathbb{E}^u(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (3.22) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to -\infty\}, \\ \mathbb{E}^s(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (3.22) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to +\infty\}, \end{aligned} \tag{3.32}$$

corresponding to the evaluation at $x \in \mathbb{R}$ of the spaces of solutions to (3.22) that decay (exponentially) as $x \to +\infty$ and as $x \to -\infty$, respectively. Following [AGJ90, Cor19], we call these sets the *unstable* and *stable bundles* respectively. For each $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if we consider $\mathbb{U}(\lambda), \mathbb{S}(\lambda), \mathbb{E}^u(x,\lambda), \mathbb{E}^s(x,\lambda)$ as points in the Grassmannian of four-dimensional subspaces of $\mathbb{R}^8$,

$$\mathrm{Gr}_4(\mathbb{R}^8) = \{V \subset \mathbb{R}^8 : \dim V = 4\},$$

which (following [Fur04, HLS17]) we equip with the metric $d(V,U) = \|P_V - P_U\|$, where P_V is the orthogonal projection onto V and $\|\cdot\|$ is any matrix norm, then we have that

$$\lim_{x\to-\infty} \mathbb{E}^u(x,\lambda) = \mathbb{U}(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s(x,\lambda) = \mathbb{S}(\lambda). \tag{3.33}$$

That is, the orthogonal projections onto $\mathbb{E}^u(x,\lambda)$ and $\mathbb{E}^s(x,\lambda)$ converge to those on $\mathbb{U}(\lambda)$ and $\mathbb{S}(\lambda)$ as $x \to -\infty$ and $x \to \infty$, respectively. This is given in [PSS97, Corollary 2].

The important feature of the system (3.22) that makes it amenable to the Maslov index is that the coefficient matrix $A(x;\lambda)$ is infinitesimally symplectic, i.e.

$$A(x;\lambda)^T J + JA(x;\lambda) = 0, \tag{3.34}$$

which follows from the symmetry of B and $C(x;\lambda)$. (J is defined in (3.47).) This is the motivation for the choice of substitutions (3.21). Consequently, (3.22) induces a flow on the manifold of *Lagrangian planes*. In particular, the stable and unstable bundles of (3.22) are Lagrangian planes of $\mathbb{R}^8$ for all x and all λ. In addition we have that λ_0 is an eigenvalue of N if and only if for any (and hence all) $x \in \mathbb{R}$ we have

$$\mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) \neq \{0\}.$$

In this case we in fact have

$$\dim \mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) = \dim\ker(N - \lambda_0 I). \tag{3.35}$$

By exploiting homotopy invariance of the Maslov index, we can determine the existence of such intersections by instead analysing the evolution of the unstable bundle $\mathbb{E}^u(x,\lambda_0)$ when $\lambda_0 = 0$. This is explained in Section 3.3.

3.3 A symplectic approach to the eigenvalue problem

In this section, we give some background material on the Maslov index before describing the homotopy argument that leads to the lower bound of Theorem 3.2. Our definition of the Maslov index follows [GPP04a,GPP04b], which involves computing the *spectral flow* (the net change in the number of nonnegative eigenvalues) of a smooth curve of symmetric matrices. We begin by discussing a general framework for such a computation.

3.3.1 Preliminaries: spectral flow and the partial signatures

We follow the discussion in [GPP04b, §2.1-2.2]. In what follows, V is a subspace of $\mathbb{R}^{2n}$ and $\mathcal{S}(V)$ is the vector space of symmetric linear operators (matrices) $T : V \to V$. Consider a smooth curve $t \mapsto L(t) \in \mathcal{S}(V)$, which has an isolated singularity at $t = t_0$, i.e. $\det L(t_0) = 0$ and $\det L(t) \neq 0$ for $0 < |t - t_0| < \varepsilon$. The following is a method to compute the jump in the number of nonnegative eigenvalues of L as t passes through t_0.

A *root function* for $L(t)$ at $t = t_0$ is a smooth map $q : [t_0 - \varepsilon, t_0 + \varepsilon] \to V$, $\varepsilon > 0$, such that $q(t_0) \in \ker(L(t_0))$. The *order* of q, ord(q), is the order of zero at $t = t_0$ of the map $t \mapsto L(t)q(t)$, i.e. the smallest positive integer k such that $\frac{d^k}{dt^k}\left(L(t)q(t)\right)\big|_{t=t_0} \neq 0$. With these notions we can define a sequence of spaces W_k and bilinear forms $B_k : W_k \times W_k \to \mathbb{R}$ for $k \geq 1$ as follows:

$$W_k := \{q_0 \in V : \text{there exists a root function } q \text{ with } \operatorname{ord}(q) \geq k \text{ and } q(t_0) = q_0\}, \tag{3.36}$$

$$B_k(q_0, r_0) := \frac{1}{k!}\left\langle \frac{d^k}{dt^k}\left(L(t)q(t)\right)\big|_{t=t_0}, r_0 \right\rangle_{\mathbb{R}^{2n}}, \qquad q_0, r_0 \in W_k, \tag{3.37}$$

where q in (3.37) is any root function with $\operatorname{ord}(q) \geq k$ and $q(t_0) = q_0$. It follows from the definition that $W_1 \subseteq \ker(L(t_0))$. It is proven in [GPP04b, Proposition 2.4] that B_k is symmetric and independent of the choice of q, and therefore well-defined. Moreover, from [GPP04b, Proposition 2.4, Corollary 2.10] we have

$$W_{k+1} \subseteq W_k \quad \text{for all } k \geq 1, \qquad \text{and} \qquad W_{k+1} = \ker B_k. \tag{3.38}$$

Notice that if B_k is nondegenerate for some k, then $W_j = \{0\}$ for all $j > k$.

The spaces W_k can be characterised as follows. Define $L_k := \frac{1}{k!}\frac{d^k}{dt^k}L(t)|_{t=t_0}$. A *generalised Jordan chain* of length $k+1$ starting at q_0 for $L(t)$ at $t = t_0$ is a sequence of nonzero vectors $\{q_0, q_1, ..., q_k\}$, $q_i \in V$ satisfying the system of $k+1$ equations

$$\begin{aligned}
&L_0 q_0 = 0,\\
&L_1 q_0 + L_0 q_1 = 0,\\
&L_2 q_0 + L_1 q_1 + L_0 q_2 = 0,\\
&\vdots\\
&\sum_{j=0}^{k} L_{k-j} q_j = 0.
\end{aligned} \tag{3.39}$$

Such a chain is called *maximal* if it cannot be extended to a chain of length $k+2$, i.e. there is no solution q_{k+1} to

$$L_{k+1}q_0 + L_k q_1 + \cdots + L_1 q_k + L_0 q_{k+1} = 0. \tag{3.40}$$

For any generalised Jordan chain $\{q_0, \ldots, q_k\}$ (not necessarily maximal), the function $q(t) := \sum_{j=0}^{k}(t-t_0)^j q_j$ is a root function with $\mathrm{ord}(q) \geq k+1$ and $q(t_0) = q_0$, since

$$\frac{d^i}{dt^i}\left(L(t)q(t)\right)|_{t=t_0} = \sum_{j=0}^{i}\binom{i}{j}L^{(i-j)}(t_0)q^{(j)}(t_0) = i!\sum_{j=0}^{i}L_{i-j}q_j = 0 \qquad \text{for all } \ i = 0, 1, \ldots, k.$$

Here we used that $q^{(j)}(t_0) = j!q_j$ and $L^{(i-j)}(t_0) = (i-j)!L_{i-j}$ in the second equality, and (3.39) in the third equality. Conversely, any root function q with $\mathrm{ord}(q) \geq k+1$ gives a generalised Jordan chain of length (at least) $k+1$ via $q_i := \frac{1}{i!}q^{(i)}(t_0)$. This shows that:

$$\begin{aligned} W_{k+1} = \{q_0 \in V :\ \exists \ \text{a generalised Jordan chain of length } k+1, \\ \text{starting at } q_0, \text{ for } L(t) \text{ at } t = t_0\}. \end{aligned} \tag{3.41}$$

Moreover, the root function q associated with any $q_0 \in W_{k+1}$ has $\mathrm{ord}(q) = k+1$ if and only if the associated Jordan chain $\{q_0, ..., q_k\}$ is maximal. Maximality of the chain holds if and only if

$$L_{k+1}q_0 + L_k q_1 + \cdots + L_1 q_k \notin \mathrm{Ran}(L_0) = \ker(L_0)^{\perp}. \tag{3.42}$$

Notice that (3.41) shows that $\ker L_0 \subseteq W_1$, since any $q_0 \in \ker L_0$ is a generalised Jordan chain of length one for $L(t)$. From our earlier observation this implies

$$W_1 = \ker L(t_0). \tag{3.43}$$

For any generalised Jordan chain $\{q_0, \ldots, q_k\}$, the bilinear form B_{k+1} is given by

$$B_{k+1}(q_0, r_0) = \sum_{j=0}^{k}\langle L_{k+1-j}q_j, r_0\rangle_{\mathbb{R}^{2n}}\,, \qquad\qquad q_0, r_0 \in W_{k+1}, \tag{3.44}$$

as can be seen from substituting the root function $q(t) = \sum_{j=0}^{k}(t-t_0)^j q_j$ into (3.37).

If the chain $\{q_0, \ldots, q_k\}$ is not maximal (i.e. it can be extended to $\{q_0, \ldots, q_{k+1}\}$ where q_{k+1} solves (3.40)), then for all $i = 0, \ldots, k$ and any $r_0 \in W_{i+1}$, we have

$$B_{i+1}(q_0, r_0) = \sum_{j=0}^{i}\langle L_{i+1-j}q_j, r_0\rangle_{\mathbb{R}^{2n}} = -\langle L_0 q_{i+1}, r_0\rangle_{\mathbb{R}^{2n}} = -\langle q_{i+1}, L_0 r_0\rangle_{\mathbb{R}^{2n}} = 0.$$

Here, the second equality follows for $i = 0, \ldots, k-1$ from (3.39) and for $i = k$ from (3.40), and we used that $r_0 \in W_{i+1} \subseteq \ker L(t_0)$. On the other hand, if the Jordan chains associated with $q_0, r_0 \in W_{k+1}$ are both of length $k+1$ and maximal, then $B_{k+1}(q_0, r_0)$ is nondegenerate. This follows from the symmetry of B_{k+1} and (3.42).

The family of bilinear forms $\{B_k\}_k$ can be used to compute the jump in the number of nonnegative eigenvalues of $L(t)$ as t increases through t_0. The following is taken from [GPP04b, Proposition 2.9]. We denote by $n_+(S), n_-(S), n_+^0(S), n_-^0(S)$ respectively the number of positive, negative, nonnegative and nonpositive eigenvalues (squares) of the symmetric matrix (symmetric bilinear form) S. For the bilinear forms defined in (3.37), the integers

$$n_-(B_k), \quad n_+(B_k), \quad n_+(B_k) - n_-(B_k), \tag{3.45}$$

for $k \geq 1$ are called, respectively, the *kth partial negative index*, the *kth partial positive index* and the *kth partial signature* of $L(t)$ at $t = t_0$. The integers in (3.45) are collectively referred to as the *partial signatures* of the curve of symmetric matrices $L(t)$ at $t = t_0$.

Proposition 3.7. *Suppose $[t_0 - \varepsilon, t_0 + \varepsilon] \mapsto L(t) \in \mathcal{S}(V)$ is a smooth curve of symmetric matrices with an isolated singularity at $t = t_0$, $\{\lambda_i(t)\}$ are the smooth curves of eigenvalues of $L(t)$, and the associated spaces W_k and bilinear forms B_k are as in (3.36),(3.37). For all nonconstant $\lambda_i(t)$ vanishing at $t = t_0$, assume the zero of $\lambda_i(t)$ at $t = t_0$ is of finite order, and that for each eigenvalue $\lambda_i(t)$, there exists a smooth family of unit eigenvectors $u_i(t)$, where the u_i are pairwise orthogonal for each t. Then the following hold:*

(i) $W_k = \mathrm{span}\{u_i(t_0) : i \in \{1, \dots, n\}$ *is such that* $\lambda_i^{(j)}(t_0) = 0$ *for all* $j < k\}$;

(ii) *if* $q \in W_k$ *is an eigenvector of* $\lambda_i(t_0)$, *where* $\lambda_i^{(j)}(t_0) = 0$ *for all* $j < k$, *then* $B_k(q, r) = \frac{1}{k!}\lambda_i^{(k)}(t_0)\langle q, r\rangle$ *for all* $r \in W_k$;

(iii) $$n_+^0(L(t_0 + \varepsilon)) - n_+^0(L(t_0)) = -\sum_{k\geq 1} n_-(B_k),$$
$$n_+^0(L(t_0)) - n_+^0(L(t_0 - \varepsilon)) = \sum_{k\geq 1} \left(n_-(B_{2k}) + n_+(B_{2k-1})\right),$$
$$n_+^0(L(t_0 + \varepsilon)) - n_+^0(L(t_0 - \varepsilon)) = \sum_{k\geq 1} \left(n_+(B_{2k-1}) - n_-(B_{2k-1})\right),$$
where each of the sums on the right hand side of the previous three equations have a finite number of nonzero terms.

Note that the negative index $n_-(B_k)$ (resp. the positive index $n_+(B_k)$) is the number of i's in $\{1, \dots, n\}$ such that $\lambda_i(t)$ has a zero of order k at $t = t_0$ and whose kth derivative is negative (resp. positive) at $t = t_0$. Note as well that to obtain the formulas in (ii), we have manipulated the corresponding formulas in [GPP04b, Proposition 2.9] using the following formula from [GPP04b, Corollary 2.11]:

$$\sum_{k\geq 1} (n_+(B_k) + n_-(B_k)) = \dim\ker(L(t_0)). \tag{3.46}$$

For some illustrative examples involving computation of the spaces W_k, the forms B_k and the behaviour the eigenvalues $\lambda_i(t)$ in some simple cases when $V = \mathbb{R}^2$ and $L(t) \in \mathbb{R}^{2\times 2}$, see [GPP04b, Examples 2.8, 2.12].

3.3.2 The Maslov index

In this section we follow the discussions in [Arn67, RS93, GPP04b]. Consider $\mathbb{R}^{2n}$ equipped with the symplectic form

$$\omega(u,v) = \langle Ju, v\rangle_{\mathbb{R}^{2n}}, \qquad J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}. \tag{3.47}$$

A *Lagrangian subspace* of $\mathbb{R}^{2n}$ is one that is n dimensional and upon which the symplectic form vanishes. We denote the Grassmannian of all Lagrangian subspaces of $\mathbb{R}^{2n}$ by

$$\mathcal{L}(n) := \{\Lambda \subset \mathbb{R}^{2n} : \dim \Lambda = n, \omega(u,v) = 0 \ \forall \ u,v \in \Lambda\}. \tag{3.48}$$

A *frame* for a Lagrangian subspace Λ of $\mathbb{R}^{2n}$ is a $2n \times n$ matrix whose columns span Λ. Such a frame has the form

$$\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \text{where } \ X^\top Y = Y^\top X, \qquad X, Y \in \mathbb{R}^{n\times n}. \tag{3.49}$$

The symmetry of $X^\top Y$ follows from the vanishing of (3.47). Such a frame is not unique; right multiplication by an invertible matrix will yield a different frame for the same space. In particular, if X is invertible then an equivalent frame is

$$\begin{pmatrix} I \\ YX^{-1} \end{pmatrix}, \qquad \text{where } \ \left(YX^{-1}\right)^\top = YX^{-1}. \tag{3.50}$$

Arnol'd [Arn67] defined a Maslov index for non-closed curves as follows. Any fixed $V \in \mathcal{L}(n)$ gives rise to a decomposition of $\mathcal{L}(n)$ via $\mathcal{L}(n) = \bigcup_{k=0}^n \mathcal{T}_k(V)$, where each stratum $\mathcal{T}_k(V) := \{W \in \mathcal{L}(n) : \dim(W \cap V) = k\}$ has codimension $k(k+1)/2$. The *train* $\mathcal{T}(V)$ of V is the set of all Lagrangian planes that intersect V nontrivially, i.e. $\mathcal{T}(V) := \bigcup_{k=1}^n \mathcal{T}_k(V)$. From the fundamental lemma of [Arn67], $\mathcal{T}_1(V)$ is two-sidedly embedded in $\mathcal{L}(n)$, that is, there exists a continuous vector field on $\mathcal{L}(n)$ that is everywhere transverse to $\mathcal{T}_1(V)$. Such a vector field therefore defines a 'positive' and a 'negative' side of $\mathcal{T}_1(V)$. For any continuous curve $\Lambda : [a,b] \to \Lambda(n)$ with endpoints lying off the train and which intersects $\mathcal{T}(V)$ only in $\mathcal{T}_1(V)$, its *Maslov index* is given by $\nu_+ - \nu_-$, where ν_+ (ν_-) is the number of points of passage of Λ from the negative to the positive side (from the positive to the negative side) of $\mathcal{T}_1(V)$. Robbin and Salamon [RS93] gave a definition in terms of *crossing forms*, which is based on an identification of the tangent space of $\mathcal{L}(n)$ with the space of quadratic forms. Their definition required neither transversality at the endpoints nor of intersections only with $\mathcal{T}_1(V)$. However, nondegeneracy of the quadratic crossing form is required; this is equivalent to the path having only transversal intersections with $\mathcal{T}(V)$. They extended the definition to *all* continuous Lagrangian paths (i.e. those for which the crossing form is degenerate) via homotopy invariance (see Proposition 3.9).

Giambò, Piccione and Portaluri [GPP04b, GPP04a] gave a formula for the Maslov index of an analytic Lagrangian path having isolated possibly nontransversal intersections with $\mathcal{T}(V)$. This is given below. In doing so, they did away with the nondegeneracy assumption of [RS93] (at least for analytic paths). To the analytic Lagrangian path they associate a locally-defined smooth curve of symmetric bilinear forms, the spectral flow of which is shown to locally compute the Maslov index.

Suppose $\Lambda : [a,b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces, and let $V \in \mathcal{L}(n)$ be fixed. Suppose further that $t = t_0$ is an isolated *crossing*, that is, $\Lambda(t_0) \cap V \neq \{0\}$, and choose any $W \in \mathcal{L}(n)$ which is transverse to both $\Lambda(t_0)$ and V. By continuity, W is transversal to $\Lambda(t)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\varepsilon > 0$ small enough, and there exists a smooth, unique family of matrices $R(t)$, viewed as operators from V into W, such that $\Lambda(t)$ is the graph of $R(t)$ for each t, i.e. $\Lambda(t) = \mathrm{graph}(R(t)) = \{q + R(t)q : q \in V\}$. This allows one to define a smooth curve of bilinear forms

$$[t_0 - \varepsilon, t_0 + \varepsilon] \ni t \mapsto \omega(R(t)\cdot, \cdot)|_{V\times V} \tag{3.51}$$

on V, which are symmetric for each t on account of $\Lambda(t)$ being Lagrangian. Indeed, for all $u, v \in V$ we have

$$\begin{aligned} \omega(R(t)u, v) &= \omega(u + R(t)u, v) \\ &= \omega(u + R(t)u, v + R(t)v) - \omega(u + R(t)u, R(t)v) \\ &= -\omega(u, R(t)v) = \omega(R(t)v, u). \end{aligned} \tag{3.52}$$

Moreover, we have

$$(\ker R(t)) \cap V = \ker\left(\omega(R(t)\cdot, \cdot)|_{V\times V}\right) = \Lambda(t) \cap V, \tag{3.53}$$

and from our assumptions the right hand side is nontrivial precisely when $t = t_0$. In this way we see that any crossing of the path Λ with the train $\mathcal{T}(V)$ corresponds to an isolated singularity of the locally-defined form (3.51).

Denote by $\pi_1(\mathcal{L}(n))$ the fundamental groupoid of $\mathcal{L}(n)$, i.e. the set of (fixed-endpoint) homotopy classes $[\Lambda]$ of paths $\Lambda : [a,b] \to \mathcal{L}(n)$, equipped with the partial operation $[\Lambda] \cdot [\xi] = [\Lambda * \xi]$, where $*$ is the concatenation of two paths $\Lambda, \xi : [a,b] \to \mathcal{L}(n)$, which is only defined if $\Lambda(b) = \xi(a)$. For all $V \in \mathcal{L}(n)$, it is proven in [GPP04b, Corollary 3.5] that there is a unique integer-valued homomorphism $\mu(\cdot; V)$ on $\pi_1(\mathcal{L}(n))$[1] such that the following holds. With our earlier choice of W, i.e. such that $W \in \mathcal{T}_0(V) \cap \mathcal{T}(\Lambda(t_0))$, if $\Lambda : [a,b] \to \mathcal{T}_0(W)$ then $\mu([\Lambda]; V)$ is given by the spectral flow of the family of forms (3.51) defined over $[a,b]$. Note (3.51) is well-defined over the entire interval in this case because $\Lambda : [a,b] \to \mathcal{T}_0(W)$. The Maslov index of *any* continuous path Λ is then defined to be $\mu([\Lambda]; V)$, and the authors prove in [GPP04b, Proposition 3.11], using Proposition 3.7, that it is computable via the partial signatures of (3.51) at each isolated crossing with $\mathcal{T}(V)$. For our purposes, it will suffice to use the latter computational tool as our definition of the Maslov index.

In the same fashion as (3.37), we define the *kth-order crossing form* by

$$\mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(q_0, r_0) = \frac{d^k}{dt^k}\,\omega\big(R(t)q(t), r_0\big)\Big|_{t=t_0}, \qquad q_0, r_0 \in W_k, \tag{3.54}$$

[1] i.e. a map $\mu : \pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ such that $\mu([\Lambda] * [\xi]) = \mu([\Lambda]) + \mu([\xi])$ for all $[\Lambda], [\xi] \in \pi(\mathcal{L}(n))$ with $\Lambda(b) = \xi(a)$

where q is a *root function* for (3.51) at $t = t_0$ with $\operatorname{ord}(q) \geq k$, i.e. a smooth map $q : [t_0 - \varepsilon, t_0 + \varepsilon] \to V$ such that $q(t_0) \in \ker JR(t_0) = \ker R(t_0)$ and $\frac{d^i}{dt^i} JR(t)q(t)|_{t=t_0} = 0$ for $i = 1, \dots, k-1$, and

$$\begin{aligned} W_k = \{q_0 \in V : \exists \text{ a generalised Jordan chain of length } k, \\ \text{starting at } q_0, \text{ for the curve of bilinear forms in (3.51) at } t = t_0\}. \end{aligned} \tag{3.55}$$

(For more details on these terms, see Section 3.3.1.) We will mostly work with the associated quadratic form

$$\mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(q_0) := \mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(q_0, q_0) \qquad q_0 \in W_k. \tag{3.56}$$

For notational convenience we will sometimes drop the subscript zero for the functions in W_k; it will be clear from the context whether q denotes a root function or a fixed vector in V. In the case that $k = 1$, we will drop the superscript and write $\mathfrak{m}_{t_0}(\Lambda, V)$. Following [RS93], a crossing $t = t_0$ will be called *regular* if $\mathfrak{m}_{t_0}$ is nondegenerate; otherwise, $t = t_0$ will be called *non-regular*. Denoting by $n_+(B)$ and $n_-(B)$ the number of positive, respectively negative, squares of the quadratic form B, we define the sequence of *partial signatures* of (3.54) (as in (3.45)):

$$n_-(\mathfrak{m}_{t_0}^{(k)}), \quad n_+(\mathfrak{m}_{t_0}^{(k)}), \quad \operatorname{sign}(\mathfrak{m}_{t_0}^{(k)}) = n_+(\mathfrak{m}_{t_0}^{(k)}) - n_-(\mathfrak{m}_{t_0}^{(k)}).$$

It is proven in [GPP04b, Lemma 3.10] that these integers are independent of the choice of W and are therefore well-defined. The Maslov index of the Lagrangian path Λ is then given as follows, as in [GPP04b, Proposition 3.11].

Definition 3.8. *Suppose $\Lambda : [a, b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces, whose intersections with $\mathcal{T}(V)$ are isolated. Its* Maslov index *is given by*

$$\begin{aligned} \operatorname{Mas}(\Lambda, V; [a, b]) = -\sum_{k\geq 1} n_-\left(\mathfrak{m}_a^{(k)}\right) + \sum_{t_0 \in (a,b)} \left(\sum_{k\geq 1} \operatorname{sign}\left(\mathfrak{m}_{t_0}^{(2k-1)}\right) \right) \\ + \sum_{k\geq 1} \left(n_+\left(\mathfrak{m}_b^{(2k-1)}\right) + n_-\left(\mathfrak{m}_b^{(2k)}\right) \right), \end{aligned} \tag{3.57}$$

where the right hand side has a finite number of nonzero terms.

Notice that at all interior crossings $t_0 \in (a, b)$, only the signatures of the crossing forms of odd order contribute; at the initial point the negative indices of crossing forms of all order contribute; while at the final point, the negative indices of the forms of even order and the positive indices of the forms of odd order contribute. From (3.46), we have that

$$\sum_{k\geq 1} \left(n_+(\mathfrak{m}_{t_0}^{(k)}) + n_-(\mathfrak{m}_{t_0}^{(k)}) \right) = \dim \Lambda(t_0) \cap V, \tag{3.58}$$

so that by taking sufficiently many higher order crossing forms, a crossing t_0 will always contribute $\dim \Lambda(t_0) \cap V$ summands (the signs of which may offset each other) to the Maslov index.

We point out that Definition 3.8 includes, as a special case, the definition given by Robbin and Salamon [RS93] in the case that all crossings are regular. To see this, we compute $\mathfrak{m}_{t_0}$ from

(3.54):

$$\mathfrak{m}_{t_0}(\Lambda, V)(q_0) = \frac{d}{dt}\omega\left(R(t)q_0, q_0\right)\big|_{t=t_0}, \qquad q_0 \in \Lambda(t_0) \cap V, \tag{3.59}$$

where we used the symmetry of $JR(t_0)$, and (3.43), (3.53) to obtain $W_1 = \Lambda(t_0) \cap V$. If $\mathfrak{m}_{t_0}$ is nondegenerate, it follows from (3.38) that $W_2 = \{0\}$ and therefore $W_k = \{0\}$ for $k \geq 3$. Thus the forms $\mathfrak{m}_{t_0}^{(k)}$ are trivial for $k \geq 2$, and from (3.58) we have $n_+(\mathfrak{m}_{t_0}) + n_-(\mathfrak{m}_{t_0}) = \dim \Lambda(t_0) \cap V$. Thus, the Maslov index of a path $\Lambda : [a, b] \to \mathcal{L}(n)$ with only regular crossings is given by

$$\operatorname{Mas}(\Lambda, V; [a, b]) = -n_-\left(\mathfrak{m}_a\right) + \sum_{t_0 \in (a,b)} \operatorname{sign}\left(\mathfrak{m}_{t_0}\right) + n_+\left(\mathfrak{m}_b\right), \tag{3.60}$$

just as in [RS93, §2].

Two special cases will be important in our analysis. The first is the instance of a non-regular crossing $t_0 = a$ at the initial point of the path $\Lambda : [a, b] \to \mathcal{L}(n)$, for which the first-order crossing form is identically zero and the second-order crossing form is nondegenerate. Then

$$n_+(\mathfrak{m}_a^{(2)}) + n_-(\mathfrak{m}_a^{(2)}) = \dim \Lambda(t_0) \cap V, \tag{3.61}$$

and from Definition 3.8 we see that, for $\varepsilon > 0$ small enough,

$$\operatorname{Mas}(\Lambda, V; [a, a + \varepsilon]) = -n_-(\mathfrak{m}_a^{(2)}), \tag{3.62}$$

just as in [CCLM23, Proposition 4.15] and [DJ11, Proposition 3.10]. Note that in this case, we have $W_2 = (\ker R(t_0)) \cap V = \ker\left(\dot{R}(t_0)\right) \cap V = \Lambda(t_0) \cap V$, and the second-order crossing form (3.54) is given by (where dot denotes d/dt)

$$\begin{aligned}
\mathfrak{m}_{t_0}^{(2)}(\Lambda, V)(q_0) &= \frac{d^2}{dt^2}\,\omega\big(R(t)q(t), q_0\big)\Big|_{t=t_0}, \\
&= \omega\big(\ddot{R}(t)q_0, q_0\big) + \omega\big(\dot{R}(t_0)\dot{q}(t_0), q_0\big) + \omega\big(R(t_0)\ddot{q}(t_0), q_0\big), \\
&= \omega\big(\ddot{R}(t)q_0, q_0\big),
\end{aligned} \tag{3.63}$$

for $q_0 \in \Lambda(t_0) \cap V$, where we used the symmetry of $JR(t_0)$ and $J\dot{R}(t_0)$.

The second special case is the instance of a non-regular interior crossing $t_0 \in (a, b)$ for which the first-order form is degenerate with nonzero rank, the second-order form is identically zero, and the third-order form is nondegenerate. Then

$$n_+(\mathfrak{m}_{t_0}^{(1)}) + n_-(\mathfrak{m}_{t_0}^{(1)}) + n_+(\mathfrak{m}_{t_0}^{(3)}) + n_-(\mathfrak{m}_{t_0}^{(3)}) = \dim \Lambda(t_0) \cap V, \tag{3.64}$$

and if t_0 is the only crossing in $[t_0 - \varepsilon, t_0 + \varepsilon]$, its contribution to the Maslov index is

$$\operatorname{Mas}(\Lambda, V; [t_0 - \varepsilon, t_0 + \varepsilon]) = \operatorname{sign} \mathfrak{m}_{t_0}^{(1)} + \operatorname{sign} \mathfrak{m}_{t_0}^{(3)}. \tag{3.65}$$

We summarise the important properties of the Maslov index for the current analysis in the following proposition, as in [GPP04b, Lemma 3.8] (see also [RS93, Theorem 2.3]).

Proposition 3.9. *The Maslov index enjoys*

1. (*Homotopy invariance.*) *If two paths* $\Lambda_1, \Lambda_2 : [a,b] \longrightarrow \mathcal{L}(n)$ *are homotopic with fixed endpoints, then*
$$\operatorname{Mas}(\Lambda_1(t), V; [a,b]) = \operatorname{Mas}(\Lambda_2(t), \Lambda_0; [a,b]). \tag{3.66}$$

2. (*Additivity under concatenation.*) *For* $\Lambda(t) : [a,c] \longrightarrow \mathcal{L}(n)$ *and* $a < b < c$,
$$\operatorname{Mas}(\Lambda(t), V; [a,c]) = \operatorname{Mas}(\Lambda(t), V; [a,b]) + \operatorname{Mas}(\Lambda(t), V; [b,c]). \tag{3.67}$$

3. (*Symplectic additivity.*) *Identify the Cartesian product* $\mathcal{L}(n) \times \mathcal{L}(n)$ *as a submanifold of* $\mathcal{L}(2n)$. *If* $\Lambda = \Lambda_1 \oplus \Lambda_2 : [a,b] \to \mathcal{L}(2n)$ *where* $\Lambda_1, \Lambda_2 : [a,b] \to \mathcal{L}(n)$, *and* $V = V_1 \oplus V_2$ *where* $V_1, V_2 \in \mathcal{L}(n)$, *then*
$$\operatorname{Mas}(\Lambda(t), V; [a,b]) = \operatorname{Mas}(\Lambda_1(t), V_1; [a,b]) + \operatorname{Mas}(\Lambda_2(t), V_2; [a,b]). \tag{3.68}$$

4. (*Zero property.*) *If* $\Lambda : [a,b] \longrightarrow \mathcal{T}_k(V)$ *for any fixed integer* k, *then*
$$\operatorname{Mas}(\Lambda(t), V; [a,b]) = 0. \tag{3.69}$$

Suppose now that we have a pair of Lagrangian paths $(\Lambda_1, \Lambda_2) : [a,b] \to \mathcal{L}(n) \times \mathcal{L}(n)$, or a *Lagrangian pair*. Using the symplectic additivity property of Proposition 3.9, it is possible to define the Maslov index of such an object (as in [GPP04b, RS93, Fur04]), where crossings are values $t_0 \in [a,b]$ such that $\Lambda_1(t_0) \cap \Lambda_2(t_0) \neq \{0\}$. Precisely, one realises the Lagrangian pair as the path $\Lambda_1 \oplus \Lambda_2$ in the doubled space $\mathbb{R}^{4n}$ equipped with the symplectic form $\Omega = \omega \times (-\omega)$, where
$$\Omega((u_1,u_2),(v_1,v_2)) = \omega(u_1,v_1) - \omega(u_2,v_2), \qquad u_1,u_2,v_1,v_2 \in \mathbb{R}^{2n}. \tag{3.70}$$
Crossings of the pair then correspond to intersections of the path $\Lambda_1 \oplus \Lambda_2 : [a,b] \to \mathbb{R}^{4n}$ with the diagonal subspace $\triangle = \{(x,x) : x \in \mathbb{R}^{2n}\} \subset \mathbb{R}^{4n}$. The resultant Maslov index,
$$\operatorname{Mas}(\Lambda_1, \Lambda_2; [a,b]) \coloneqq \operatorname{Mas}(\Lambda_1 \oplus \Lambda_2, \triangle; [a,b]), \tag{3.71}$$
is thus a signed count of the intersections of Λ_1 and Λ_2 which, loosely speaking, measures the winding of Λ_1 *relative to* Λ_2.

The right hand side of (3.71) is computed with Definition 3.8. To that end, using Ω as the symplectic form in (3.54) for the path $\Lambda_1 \oplus \Lambda_2$, we define the *kth-order relative crossing form* of the Lagrangian pair (Λ_1, Λ_2) to be the quadratic form
$$\mathfrak{m}_{t_0}^{(k)}(\Lambda_1,\Lambda_2)(q) \coloneqq \mathfrak{m}_{t_0}^{(k)}(\Lambda_1,\Lambda_2(t_0))(q) - \mathfrak{m}_{t_0}^{(k)}(\Lambda_2,\Lambda_1(t_0))(q), \quad q \in W^k, \tag{3.72}$$
where $W_k \subseteq \Lambda_1(t_0) \cap \Lambda_2(t_0)$. Using these forms in Definition 3.8 thus allows us to compute the Maslov index of the pair $(\Lambda_1, \Lambda_2) : [a,b] \to \mathcal{L}(n) \times \mathcal{L}(n)$. In the case that $\Lambda_2 = V$ is constant, the Maslov index of the pair reduces to the Maslov index of the single path Λ_1, with respect to the reference plane V.

The Maslov index is invariant for Lagrangian pairs that are *stratum homotopic*. This result will be needed in our analysis, and we give a proof below. The result for single paths can be found in[RS93, Theorem 2.4]. Suppose the pairs $(\Lambda_1, \Lambda_2) : [a,b] \to \mathcal{L}(n)$ and $(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) : [a,b] \to \mathcal{L}(n)$

are stratum homotopic, i.e. there exist continuous mappings $H_1, H_2 : [0,1] \times [a,b] \to \mathcal{L}(n)$ such that

$$H_1(0,\cdot) = \Lambda_1(\cdot), \qquad H_2(0,\cdot) = \Lambda_2(\cdot)$$
$$H_1(1,\cdot) = \widetilde{\Lambda}_1(\cdot), \qquad H_2(1,\cdot) = \widetilde{\Lambda}_2(\cdot),$$

for which $\dim(H_1(s,a) \cap H_2(s,a))$ and $\dim(H_1(s,b) \cap H_2(s,b))$ are constant with respect to $s \in [0,1]$. (The name "stratum homotopy" derives from the fact that

$$H_1(s,a) \oplus H_2(s,a) \in \mathcal{T}_{k_1}(\triangle), \qquad H_1(s,b) \oplus H_2(s,b) \in \mathcal{T}_{k_2}(\triangle),$$

for all $s \in [0,1]$ and fixed integers k_1, k_2.) Then we have:

Lemma 3.10.

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a,b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a,b]). \tag{3.73}$$

Proof. Consider the continuous mapping $H = H_1 \oplus H_2 : [0,1] \times [a,b] \to \mathcal{L}(n) \times \mathcal{L}(n)$. By continuity of H and homotopy invariance (i.e. property (3) of Proposition 3.9), we have

$$\mathrm{Mas}(H(0,\cdot), \triangle; [a,b]) + \mathrm{Mas}(H(\cdot,b), \triangle; [0,1]) \\ - \mathrm{Mas}(H(1,\cdot), \triangle; [a,b]) - \mathrm{Mas}(H(\cdot,a), \triangle; [0,1]) = 0. \tag{3.74}$$

Using (3.71) we have

$$\mathrm{Mas}(H(0,\cdot), \triangle; [a,b]) = \mathrm{Mas}(\Lambda_1, \Lambda_2; [a,b]), \quad \mathrm{Mas}(H(1,\cdot), \triangle; [a,b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a,b]).$$

By assumption $\dim(H(\cdot,a) \cap \triangle) = \dim(H_1(\cdot,a) \cap H_2(\cdot,a))$ and $\dim(H(\cdot,b) \cap \triangle) = \dim(H_1(\cdot,b) \cap H_2(\cdot,b))$ are constant, so by property (4) of Proposition 3.9 the Maslov indices of the second and fourth terms in (3.74) are zero. Equation (3.73) follows. ☐

For a Lagrangian pair, when the first-order form $\mathfrak{m}_{t_0}(\Lambda_1, \Lambda_2)$ of (3.72) at $t = a$ is identically zero, and the second order form $\mathfrak{m}_a^{(2)}(\Lambda_1, \Lambda_2)$ is nondegenerate, equation (3.62) becomes

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a, a+\varepsilon]) = -n_-(\mathfrak{m}_a^{(2)}(\Lambda_1, \Lambda_2)). \tag{3.75}$$

This formula will be needed in our application to the eigenvalue problem (3.12). In particular, the crossing corresponding to the zero eigenvalue of the operator N is not regular in the λ direction, and the conditions for (3.75) are met under the assumption that $\mathcal{I}_1, \mathcal{I}_2 \neq 0$.

We will call a crossing $t = t_0$ *positive* if

$$\sum_{k \geq 1} \left(n_+(\mathfrak{m}_{t_0}^{(2k-1)}) \right) = \dim \Lambda(t_0) \cap V, \tag{3.76}$$

and *negative* if

$$\sum_{k \geq 1} \left(n_-(\mathfrak{m}_{t_0}^{(2k-1)}) \right) = \dim \Lambda(t_0) \cap V. \tag{3.77}$$

In light of Definition 3.8, if t_0 is a positive interior crossing, or a positive crossing at the final point $t_0 = b$, then it contributes $\dim \Lambda(t_0) \cap V$ to the Maslov index. Similarly, if t_0 is a negative interior crossing, or a negative crossing at the initial point $t_0 = a$, then its contribution is $-\dim \Lambda(t_0) \cap V$. Note, however, that with this convention, the final crossing $t_0 = b$ may still contribute $\dim \Lambda(b) \cap V$ if it is not positive, and the initial point $t_0 = a$ may still contribute $-\dim \Lambda(a) \cap V$ if it is not negative.

3.3.3 Lagrangian pairs and the Maslov box

We first discuss the regularity and Lagrangian property of the stable and unstable bundles. Recall $\mathbb{E}^s(x,\lambda)$ and $\mathbb{E}^u(x,\lambda)$ defined in (3.32) for $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. We extend $\mathbb{E}^s$ to $x = +\infty$ and $\mathbb{E}^u$ to $x = -\infty$ by setting

$$\mathbb{E}^s(+\infty,\lambda) := \mathbb{S}(\lambda), \qquad \mathbb{E}^u(-\infty,\lambda) := \mathbb{U}(\lambda). \tag{3.78}$$

Thus by (3.33), $\mathbb{E}^s$ and $\mathbb{E}^u$ are continuous on $(-\infty,\infty] \times \mathbb{R}$ and $[-\infty,\infty) \times \mathbb{R}$ respectively. Furthermore, since the right hand side of (3.22) is analytic in λ and x, it follows that the solution spaces $\mathbb{E}^s$ and $\mathbb{E}^u$ are analytic on $(x,\lambda) \in \mathbb{R} \times \mathbb{R}$ (note that $x = \pm\infty$ is excluded). We remark here that the mapping

$$\lambda \mapsto \lim_{x\to\infty} \mathbb{E}^u(x;\lambda) \tag{3.79}$$

is discontinuous at eigenvalues $\lambda \in \mathrm{Spec}(N)$. Indeed, if $\lambda \notin \mathrm{Spec}(N)$, then $\lim_{x\to\infty} \mathbb{E}^u(x;\lambda) = \mathbb{U}(\lambda)$ (again as points on the Grassmannian $\mathrm{Gr}_4(\mathbb{R}^8)$), while if $\lambda \in \mathrm{Spec}(N)$ is an eigenvalue then $\lim_{x\to\infty} \mathbb{E}^u(x;\lambda) \cap \mathbb{S}(\lambda) \neq \{0\}$. Now since $\mathbb{U}(\lambda) \cap \mathbb{S}(\lambda) = \{0\}$ i.e. $\mathbb{U}(\lambda) \in \mathcal{T}_0(\mathbb{S}(\lambda))$, and $\mathcal{T}_0(\mathbb{S}(\lambda))$ is an open subset of $\mathcal{L}(n)$ with boundary $\mathcal{T}(\mathbb{S}(\lambda))$, it follows that $\mathbb{U}(\lambda)$ is bounded away from $\mathcal{T}(\mathbb{S}(\lambda))$. For more details see the Appendix in [HLS18].

Remark 3.11. The Maslov index is defined for Lagrangian paths over compact intervals. Following [HLS18] we will sometimes compactify $\mathbb{R}$ via the change of variables

$$x = \ln\left(\frac{1+\tau}{1-\tau}\right), \qquad \tau \in [-1,1]. \tag{3.80}$$

(Similar transformations are used in [BCJ+18, AGJ90].) Notationally we will use a hat to indicate such a change has been made, for example,

$$\widehat{\mathbb{E}}^{s,u}(\tau,\cdot) := \mathbb{E}^{s,u}\left(\ln\left(\frac{1+\tau}{1-\tau}\right),\cdot\right), \quad \tau \in [-1,1]. \tag{3.81}$$

In this case, (3.78) implies that $\widehat{\mathbb{E}}^u(-1,\lambda) = \mathbb{U}(\lambda)$ and $\widehat{\mathbb{E}}^s(1,\lambda) = \mathbb{S}(\lambda)$.

Lemma 3.12. *The spaces $\mathbb{E}^u(x;\lambda)$ and $\mathbb{E}^s(x;\lambda)$ are Lagrangian subspaces of $\mathbb{R}^8$ for all $x \in [-\infty,\infty]$ and $\lambda \in \mathbb{R}$.*

Proof. First, recall that $\dim \mathbb{U}(\lambda) = \dim \mathbb{S}(\lambda) = 4$ (we showed in (3.78) that $A_\infty(\lambda)$ is hyperbolic with four eigenvalues of positive real part and four of negative real part.) It follows from the continuity of $\mathbb{E}^u$ on $[-\infty,\infty) \times \mathbb{R}$ that $\dim \mathbb{E}^u(x,\lambda) = 4$ for all $(x,\lambda) \in [-\infty,\infty) \times \mathbb{R}$. A similar argument shows $\dim \mathbb{E}^s(x,\lambda) = 4$ for $(x,\lambda) \in (-\infty,\infty] \times \mathbb{R}$.

Next, for $x \in \mathbb{R}$, let $\mathbf{w}_1(x;\lambda), \mathbf{w}_2(x;\lambda) \in \mathbb{E}^u(x;\lambda)$. We have:

$$\begin{aligned}
\omega(\mathbf{w}_1(x;\lambda), \mathbf{w}_2(x;\lambda)) &= \langle J\mathbf{w}_1(x;\lambda), \mathbf{w}_2(x;\lambda)\rangle, \\
&= \int_{-\infty}^{x} \frac{d}{ds}\langle J\mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\rangle ds, \\
&= \int_{-\infty}^{x} \langle JA(s;\lambda)\mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\rangle + \langle J\mathbf{w}_1(s;\lambda), A(s;\lambda)\mathbf{w}_2(s;\lambda)\rangle ds, \\
&= \int_{-\infty}^{x} \left\langle \left(A(s;\lambda)^\top J + JA(s;\lambda)\right) \mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\right\rangle ds, \\
&= 0,
\end{aligned}$$

where we used (3.34), i.e. that $A(x;\lambda)$ is infinitesimally symplectic. The proof for $\mathbb{E}^s(x;\lambda)$ is similar, but the integral is taken over $[x,\infty)$. We have shown that $\mathbb{E}^u$ and $\mathbb{E}^s$ are Lagrangian on $\mathbb{R}\times\mathbb{R}$. That this property extends to $x = \pm\infty$ follows the closedness of $\mathcal{L}(n)$ as a submanifold of the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$. (Note this latter property follows from the continuity of the symplectic form ω.) ☐

We are now ready to give the homotopy argument that leads to the lower bound of Theorem 3.2. Consider the following path of Lagrangian pairs

$$\Gamma \ni (x,\lambda) \mapsto (\mathbb{E}^u(x,\lambda), \mathbb{E}^s(\ell,\lambda)) \in \mathcal{L}(4) \times \mathcal{L}(4), \tag{3.82}$$

where $\ell \gg 1$ needs to be chosen large enough so that

$$\mathbb{U}(\lambda) \cap \mathbb{E}^s(x,\lambda) = \{0\} \quad \text{for all} \quad x \geq \ell \tag{3.83}$$

(see Remark 3.13). Here $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where the Γ_i are the contours

$$\begin{aligned}
&\Gamma_1 : x \in [-\infty,\ell], \quad \lambda = 0, \quad \Gamma_3 : x \in [-\infty,\ell], \quad \lambda = \lambda_\infty, \\
&\Gamma_2 : x = \ell, \quad \lambda \in [0,\lambda_\infty], \quad\ \ \Gamma_4 : x = -\infty, \quad \lambda = \lambda \in [0,\lambda_\infty],
\end{aligned} \tag{3.84}$$

in the λx-plane (see Fig. 3.1). The set Γ has been referred to by some as the *Maslov box* [HLS18, Cor19], although the associated homotopy argument (outlined below) can be seen in as far back as the works of Bott [Bot56], Edwards [Edw64], Arnol'd [Arn67] and Duistermaat [Dui76]. Notice that along Γ_1 and Γ_3, the second entry $\mathbb{E}^s(\ell,\lambda)$ of (3.82) is fixed. The Maslov index of (3.82) along these pieces thus reduces to the Maslov index for a single path with respect to a fixed reference plane. Along Γ_2 and Γ_4, however, we have a genuine Lagrangian pair.

Crossings of (3.82) are thus points $(x,\lambda) \in \Gamma$ such that

$$\mathbb{E}^u(x,\lambda) \cap \mathbb{E}^s(\ell,\lambda) \neq \{0\}.$$

Recalling that λ is an eigenvalue of N if and only if $\mathbb{E}^u(x,\lambda) \cap \mathbb{E}^s(x,\lambda) \neq \{0\}$ for all $x \in \mathbb{R}$, it follows that the λ-values of the crossings along Γ_2 are exactly the eigenvalues of N. In particular, because $0 \in \operatorname{Spec}(N)$ there will be a crossing at $(x,\lambda) = (0,\ell)$. From Hypothesis 3.1 we have

$\ker(L_-) = \text{span}\{\phi\}$ and $\ker(L_+) = \text{span}\{\phi\}$. The corresponding solutions of (3.22),

$$\boldsymbol{\phi}(x) := \begin{pmatrix} 0 \\ \phi''(x) + \sigma_2\phi(x) \\ 0 \\ -\phi(x) \\ 0 \\ -\phi'(x) \\ 0 \\ \phi'''(x) \end{pmatrix}, \qquad \boldsymbol{\varphi}(x) := \begin{pmatrix} \phi'''(x) + \sigma_2\phi'(x) \\ 0 \\ \phi'(x) \\ 0 \\ \phi''(x) \\ 0 \\ \phi''''(x) \\ 0 \end{pmatrix}, \tag{3.85}$$

(obtained from (3.21) with $v = \phi$ and $u = \phi'$ respectively) will therefore satisfy $\boldsymbol{\phi}(x), \boldsymbol{\varphi}(x) \in \mathbb{E}^u(x; 0) \cap \mathbb{E}^s(x; 0)$ for all $x \in \mathbb{R}$.

Remark 3.13. That the path (3.79) is discontinuous in λ prohibits taking Γ_2 to be at $x = +\infty$. Taking Γ_2 to be at $x = \ell$ for ℓ large enough avoids this issue. Chen and Hu [CH07] showed that by taking ℓ large enough so that (3.83) holds, the Maslov index of (3.82) along Γ_1 is independent of the choice of ℓ. For more details, see [CH07, Cor19].

Crossings along Γ_1, i.e. points $(x, \lambda) = (x_0, 0)$ such that

$$\mathbb{E}^u(x_0, 0) \cap \mathbb{E}^s(\ell, 0) \neq \{0\}, \tag{3.86}$$

are called *conjugate points*. Recall that when $\lambda = 0$ the eigenvalue equations (3.12) decouple into two independent equations for the operators L_+ and L_-. Similarly, when $\lambda = 0$ the first order system (3.22) decouples into two independent systems for the u and v variables. In Section 3.4 the eigenvalue problems for the operators L_+ and L_- will be written as first order systems; the stable and unstable bundles for the L_+ system will be denoted by $\mathbb{E}^s_+(x, \lambda)$ and $\mathbb{E}^u_+(x, \lambda)$, respectively, while the stable and unstable bundles for the L_- system will be denoted by $\mathbb{E}^{s,u}_-(x, \lambda)$. For the system (3.22), as a result of the decoupling at $\lambda = 0$ we have

$$\mathbb{E}^u(x, 0) = \mathbb{E}^u_+(x, 0) \oplus \mathbb{E}^u_-(x, 0) \quad \text{and} \quad \mathbb{E}^s(x, 0) = \mathbb{E}^s_+(x, 0) \oplus \mathbb{E}^s_-(x, 0), \tag{3.87}$$

so that

$$\begin{aligned} \{x \in \mathbb{R} : \mathbb{E}^u(x, 0) \cap \mathbb{E}^s(\ell, 0) \neq \{0\}\} = \\ \{x \in \mathbb{R} : \mathbb{E}^u_+(x, 0) \cap \mathbb{E}^s_+(\ell, 0) \neq \{0\}\} \cup \{x \in \mathbb{R} : \mathbb{E}^u_-(x, 0) \cap \mathbb{E}^s_-(\ell, 0) \neq \{0\}\}. \end{aligned} \tag{3.88}$$

The precise notion of the direct sums in (3.87) will be given in Section 3.5. When dealing with conjugate points, we will show in Section 3.4 that it suffices to use the stable subspace $\mathbb{S}(0)$ (instead of $\mathbb{E}^s(\ell, 0)$) as the reference plane to do computations. That $\mathbb{S}(0) = \mathbb{S}_+(0) \oplus \mathbb{S}_-(0)$, where $\mathbb{S}_\pm(0)$ is the stable subspace of the asymptotic first order system for the eigenvalue problem for $L_\pm$, leads to the following classification of conjugate points.

Definition 3.14. *An* L_+ conjugate point *is a point* $(x, \lambda) = (x_0, 0)$ *such that* $\mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) \neq \{0\}$. *An* L_- conjugate point *is similarly defined via* $\mathbb{E}^u_-(x_0, 0) \cap \mathbb{S}_-(0) \neq \{0\}$.

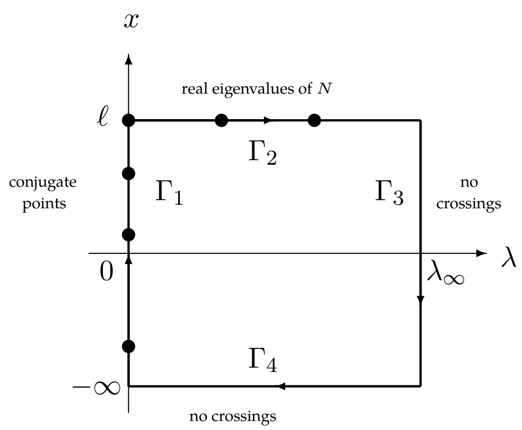


Figure 3.1: Maslov box in the λx-plane, with edges oriented in a clockwise fashion. The crossing at the top left corner $(0, \ell)$ corresponds to the zero eigenvalue of N. Noting that $\lambda \in \mathbb{R}$ is a spectral parameter, and therefore lives on the real axis in $\mathbb{C}$, it is natural to place λ on the horizontal axis.

Since the solid rectangle $[-\infty, \ell] \times [0, \lambda_\infty]$ is contractible and the map (3.82) is continuous, the image of the boundary of the rectangle in $\mathcal{L}(4) \times \mathcal{L}(4)$ is homotopic to a fixed point. From homotopy invariance (Proposition 3.9), it follows that

$$\mathrm{Mas}(\mathbb{E}^u(\cdot,\cdot), \mathbb{E}^s(\cdot,\cdot); \Gamma) = 0. \tag{3.89}$$

By additivity under concatenation, we can decompose the left hand side into the contributions coming from the constituent sides of the Maslov box, i.e.

$$\begin{aligned} &\mathrm{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty,\ell]) + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\lambda_\infty]) \\ &\quad - \mathrm{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty), \mathbb{E}^s(\ell,\lambda_\infty); [-\infty,\ell]) - \mathrm{Mas}(\mathbb{E}^u(-\infty,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\lambda_\infty]) = 0. \end{aligned} \tag{3.90}$$

Note we have included minus signs for the last two terms in order to be consistent with the clockwise orientation of the Maslov box (see Fig. 3.1). We will show in Section 3.5 that in fact these last two Maslov indices are zero. A distinguished quantity will be the contribution to the Maslov index of the conjugate point $(x, \lambda) = (\ell, 0)$ at the top left corner of the Maslov box,

$$\mathfrak{c} := \mathrm{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [\ell-\varepsilon,\ell]) + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\varepsilon]), \tag{3.91}$$

where $\varepsilon > 0$ is small. This is because the crossing $(\ell, 0)$ is non-regular in λ, and hence higher order crossing forms are needed to compute the second term in (3.91). It follows once more from additivity under concatenation that

$$\mathrm{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty,\ell-\varepsilon]) + \mathfrak{c} + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [\varepsilon,\lambda_\infty]) = 0. \tag{3.92}$$

We will compute the first term of (3.92) by counting L_+ and L_- conjugate points. By bounding the third term, computing $\mathfrak{c}$ and rearranging, we will arrive at the statement of Theorem 3.2. Before doing so, we turn our attention to the computation of the Morse indices of L_+ and L_- via the Maslov index.

3.4 Spectral counts for the operators L_+ and L_-

In this section we focus on the spectral problems for the operators L_+ and L_-. Specifically, for each operator we prove that the Morse index is equal to the number of conjugate points on $\mathbb{R}$. Proposition 3.15 is proven under two genericity conditions which will be formulated later on.

Proposition 3.15. *Assume Hypotheses 3.16 and 3.17. The number of positive eigenvalues of L_+ is equal to the number of L_+-conjugate points on $\mathbb{R}$ (up to multiplicity),*

$$P = \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0)\right). \tag{3.93}$$

A similar assertion holds for L_-.

We will prove the proposition in a series of lemmas, focusing on the L_+ operator; the spectral count for L_- follows similarly with minor adjustments. Many of the ideas here have already been discussed in §3, and so in the interest of expediency we present only the main arguments. In what follows, we use a subscript $+$ or $-$ to indicate that objects pertain to the eigenvalue problem for L_+ or L_-.

The eigenvalue equation for L_+,

$$-u'''' - \sigma_2 u'' - \beta u + 3\phi^2 u = \lambda u, \qquad u \in H^4(\mathbb{R}), \tag{3.94}$$

can be reduced to the following first order system via the u substitutions in (3.21),

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & \sigma_2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -\sigma_2 & 0 & 0 \\ -\sigma_2 & \alpha(x)-\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{3.95}$$

where $\alpha(x) = 3\phi(x)^2 - \beta + 1$. Similar to (3.22), we write this system as

$$\mathbf{u}_x = A_+(x,\lambda)\mathbf{u}, \tag{3.96}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^\top$ and

$$A_+(x,\lambda) = \begin{pmatrix} 0 & B_+ \\ C_+(x,\lambda) & 0 \end{pmatrix}, \quad B_+ = \begin{pmatrix} \sigma_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_+(x,\lambda) = \begin{pmatrix} 1 & -\sigma_2 \\ -\sigma_2 & \alpha(x)-\lambda \end{pmatrix}.$$

Likewise, the eigenvalue equation for L_-,

$$-v'''' - \sigma_2 v'' - \beta v + \phi^2 v = \lambda v, \qquad v \in H^4(\mathbb{R}), \tag{3.97}$$

can be reduced to the following first order system via the v substitutions in (3.21),

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & -\sigma_2 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -\sigma_2 & 0 & 0 \\ -\sigma_2 & \eta(x)+\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{3.98}$$

where $\eta(x) = -\phi(x)^2 + \beta - 1$. We write this as

$$\mathbf{v}_x = A_-(x,\lambda)\mathbf{v}, \tag{3.99}$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)^\top$ and

$$A_-(x,\lambda) = \begin{pmatrix} 0 & B_- \\ C_-(x,\lambda) & 0 \end{pmatrix}, \quad B_- = \begin{pmatrix} -\sigma_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_-(x,\lambda) = \begin{pmatrix} -1 & -\sigma_2 \\ -\sigma_2 & \eta(x)+\lambda \end{pmatrix}.$$

The coefficient matrices $A_\pm(x,\lambda)$ are infinitesimally symplectic, satisfying equation (3.34). In order to be consistent with (3.22) at $\lambda = 0$, we have used the same substitutions (3.21) to reduce (3.94) and (3.97) to (3.95) and (3.98) respectively. Notice that λ appears with a different sign in (3.95) and (3.98), due to the substitutions for u_2 and u_3 in (3.21) having different signs to the corresponding substitutions for v_2 and v_3. This will be the reason for the difference in sign of the Maslov indices in Lemma 3.18.

The asymptotic matrices $A_+(\lambda) := \lim_{x\to\pm\infty} A_+(x,\lambda)$ and $A_-(\lambda) := \lim_{x\to\pm\infty} A_-(x,\lambda)$ each have two eigenvalues with negative real part and two with positive real part. We denote the associated stable and unstable subspaces by $\mathbb{S}_\pm(\lambda)$ and $\mathbb{U}_\pm(\lambda)$. Reasoning as in Section 3.3.3, associated with each of the systems (3.95) and (3.98) are stable and unstable bundles,

$$\begin{aligned} \mathbb{E}^u_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (3.95) and } \mathbf{u}(x;\lambda) \to 0 \text{ as } x \to -\infty\}, \\ \mathbb{E}^s_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (3.95) and } \mathbf{u}(x;\lambda) \to 0 \text{ as } x \to +\infty\}, \\ \mathbb{E}^u_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (3.98) and } \mathbf{v}(x;\lambda) \to 0 \text{ as } x \to -\infty\}, \\ \mathbb{E}^s_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (3.98) and } \mathbf{v}(x;\lambda) \to 0 \text{ as } x \to +\infty\}, \end{aligned} \tag{3.100}$$

which, when considered as points on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$, converge to the stable and unstable subspaces at $\pm\infty$ as follows,

$$\lim_{x\to-\infty} \mathbb{E}^u_+(x,\lambda) = \mathbb{U}_+(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s_+(x,\lambda) = \mathbb{S}_+(\lambda),$$
$$\lim_{x\to-\infty} \mathbb{E}^u_-(x,\lambda) = \mathbb{U}_-(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s_-(x,\lambda) = \mathbb{S}_-(\lambda).$$

That $\mathbb{E}^u_+(x,\lambda), \mathbb{E}^u_-(x,\lambda), \mathbb{E}^s_+(x,\lambda), \mathbb{E}^s_-(x,\lambda)$ are Lagrangian subspaces of $\mathbb{R}^4$, with the mappings $(x,\lambda) \mapsto \mathbb{E}^u_\pm(x,\lambda)$ being continuous on $[-\infty,\infty) \times \mathbb{R}$ and $(x,\lambda) \mapsto \mathbb{E}^{u,s}_\pm(x,\lambda)$ analytic on $\mathbb{R} \times \mathbb{R}$, follows from the same arguments as in Section 3.3.3. We omit the proofs.

In order to show Proposition 3.15, we need to write down frames for $\mathbb{S}_\pm(0)$ that we can do computations with. To that end, first note that the asymptotic matrices $A_\pm(0)$ satisfy $\mathrm{Spec}(A_+(0)) =$

$\text{Spec}(A_-(0)) = \{\pm\mu_1, \pm\mu_2\}$, where

$$\mu_1 = \frac{\sqrt{-\sigma_2 - \sqrt{1-4\beta}}}{\sqrt{2}}, \qquad \mu_2 = \frac{\sqrt{-\sigma_2 + \sqrt{1-4\beta}}}{\sqrt{2}}. \tag{3.101}$$

Under the assumption (3.8), we have $\mu_2 = \bar{\mu}_1$ whenever $\beta \geq 1/4$ (for both $\sigma_2 = 1$ and $\sigma_2 = -1$), and $\mu_1, \mu_2 \in \mathbb{R}$ when $\sigma_2 = -1$ and $0 < |\beta| \leq 1/4$. The corresponding eigenvectors are given by

$$\mathbf{u}_1 = \begin{pmatrix} \mu_2^2 \\ -1 \\ \mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} \mu_1^2 \\ -1 \\ \mu_2 \\ \mu_2^3 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_1 = \begin{pmatrix} \mu_2^2 \\ 1 \\ -\mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} \mu_1^2 \\ 1 \\ -\mu_2 \\ \mu_2^3 \end{pmatrix}, \tag{3.102}$$

where $\ker(A_+(0) + \mu_i) = \text{span}\{\mathbf{u}_i\}$ and $\ker(A_-(0) + \mu_i) = \text{span}\{\mathbf{v}_i\}$, $i = 1, 2$. Notice that the vectors $\mathbf{u}_i, \mathbf{v}_i$ for $i = 1, 2$ are complex-valued if $\beta \geq 1/4$. We collect these vectors into the columns of two frames, which we denote with 2×2 blocks P_i, M_i, $i = 1, 2$, via

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} := \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ -1 & -1 \\ \mu_1 & \mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}, \qquad \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} := \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ 1 & 1 \\ -\mu_1 & -\mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}. \tag{3.103}$$

All of the matrices P_i, M_i are invertible under (3.8) and (3.9). Right multiplying each frame in (3.103) by the inverse of its upper 2×2 block yields the following *real* frame for $\mathbb{S}_\pm(0)$,

$$\mathbf{S}_\pm = \begin{pmatrix} I \\ S_\pm \end{pmatrix}, \qquad S_\pm = \frac{1}{\sqrt{2\sqrt{\beta} - \sigma_2}} \begin{pmatrix} \mp 1 & \sigma_2 - \sqrt{\beta} \\ \sigma_2 - \sqrt{\beta} & \pm(\sqrt{\beta}\sigma_2 + \beta - 1) \end{pmatrix}, \tag{3.104}$$

where $S_+ = P_2 P_1^{-1}$ and $S_- = M_2 M_1^{-1}$.

An important relation exists between $S_\pm$ and the blocks of the asymptotic matrix $A_\pm(0)$ that will be needed in our analysis. Define $C_\pm(x) := C_\pm(x, 0)$ and

$$\widehat{C}_+(x) := \begin{pmatrix} 0 & 0 \\ 0 & 3\phi(x)^2 \end{pmatrix}, \quad \widehat{C}_-(x) := \begin{pmatrix} 0 & 0 \\ 0 & -\phi(x)^2 \end{pmatrix}, \quad \widetilde{C}_\pm := \begin{pmatrix} \pm 1 & -\sigma_2 \\ -\sigma_2 & \mp(\beta - 1) \end{pmatrix}, \tag{3.105}$$

so that $C_\pm(x) = \widehat{C}_\pm(x) + \widetilde{C}_\pm$. Because the columns of the frames in (3.103) are eigenvectors of $A_\pm(0)$, we have

$$\begin{pmatrix} 0 & B_+ \\ \widetilde{C}_+ & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} D_+, \qquad D_+ = \text{diag}\{-\mu_1, -\mu_2\}, \tag{3.106}$$

with a similar equation holding for $A_-(0)$ and the frame (M_1, M_2). That is, $B_+ P_2 = P_1 D_+$ and $\widetilde{C}_+ P_1 = P_2 D_+$. It follows that

$$\widetilde{C}_+ = P_2 D_+ P_1^{-1} = \left(P_2 P_1^{-1}\right)\left(P_1 D_+ P_2^{-1}\right)\left(P_2 P_1^{-1}\right) = S_+ B_+ S_+. \tag{3.107}$$

It can be similarly shown that

$$\widetilde{C}_- = S_- B_- S_-. \tag{3.108}$$

The first intermediate result that will be used in the proof of Proposition 3.15 is Lemma 3.18, which proves sign-definiteness of the L_+ and L_- conjugate points on Γ_1. For it, we will require two genericity conditions. For details on how the first may be removed, see Remark 3.22.

Hypothesis 3.16. For any $x_0 \in \mathbb{R}$ where $\mathbb{E}^u_\pm(x_0, 0) \cap \mathbb{S}_\pm(0) \neq \{0\}$, we assume $\phi(x_0) \neq 0$.

Denote a frame for the unstable bundle $\mathbb{E}^u_\pm(x, 0)$ by

$$\mathbf{U}_\pm(x) = \begin{pmatrix} X_\pm(x) \\ Y_\pm(x) \end{pmatrix}, \qquad X_\pm(x), Y_\pm(x) \in \mathbb{R}^{2\times 2}. \tag{3.109}$$

We will assume that in the event of a one dimensional crossing on Γ_1, the intersection of the unstable bundle with the stable subspace does not perfectly align with the span of the first column of the frame $\mathbf{S}_\pm$.

Hypothesis 3.17. Suppose $\dim \mathbb{E}^u_\pm(x_0, 0) \cap \mathbb{S}_\pm(0) = 1$. Then there exist vectors $k = (a, b) \in \mathbb{R}^2$ and $h = (c, d) \in \mathbb{R}^2$ so that $\mathbf{U}_\pm(x_0)h = \mathbf{S}_\pm k$. We assume that $a \neq 0$.

Lemma 3.18. *Assume Hypotheses 3.16 and 3.17. Each crossing $x = x_0 \in \mathbb{R}$ of the Lagrangian path $x \mapsto (\mathbb{E}^u_+(x, 0), \mathbb{S}_+(0))$ is negative. Thus*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0); [-\infty, \infty)) = -\sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_+(x, 0) \cap \mathbb{S}_+(0)\right). \tag{3.110}$$

Similarly, each crossing $x = x_0 \in \mathbb{R}$ of $x \mapsto \left(\mathbb{E}^u_-(x, 0), \mathbb{S}_-(0)\right)$ is positive, and we have

$$\operatorname{Mas}(\mathbb{E}^u_-(\cdot, 0), \mathbb{S}_-(0); [-\infty, \infty)) = \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_-(x, 0) \cap \mathbb{S}_-(0)\right). \tag{3.111}$$

Remark 3.19. In the above lemma (and throughout), by having the domain of the Lagrangian paths $x \mapsto (\mathbb{E}^u_\pm(x, 0), \mathbb{S}_\pm(0))$ as $x \in [-\infty, \infty)$, we mean that $\tau \in [-1, 1-\varepsilon]$ for the compactified path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau, 0)$ for some small $\varepsilon > 0$ (see Remark 3.11). Note however that the initial point $\tau = -1$ ($x = -\infty$) is never a conjugate point because $\mathbb{U}_+(0) \cap \mathbb{S}_+(0) = \{0\}$. On the other hand, $\tau = +1$ ($x = \infty$) is always a conjugate point, because $\mathbb{E}^u_\pm(+\infty, 0) \in \mathcal{T}_1(\mathbb{S}_\pm(0))$ on account of Hypothesis 3.1; nonetheless, because crossings are isolated (c.f. Lemma 3.24), we can make $\varepsilon > 0$ as small as we like.

The proof of Lemma 3.18 will focus on the L_+ problem, with the modifications needed for the L_- problem listed at the end. In order to compute the partial signatures of Definition 3.8, we will explicitly construct the matrix family $R(x)$ defining the curve of symmetric bilinear forms $\omega(R(x)\cdot, \cdot)\big|_{\mathbb{S}_+(0)\times\mathbb{S}_+(0)}$ in (3.51). This is given in Lemma 3.20. Recall that for each x near x_0, $R(x)$ is the unique matrix, when viewed as an operator from $\mathbb{S}_+(0)$ into $\mathbb{S}_+(0)^\perp$, whose graph is the Lagrangian plane $\mathbb{E}^u_+(x, 0)$. For ease of presentation we will drop the subscript $+$ on the frame $\mathbf{U}_+(x)$ for the unstable bundle, which we denote by

$$\mathbf{U}(x) = \begin{pmatrix} X(x) \\ Y(x) \end{pmatrix}. \tag{3.112}$$

Lemma 3.20. *Suppose $x = x_0 \in \mathbb{R}$ is a conjugate point. For all x near x_0, the curve of matrices $x \mapsto R(x) \in \mathbb{R}^{4\times 4}$,*

$$R(x) = \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top - \mathbf{S}_+\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top, \tag{3.113}$$

is analytic and satisfies $\mathbb{E}_+^u(x,0) = \mathrm{graph}(R(x)) = \{q + R(x)q : q \in \mathbb{S}_+(0)\}$.

Proof. First, note that by continuity, $\mathbb{E}_+^u(x,0)$ and $\mathbb{S}_+(0)^\perp$ are transverse for all x near x_0. It follows that $\mathbf{S}_+^\top\mathbf{U}(x) = X(x) + S_+Y(x)$ is invertible for all x near x_0. Indeed, transversality of $\mathbb{E}_+^u(x,0)$ and $\mathbb{S}_+(0)^\perp$ implies that the 4×4 matrix whose columns consist of bases for these spaces is invertible. A frame for $\mathbb{S}_+(0)^\perp$ is given by $J(I,S_+) = (-S_+, I)$. Using Schur's formula, we therefore have

$$0 \neq \det\begin{pmatrix} X(x) & -S_+ \\ Y(x) & I\end{pmatrix} = \det\left(X(x) + S_+Y(x)\right).$$

Analyticity of $x \mapsto R(x)$ now follows from the analyticity of $x \mapsto \mathbf{U}(x)$, the entries of which are solutions to (3.95).

Now for any $q \in \mathbb{S}_+(0)$ we have $q = \mathbf{S}_+k_0$ for some $k_0 \in \mathbb{R}^2$. Then

$$\begin{aligned} q + R(x)q &= \mathbf{S}_+k + \mathbf{U}(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+k - \mathbf{S}_+\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+k, \\ &= \mathbf{U}(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+k \in \mathbb{E}_+^u(x,0). \end{aligned}$$

For the opposite inclusion, for any $v \in \mathbb{E}_+^u(x,0)$ we may write $v = \mathbf{U}(x)h$ for some $h \in \mathbb{R}^2$. Now set $k = \left(\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}h$. Then

$$v = \mathbf{U}(x)h = \mathbf{S}_+k + \mathbf{U}(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+k - \mathbf{S}_+\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+k,$$

and setting $q = \mathbf{S}_+k$ we have $v = q + R(x)q \in \mathrm{graph}(R(x))$. □

Proof of Lemma 3.18. We will prove that crossings of the path $x \mapsto (\mathbb{E}_+^u(x,0), \mathbb{S}_+(0))$ are negative in two cases: (1) $\dim \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0) = 1$ and Hypothesis 3.17 holds, and (2) $\dim \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0) = 2$.

For the first case, we need to show that the first order form $\mathfrak{m}_{x_0}$ is negative definite. From (3.59) we have

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q) = \frac{d}{dx}\omega\left(R(x)q, q\right)\big|_{x=x_0}, \tag{3.114}$$

where $q \in \mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)$ is fixed, and $R(x)$ is given in Lemma 3.20. Note that for any $q \in \mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)$ we may write $q = \mathbf{U}(x_0)h = \mathbf{S}_+k$ for some $h = (c,d) \in \mathbb{R}^2$, $k = (a,b) \in \mathbb{R}^2$.

We will require the first derivatives of the matrices $X(x)$, $Y(x)$ and $R(x)$. Since the columns of the frame $\mathbf{U}(x) = (X(x), Y(x))$ satisfy (3.95), we have

$$X'(x) = B_+Y(x), \qquad Y'(x) = C_+(x)X(x). \tag{3.115}$$

(Recall $C_+(x) = C_+(x,0)$.) We also have

$$R'(x) = \mathbf{U}'(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top - \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top \mathbf{U}'(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top. \tag{3.116}$$

Denoting

$$R_0 = \left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top \mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top, \tag{3.117}$$

we now compute:

$$\begin{aligned}
\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(q) &= \omega(R'(x_0)q,q) = \langle JR'(x_0)\mathbf{S}_+k, \mathbf{S}_+k\rangle_{\mathbb{R}^4}, \\
&= \left\langle J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top \mathbf{S}_+k, \mathbf{S}_+k\right\rangle_{\mathbb{R}^4} \\
&\qquad + \langle J\mathbf{U}(x_0)R_0\,\mathbf{S}_+k, \mathbf{S}_+k\rangle_{\mathbb{R}^4}, \\
&= \left\langle J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top \mathbf{U}(x_0)h, \mathbf{U}(x_0)h\right\rangle_{\mathbb{R}^4} \\
&\qquad + \langle \mathbf{U}(x_0)^\top J\mathbf{U}(x_0)R_0\,\mathbf{U}(x_0)h, h\rangle_{\mathbb{R}^4}, \\
&= \langle J\mathbf{U}'(x_0)h, \mathbf{U}(x_0)h\rangle_{\mathbb{R}^4}, \\
&= -\langle C_+(x_0)X(x_0)h, X(x_0)h\rangle_{\mathbb{R}^2} + \langle B_+Y(x_0)h, Y(x_0)h\rangle_{\mathbb{R}^2}, \\
&= \langle (-C_+(x_0) + S_+B_+S_+)\,k, k\rangle_{\mathbb{R}^2},
\end{aligned}$$

where $\mathbf{U}(x_0)^\top J\mathbf{U}(x_0) = -X(x_0)^\top Y(x_0) + Y(x_0)^\top X(x_0) = 0$ because $\mathbf{U}(x_0)$ is the frame for a Lagrangian plane, and we used (3.105) and the symmetry of S_+. (Recall that $q = \mathbf{U}(x_0)h = \mathbf{S}_+k$.) Recalling (3.105) and (3.107), we have

$$C_+(x) - S_+B_+S_+ = \widehat{C}_+(x_0) + \widetilde{C}_+ - S_+B_+S_+ = \widehat{C}_+(x_0), \tag{3.118}$$

and therefore, under Hypotheses 3.16 and 3.17,

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(q) = -\langle \widehat{C}_+(x_0)k, k\rangle_{\mathbb{R}^2} = -3\phi(x_0)^2 b^2 < 0. \tag{3.119}$$

Hence $n_-(\mathfrak{m}_{x_0}) = 1$, and crossings are negative in this case. By (3.65) their contribution to the Maslov index is $-\dim(\mathbb{E}_+^u(\cdot,0)\cap\mathbb{S}_+(0)) = -1$.

Next, we treat the case $\dim \mathbb{E}_+^u(x,0)\cap\mathbb{S}_+(0) = 2$. We have already seen that $\mathfrak{m}_{x_0}$ is degenerate (but not identically zero), and thus we cannot possibly have $n_-(\mathfrak{m}_{x_0}) = 2$. Therefore, recalling that a crossing is negative if (3.77) holds, our goal will be to show that $n_-(\mathfrak{m}_{x_0}) = 1$, $\mathfrak{m}_{x_0}^{(2)}$ is identically zero, and $n_-(\mathfrak{m}_{x_0}^{(3)}) = 1$.

By definition, we have

$$\mathfrak{m}_{x_0}^{(k)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(q_0) = \frac{d^k}{dx^k}\,\omega\big(R(x)q(x), q_0\big)\Big|_{x=x_0}, \qquad q_0 \in W_k, \tag{3.120}$$

where

$$\begin{aligned} W_k = \{q_0 \in \mathbb{S}_+(0) :\ \exists\ &\text{a generalised Jordan chain of length } k, \\ &\text{starting at } q_0\text{, for the curve of matrices } JR(x) \text{ at } x = x_0\}. \end{aligned} \tag{3.121}$$

To compute the forms for $k = 1, 2, 3$, we will work instead with the smooth curve of symmetric matrices

$$[x_0 - \varepsilon, x_0 + \varepsilon] \ni x \mapsto L(x) := \mathbf{S}_+^\top J R(x) \mathbf{S}_+ \in \mathbb{R}^{2\times 2}. \tag{3.122}$$

If there exists a generalised Jordan chain $\{k_i\}_i$ for the curve $L(x)$ at $x = x_0$, then $\{q_i\}_i = \{\mathbf{S}_+ k_i\}_i$ is a generalised Jordan chain for the family $x \mapsto JR(x) : \mathbb{S}_+(0) \to \mathbb{S}_+(0)$ at $x = x_0$. We can thus write the crossing forms as

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L'(x_0)k_0, k_0\rangle_{\mathbb{R}^2}, \tag{3.123a}$$
$$\mathfrak{m}_{x_0}^{(2)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L''(x_0)k_1, k_0\rangle_{\mathbb{R}^2} + \langle L'(x_0)k_0, k_0\rangle_{\mathbb{R}^2}, \tag{3.123b}$$
$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L'''(x_0)k_2, k_0\rangle_{\mathbb{R}^2} + \langle L''(x_0)k_1, k_0\rangle_{\mathbb{R}^2} + \langle L'(x_0)k_0, k_0\rangle_{\mathbb{R}^2}. \tag{3.123c}$$

Let us first compute the derivatives $L'(x_0), L''(x_0), L'''(x_0)$. Differentiating (3.115),

$$X''(x) = B_+C_+(x)X(x), \qquad Y''(x) = C_+'(x)X(x) + C_+(x)B_+Y(x),$$

and

$$\begin{aligned} X'''(x) &= B_+C_+'(x)X(x) + B_+C_+(x)B_+Y(x), \\ Y'''(x) &= C_+''(x)X(x) + 2C_+'(x)B_+Y(x) + C_+(x)B_+C_+(x)X(x). \end{aligned}$$

Since $\dim \mathbb{E}_+^u(x_0,0) \cap \mathbb{S}_+(0) = 2$, we have $\mathbb{E}_+^u(x_0,0) = \mathbb{S}_+(0)$, and from (3.53),

$$\ker \omega(R(x)\cdot,\cdot) = \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0) = \mathbb{S}_+(0). \tag{3.124}$$

Moreover, $\mathbf{S}_+ = (I, S_+)$ and $\mathbf{U}(x_0) = (X(x_0), Y(x_0))$ are frames for the same Lagrangian plane, meaning there exists an invertible 2×2 matrix F so that $\mathbf{U}(x_0) = \mathbf{S}_+F$. Looking at the upper 2×2 block of this equation, this means that $X(x_0) = F$ is invertible, and therefore $X(x)$ is invertible for nearby x. Right multiplying by $X(x)^{-1}$, we can thus take

$$\mathbf{U}(x) = \begin{pmatrix} I \\ U(x) \end{pmatrix}, \qquad U(x) := Y(x)X(x)^{-1}, \tag{3.125}$$

to be a frame for $\mathbb{E}_+^u(x,0)$, where now $\mathbf{U}(x_0) = \mathbf{S}_+$ and $U(x_0) = Y(x_0)X(x_0)^{-1} = S_+$. The first derivative of $U(x)$ is given by

$$U'(x) = Y'(x)X(x)^{-1} - Y(x)X(x)^{-1}X'(x)X(x)^{-1} = C_+(x) - U(x)B_+U(x),$$

hence

$$U'(x_0) = C_+(x) - S_+B_+S_+ = \widehat{C}_+(x_0), \tag{3.126}$$

recalling (3.118). Using (3.126) and (3.107), the second and third derivatives are shown to be

$$U''(x_0) = C_+'(x_0) - \widehat{C}_+(x_0)B_+S_+ - S_+B_+\widehat{C}_+(x_0), \tag{3.127}$$

$$\begin{aligned} U'''(x_0) = {} & C_+''(x_0) - 2\widehat{C}_+(x_0)B_+\widehat{C}_+(x_0) + 2S_+B_+\widehat{C}_+(x_0)B_+S_+ \\ & - C_+'(x_0)B_+S_+ - S_+B_+C_+'(x_0) + \widehat{C}_+(x_0)B_+\widetilde{C}_+ + \widetilde{C}_+B_+\widehat{C}_+(x_0). \end{aligned} \tag{3.128}$$

We are ready to compute derivatives of $L(x)$. Using (3.116) and (3.117), and that $\mathbf{U}(x_0) = \mathbf{S}_+$, $\mathbf{S}_+^\top J \mathbf{S}_+ = 0$ and $\mathbf{U}'(x) = (0, U'(x))$, we have

$$L'(x_0) = \mathbf{S}_+^\top J R'(x_0) \mathbf{S}_+ = \mathbf{S}_+^\top J \mathbf{U}'(x_0) \left(\mathbf{S}_+^\top \mathbf{U}(x_0) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ - \mathbf{S}_+^\top J \mathbf{U}(x_0) R_0 \mathbf{S}_+ = -\widehat{C}_+(x_0).$$

Differentiating (3.116),

$$\begin{aligned} R''(x) = \mathbf{U}''(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top - 2\mathbf{U}'(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top \\ + \mathbf{U}(x) \frac{d^2}{dx^2} \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top, \end{aligned}$$

thus

$$\begin{aligned} L''(x_0) = \mathbf{S}_+^\top J R''(x_0) \mathbf{S}_+ &= \mathbf{S}_+^\top J \mathbf{U}''(x_0) \left(\mathbf{S}_+^\top \mathbf{U}(x_0) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ \\ &\quad - 2\mathbf{S}_+^\top J \mathbf{U}'(x_0) \left(\mathbf{S}_+^\top \mathbf{U}(x_0) \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x_0) \left(\mathbf{S}_+^\top \mathbf{U}(x_0) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ \\ &\quad - \mathbf{S}_+^\top J \mathbf{U}(x_0) \frac{d^2}{dx^2} \left(\mathbf{S}_+^\top \mathbf{U}(x_0) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}, \\ &= \mathbf{S}_+^\top J \mathbf{U}''(x_0) - 2\mathbf{S}_+^\top J \mathbf{U}'(x_0) \left(\mathbf{S}_+^\top \mathbf{S}_+ \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x_0), \\ &= -U''(x_0) + 2U'(x_0)(I + S_+^2)^{-1} S_+ U'(x_0). \end{aligned} \tag{3.129}$$

Differentiating again,

$$\begin{aligned} R'''(x) = \mathbf{U}'''(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top - 3\mathbf{U}''(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x) \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top, \\ + 3\mathbf{U}'(x) \frac{d^2}{dx^2} \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top + \mathbf{U}(x) \frac{d^3}{dx^3} \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top, \end{aligned}$$

hence

$$\begin{aligned} L'''(x_0) = \mathbf{S}_+^\top J R'''(x_0) \mathbf{S}_+ &= \mathbf{S}_+^\top J \mathbf{U}'''(x_0) - 3\mathbf{S}_+^\top J \mathbf{U}''(x_0) \left(\mathbf{S}_+^\top \mathbf{S}_+ \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x_0), \\ &\quad + 3\mathbf{S}^\top J \mathbf{U}'(x_0) \frac{d^2}{dx^2} \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ |_{x=x_0} \\ &= -U'''(x_0) + 3U''(x_0) \left(\mathbf{S}_+^\top \mathbf{S}_+ \right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x_0) \\ &\quad - 3U'(x_0) \frac{d^2}{dx^2} \left(\mathbf{S}_+^\top \mathbf{U}(x) \right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ |_{x=x_0}. \end{aligned}$$

Some algebra shows that

$$\begin{aligned} L'''(x_0) = -U'''(x_0) + 3U''(x_0) \left(I + S_+^2 \right)^{-1} S_+ U'(x_0) + 3U'(x_0)(I + S_+ U(x_0))^{-1} S_+ U''(x_0) \\ - 6U'(x_0)(I + S_+ U(x_0))^{-1} S_+ U'(x_0)(I + S_+ U(x_0))^{-1} S_+ U'(x_0). \end{aligned} \tag{3.130}$$

Let us examine the above expressions more closely. For $L''(x_0)$ we have

$$U''(x_0) = C'_+(x_0) - \widehat{C}_+(x_0)B_+S_+ - S_+B_+\widehat{C}_+(x_0) = \begin{pmatrix} 0 & \frac{3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix}. \tag{3.131}$$

Noting that $U'(x_0) = \widehat{C}(x_0)$ and

$$\widehat{C}_+(x_0)M\widehat{C}_+(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \tag{3.132}$$

for any 2×2 matrix M, the second term of (3.129) is of the form of (3.132). For $L'''(x_0)$, the first two terms of $U'''(x_0)$ in (3.128) and the last term of $L'''(x_0)$ in (3.130) all have the form of (3.132). The third term of $U'''(x_0)$ has the form

$$2S_+B_+\widehat{C}_+(x_0)B_+S_+ = \begin{pmatrix} \frac{6\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}. \tag{3.133}$$

The remaining terms in $U'''(x_0)$, i.e.

$$-C'_+(x_0)B_+S_+ - S_+B_+C'_+(x_0) + \widehat{C}_+(x_0)B_+\widetilde{C}_+ + \widetilde{C}_+B_+\widehat{C}_+(x_0), \tag{3.134}$$

as well as the second and third terms of $L''(x_0)$ in (3.130), can all be shown to have the form

$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}. \tag{3.135}$$

In summary, we have

$$L'(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & -3\phi(x_0)^2 \end{pmatrix}, \; L''(x_0) = \begin{pmatrix} 0 & \frac{-3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{-3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix}, \; L'''(x_0) = \begin{pmatrix} \frac{-6\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}. \tag{3.136}$$

The expressions (3.136) are sufficient to determine the partial signatures of (3.123). To do so, we need to compute any generalised Jordan chains for the curve $L(x)$. Define $k_i = (a_i, b_i)^\top \in \mathbb{R}^2$ for $i = 0, 1, 2, 3$. That $\dim(\mathbb{E}^u_+(x, 0) \cap \mathbb{S}_+(0)) = 2$ means that

$$\ker L(x_0) = \mathbb{R}^2, \tag{3.137}$$

and therefore $\{k_0\}$ is a chain of length one for any $k_0 \in \ker L(x_0)$. Next, there exists solutions $k_1 = (a_1, b_1)^\top$ to

$$L(x_0)k_1 + L'(x_0)k_0 = \begin{pmatrix} 0 \\ 3\, b_0\, \phi(x_0)^2 \end{pmatrix} = 0 \tag{3.138}$$

if and only if $b_0 = 0$. Hence, $\{k_0, k_1\}$ is a chain of length two if and only if $b_0 = 0$. Now taking $k_0 = (a_0, 0)^\top$, there exists solutions k_2 to

$$L''(x_0)k_0 + L'(x_0)k_1 + L(x_0)k_2 = \begin{pmatrix} 0 \\ 3\, b_1\, \phi(x_0)^2 + \frac{3\, a_0 \phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \end{pmatrix} = 0 \tag{3.139}$$

if and only if $b_1 = -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}}$. Thus $\{k_0, k_1, k_2\}$ is a chain of length three if and only if $b_0 = 0$ and $b_1 = -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}}$. Finally, note that for nontrivial $k_0 = (a_0, 0)^\top$ there are no solutions k_3 to

$$L'''(x_0)k_0 + L''(x_0)k_1 + L'(x_0)k_2 + L(x_0)k_3 = \begin{pmatrix} \frac{3a_0\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} \\ * \end{pmatrix} = 0. \tag{3.140}$$

In other words, the chain $\{k_0, k_1, k_2\}$ is maximal. We are ready to compute the partial signatures. For $k_0 = (a_0, b_0)^\top \in \ker L(x_0)$, we have

$$\mathfrak{m}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = \langle L'(x_0)k_0, k_0\rangle_{\mathbb{R}^2} = -3b_0^2\phi(x_0)^2 < 0, \tag{3.141}$$

while for $k_0 = (a_0, 0)^\top$ and $k_1 = (a_1, -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}})^\top$, we have

$$\mathfrak{m}^{(2)}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = \langle L''(x_0)k_0 + L'(x_0)k_1, k_0\rangle_{\mathbb{R}^2} = 0, \tag{3.142}$$

$$\mathfrak{m}^{(3)}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = \langle L'''(x_0)k_0 + L''(x_0)k_1 + L'(x_0)k_2, k_0\rangle = -\frac{3a_0^2\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} < 0. \tag{3.143}$$

The right hand sides of (3.141) and (3.143) are negative due to Hypothesis 3.16 and the assumptions (3.8) (which implies that $2\sqrt{\beta} - \sigma_2 > 0$). We have just shown that $n_-(\mathfrak{m}_{x_0}) = n_-(\mathfrak{m}^{(3)}_{x_0}) = 1$ and $n_+(\mathfrak{m}_{x_0}) = n_+(\mathfrak{m}^{(3)}_{x_0}) = n_\pm(\mathfrak{m}^{(2)}_{x_0}) = 0$. Therefore each crossing $x_0 \in \mathbb{R}$ where $\dim(\mathbb{E}^u_+(\cdot, 0) \cap \mathbb{S}_+(0)) = 2$ is negative, because in such cases

$$\dim(\mathbb{E}^u_+(\cdot, 0) \cap \mathbb{S}_+(0)) = n_-(\mathfrak{m}_{x_0}) + n_-(\mathfrak{m}^{(3)}_{x_0}). \tag{3.144}$$

By (3.65) the contribution of each crossing $x_0 \in \mathbb{R}$ is therefore $-\dim(\mathbb{E}^u_+(\cdot, 0) \cap \mathbb{S}_+(0))$. This completes the proof for the L_+ problem.

The proof for the L_- problem is similar. The case of one-dimensional crossings under Hypothesis 3.17 is almost identical, while for the case of two-dimensional crossings we'll have

$$L'(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(x_0)^2 \end{pmatrix}, \quad L''(x_0) = \begin{pmatrix} 0 & \frac{\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix}, \quad L'''(x_0) = \begin{pmatrix} \frac{2\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}.$$

Computing the generalised Jordan chains as above leads to

$$\mathfrak{m}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = b_0^2\phi(x_0)^2 > 0,$$
$$\mathfrak{m}^{(2)}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = 0,$$
$$\mathfrak{m}^{(3)}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(q_0) = \frac{a_0^2\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} > 0,$$

for some $a_0, b_0 \in \mathbb{R}$, with positivity under Hypothesis 3.16. Each crossing $x \in \mathbb{R}$ of the path $x \mapsto \mathbb{E}^u_-(x,0) \cap \mathbb{S}_-(0)$ thus contributes $\dim \mathbb{E}^u_-(x,0) \cap \mathbb{S}_-(0)$ to its Maslov index. $\square$

Remark 3.21. That the matrix $L'(x_0)$ is degenerate, i.e. that crossings $x_0 \in \mathbb{R}$ are non-regular, is the reason for using the partial signatures approach of [GPP04b] to compute the Maslov index.

Remark 3.22. If Hypothesis 3.16 fails, i.e. for any crossing $x_0 \in \mathbb{R}$ such that $\phi(x_0) = 0$, the forms $\mathfrak{m}_{x_0}$ and $\mathfrak{m}^{(3)}_{x_0}$ are degenerate, and higher order crossing forms are needed.

Remark 3.23. Proposition 3.15 will also hold for any power-law fourth-order NLS equation, i.e. (3.19) for any $p \in \mathbb{N}$. In these cases the crossing forms $\mathfrak{m}^{(k)}_{x_0}(q_0)$ will be the same as those above, but scaled by a positive constant, and with $\phi(x_0)^2$ replaced by $\phi(x_0)^{2p}$. The signs are therefore preserved.

The following lemma shows that crossings along Γ_1 are isolated.

Lemma 3.24. *There are finitely-many isolated intersections of the path* $[-1,1] \ni \tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$ *with the trains* $\mathcal{T}(\mathbb{S}_\pm(0))$ *and* $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$.

Proof. First, note that because $\mathbb{U}(0)_\pm \cap \mathbb{S}_\pm(0) = \{0\}$ and $\lim_{\tau\to-1^+} \widehat{\mathbb{E}}^u_\pm(\tau,0) = \mathbb{U}_\pm(0)$, by continuity there exists a $\hat{\tau}$ close to -1 such that $\widehat{\mathbb{E}}^u_\pm(\tau,0) \cap \mathbb{S}_\pm(0) = \{0\}$ for all $\tau \in [-1,\hat{\tau}]$. Now consider the compactly-defined path $\tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$, $\tau \in [-\hat{\tau},\tau_\ell] \subset (-1,1)$. Since the elements of $\widehat{\mathbb{E}}^u_\pm(\cdot,0)$ are solutions to a differential equation and therefore analytic on $(-1,1)$, we can form an analytic path of frames $\tau \mapsto \widehat{\mathbf{U}}_\pm(\tau)$ on $[-\hat{\tau},\tau_\ell]$. Now collecting the columns of $\mathbf{U}_\pm(\tau)$ and the columns of a frame for $\mathbb{E}^s_\pm(\ell,0)$ into a 4×4 matrix $F(\tau)$, the function $\tau \mapsto \det D(\tau)$ is real-valued and analytic on $[-\hat{\tau},\tau_\ell]$. It therefore has finitely-many isolated zeroes, which correspond to intersections of $\tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$. It will follow from the perturbative arguments in the proof of Lemma 3.25 that the crossings of $\tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$ over $\tau \in [-1,\tau_\ell]$ and the crossings of $\tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{S}_\pm(0))$ over $\tau \in [-1,1]$ are in one-to-one correspondence. This completes the proof. $\square$

Lemma 3.25. *For the Lagrangian path* $x \mapsto \big(\mathbb{E}^u_+(x,0), \mathbb{E}^s_+(\ell,0)\big)$ *we have*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty, \ell-\varepsilon]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty)) \tag{3.145}$$

for $\varepsilon > 0$ *small enough. A similar statement holds for the path* $x \mapsto (\mathbb{E}^u_-(x,0), \mathbb{E}^s_-(\ell,0))$.

Proof. First, we show that

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty, \ell]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty]). \tag{3.146}$$

In order to do so, it will be convenient to compactify $\mathbb{R}$ via the change of variables in Remark 3.11. Thus, defining

$$\widehat{\mathbb{E}}^{s,u}_\pm(\tau,0) := \mathbb{E}^{s,u}_\pm\left(\ln\left(\frac{1+\tau}{1-\tau}\right), 0\right), \tag{3.147}$$

(3.146) is equivalent to

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,\tau_\ell]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1]), \tag{3.148}$$

where $\ell = \ln((1+\tau_\ell)/(1-\tau_\ell))$, i.e. $\tau_\ell = (e^\ell - 1)/(e^\ell + 1)$, and we have used that $\widehat{\mathbb{E}}^s_+(1,0) = \mathbb{E}^s_+(+\infty, 0) := \mathbb{S}_+(0)$. Rescaling further, we can map $[-1,1]$ to $[-1,\tau_\ell]$ via the function

$$g(\tau) = \left(\frac{1+\tau_\ell}{2}\right)\tau + \left(\frac{\tau_\ell - 1}{2}\right),$$

where $g(-1) = -1$ and $g(1) = \tau_\ell$. This allows us to write both Lagrangian paths in (3.148) over $[-1,1]$, i.e.

$$\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,1]) = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1]). \tag{3.149}$$

To prove (3.149), we set

$$\Lambda_1(s,\tau) := \widehat{\mathbb{E}}^u_+(\tau + (g(\tau)-\tau)s, 0), \qquad \Lambda_2(s,\tau) := \widehat{\mathbb{E}}^s_+(1 + (\tau_\ell - 1)s, 0). \tag{3.150}$$

(Λ_2 is independent of τ.) Both maps $(s,\tau) \to \Lambda_{1,2}(s,\tau)$ are continuous on $[0,1]\times[-1,1]$. In addition,

$$\Lambda_1(s,-1) = \widehat{\mathbb{E}}^u_+(-1,0) = \mathbb{U}_+(0), \qquad \Lambda_2(s,-1) = \widehat{\mathbb{E}}^s_+(1 + (\tau_\ell - 1)s, 0),$$

where we used that $g(-1) = -1$. Since $\mathbb{U}_+(0) \cap \mathbb{E}^s_+(x,0) = \{0\}$ for all $x \geq \ell$ (see (3.83)) and $\mathbb{U}_+(0) \cap \mathbb{S}_+(0) = \{0\}$, we have $\mathbb{U}_+(0) \cap \widehat{\mathbb{E}}^s_+(\tau,0) = \{0\}$ for all $\tau \in [\tau_\ell, 1]$, and hence

$$\Lambda_1(s,-1) \cap \Lambda_2(s,-1) = \{0\}$$

for all $s \in [0,1]$. Furthermore,

$$\Lambda_1(s,1) = \widehat{\mathbb{E}}^u_+(1 + (\tau_\ell - 1)s, 0), \qquad \Lambda_2(s,1) = \widehat{\mathbb{E}}^s_+(1 + (\tau_\ell - 1)s, 0),$$

and therefore

$$\dim \Lambda_1(s,1) \cap \Lambda_2(s,1) = 1$$

for all $s \in [0,1]$ by Hypothesis 3.1. Equation (3.149) (and thus (3.146)) now follows from Lemma 3.10.

By additivity under concatenation (see Proposition 3.9), we can write (3.149) as

$$\begin{aligned}
&\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,1-\varepsilon]) + \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1]) \\
&\quad = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1-\varepsilon_0]) + \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon_0,1])
\end{aligned} \tag{3.151}$$

for $\varepsilon, \varepsilon_0 > 0$ small. Because crossings of the path $x \mapsto \mathbb{E}^u_+(x,0)$ with $\mathcal{T}(\mathbb{S}_+(0))$ and $\mathcal{T}(\mathbb{E}^s_+(\ell,0))$ are isolated (see Lemma 3.24), we can choose $\varepsilon, \varepsilon_0 > 0$ small enough so that $\tau = 1$ is the only crossing in the intervals $[1-\varepsilon, 1]$ and $[1-\varepsilon_0, 1]$ for the paths in (3.151). To prove Lemma 3.25, it thus suffices to show that

$$\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1]) = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon_0,1]), \tag{3.152}$$

i.e. that the conjugate points occurring at the final points of each of the paths

$$\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(g(\tau),0), \widehat{\mathbb{E}}^s(\tau_\ell,0)\right), \quad \tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \mathbb{S}_+(0)\right), \quad \tau \in [-1,1], \tag{3.153}$$

have the same contribution to their respective Maslov indices. To this end, notice that the arguments of the unstable bundles appearing in (3.153) are arbitrarily close: by choosing ℓ large enough, so that $\tau_\ell = 1-\delta$ for $\delta > 0$ small enough, we have

$$|g(\tau) - \tau| = \left(\frac{1-\tau_\ell}{2}\right)(\tau+1) \leq \delta$$

uniformly for $\tau \in [-1,1]$. Thus, the paths in (3.153) are arbitrarily small perturbations of one another. In addition, since $\mathbb{E}^s(\tau,0)$ can be taken as close to $\mathbb{S}_+(0)$ (as points in $\mathcal{L}(2)$) as we like, the trains $\mathcal{T}(\mathbb{E}^s(\tau,0))$ and $\mathcal{T}(\mathbb{S}_+(0))$ are also arbitrarily small perturbations of one another. From these two facts, it follows that the paths in (3.153) approach the trains $\mathcal{T}(\mathbb{E}^s(\tau,0))$ and $\mathcal{T}(\mathbb{S}_+(0))$ from the same direction as $\tau \to 1^-$. The contributions of the associated conjugate points to their respective Maslov indices are therefore the same, i.e. (3.152) holds, and by (3.151) we have

$$\text{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1, 1-\varepsilon]) = \text{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1, 1-\varepsilon_0]).$$

Recalling Remark 3.19, this is exactly (3.145) (for a different but still arbitrarily small ε). The proof for the L_- problem is similar. □

We remark here that Lemma 3.18 does not apply to the conjugate point at $\tau = 1$ ($x = +\infty$). This is because the functions in the unstable bundle used in the crossing form calculations either blow up to infinity or decay to zero there. Nonetheless, recalling the definition given by Arnol'd (see Section 3.3.2), we can still compute the Maslov indices in (3.152). Undoing the scaling by g, the paths in (3.153) are given by

$$\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0)\right), \ \tau \in [-1,\tau_\ell], \qquad \tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \mathbb{S}_+(0)\right), \ \tau \in [-1,1]. \tag{3.154}$$

We know from Hypothesis 3.1 that the final crossing of each path in (3.154) is one-dimensional. In particular, we have $\widehat{\mathbb{E}}^u_+(\tau_\ell,0) \in \mathcal{T}_1(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$. From the arguments in the proof of Lemma 3.25, $\widehat{\mathbb{E}}^u_+(\tau_\ell,0)$ is therefore arbitrarily close to $\mathcal{T}_1(\mathbb{S}_+(0))$. Lemma 3.18 implies that at all interior one-dimensional crossings $\tau \in (-1,1)$, the path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ passes through $\mathcal{T}_1(\mathbb{S}_+(0))$ in the negative direction (i.e. from the positive to the negative side of $\mathcal{T}_1(\mathbb{S}_+(0))$). It follows that at $\tau_\ell \in (-1,1)$, the path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ must arrive at $\mathcal{T}(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$ in the negative direction as $\tau \to \tau_\ell^-$. The final crossings of the paths in (3.153) are thus both negative. By our convention the final crossings may only contribute positively, and therefore

$$\text{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon, 1]) = \text{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [\ell-\varepsilon, \ell]) = 0. \tag{3.155}$$

Lemma 3.26. *Each crossing $\lambda = \lambda_0$ of the path of Lagrangian pairs $\lambda \mapsto (\mathbb{E}^u_+(\ell,\lambda), \mathbb{E}^s_+(\ell,\lambda))$ is positive. Thus,*

$$\text{Mas}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot); [\varepsilon, \lambda_\infty]) = P. \tag{3.156}$$

for $\varepsilon > 0$ small enough. Similarly, each crossing $\lambda = \lambda_0$ of the path $\lambda \mapsto (\mathbb{E}^u_-(\ell,\lambda), \mathbb{E}^s_-(\ell,\lambda))$ is negative, and we have

$$\operatorname{Mas}(\mathbb{E}^u_-(\ell,\cdot), \mathbb{E}^s_-(\ell,\cdot); [\varepsilon, \lambda_\infty]) = -Q. \tag{3.157}$$

Proof. We begin with the first two statements. We proceed by computing the relative crossing form of Robbin and Salamon [RS93] at each crossing $\lambda = \lambda_0$, given by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot))(q) = \mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) - \mathfrak{m}_{\lambda_0}(\mathbb{E}^s_+(\ell,\cdot), \mathbb{E}^u_+(\ell,\lambda_0))(q), \tag{3.158}$$

where $q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0)$ is fixed. We compute each of the crossing forms on the right hand side separately.

For the first, we consider the path $\lambda \mapsto \mathbb{E}^u_+(\ell,\lambda)$ over $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ for $\varepsilon > 0$ small with reference plane $\mathbb{E}^s_+(\ell,\lambda_0)$. We have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^u_+(\lambda)q, q)\big|_{\lambda=\lambda_0}, \quad q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0), \tag{3.159}$$

where $R^u_+(\lambda) : \mathbb{E}^s_+(\ell,\lambda_0) \to \mathbb{E}^s_+(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^u_+(\ell,\lambda) = \operatorname{graph}(R^u_+(\lambda)) = \{q + R^u_+(\lambda)q : q \in \mathbb{E}^s_+(\ell,\lambda_0)\}$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Fixing $q \in \mathbb{E}^s_+(\ell,\lambda_0) \cap \mathbb{E}^u_+(\ell,\lambda_0)$, let $h(\lambda) = q + R^u_+(\lambda)q \in \mathbb{E}^u_+(\ell,\lambda)$. From the definition of $\mathbb{E}^u_+(\ell,\lambda)$, there exists a one-parameter family of solutions $\lambda \mapsto \mathbf{u}(\cdot;\lambda)$ to (3.95) satisfying $\mathbf{u}(x;\lambda) \to 0$ as $x \to -\infty$, such that $h(\lambda) = \mathbf{u}(\ell;\lambda)$. Moreover, $h(\lambda_0) = q = \mathbf{u}(\ell;\lambda_0)$ because $q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0) = \big(\ker R^u_+(\lambda_0)\big) \cap \mathbb{E}^s_+(\ell,\lambda_0)$. This allows us to write

$$\begin{aligned}\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) &= \frac{d}{d\lambda}\omega(R^u_+(\lambda)q, q)\big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(q + R^u_+(\lambda)q, q)\big|_{\lambda=\lambda_0},\\ &= \omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell,\lambda), \mathbf{u}(\ell,\lambda_0)\Big)\big|_{\lambda=\lambda_0}.\end{aligned}$$

Now

$$\begin{aligned}\omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell;\lambda), \mathbf{u}(\ell;\lambda)\Big) &= \int_{-\infty}^{\ell} \partial_x\, \omega(\partial_\lambda \mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda))\, dx,\\ &= \int_{-\infty}^{\ell} \omega\left(\partial_\lambda\left[A_+(x;\lambda)\mathbf{u}(x;\lambda)\right], \mathbf{u}(x;\lambda)\right)\\ &\qquad + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\, dx,\\ &= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda))\\ &\qquad + \omega(A_+(x;\lambda)\partial_\lambda \mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda))\\ &\qquad\qquad + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\, dx,\\ &= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda))\\ &\qquad + \big\langle [A_+(x;\lambda)^\top J + JA_+(x;\lambda)]\partial_\lambda \mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda)\big\rangle\, dx,\\ &= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda), \mathbf{u}(x;\lambda))\, dx,\end{aligned} \tag{3.160}$$

where we used that $\lim_{x\to-\infty}\mathbf{u}(x;\lambda)=0$ in the first line and (3.34) in the last line. Since

$$\partial_\lambda A_+(x;\lambda)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-1&0&0\end{pmatrix},\tag{3.161}$$

and $\mathbf{u}=(u_1,u_2,u_3,u_4)^\top$, evaluating the last line of (3.160) at $\lambda=\lambda_0$ we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\lambda_0))(q)=\int_{-\infty}^{\ell}u_2(x;\lambda_0)^2\,dx.\tag{3.162}$$

For the second term of the relative crossing form we use a similar argument. We have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s_+(\ell,\cdot),\mathbb{E}^u_+(\ell,\lambda_0))(q)=\frac{d}{d\lambda}\omega(R^s_+(\lambda)q,q)\big|_{\lambda=\lambda_0},\quad q\in\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0),\tag{3.163}$$

where $R^s_+(\lambda):\mathbb{E}^u_+(\ell,\lambda_0)\to\mathbb{E}^u_+(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^s_+(\ell,\lambda)=\operatorname{graph}(R^s_+(\lambda))=\{q+R^s_+(\lambda)q:q\in\mathbb{E}^s_+(\ell,\lambda_0)\}$. For the *same* fixed $q\in\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0)$ as in the paragraph following (3.159), we can construct a curve $g(\lambda)=q+R^s_+(\lambda)q\in\mathbb{E}^s_+(\ell,\lambda)$ for which there exists a one-parameter family of solutions $\lambda\mapsto\widetilde{\mathbf{u}}(\cdot;\lambda)$ to (3.95) such that $g(\lambda)=\widetilde{\mathbf{u}}(\ell;\lambda)$ and $g(\lambda_0)=q=\widetilde{\mathbf{u}}(\ell;\lambda_0)$. Arguing as previously, but noting that now $\widetilde{\mathbf{u}}(x;\lambda)\to0$ as $x\to+\infty$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,\lambda_0))(q)=\omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{w}}(\ell;\lambda),\widetilde{\mathbf{w}}(\ell;\lambda)\Big)\big|_{\lambda=\lambda_0}=-\int_{\ell}^{\infty}\widetilde{u}_2(x;\lambda_0)^2\,dx\tag{3.164}$$

(where $\widetilde{\mathbf{u}}=(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{u}_4)^\top$). Importantly, by uniqueness of solutions we have $\widetilde{\mathbf{u}}(\cdot;\lambda_0)=\mathbf{u}(\cdot;\lambda_0)$, so that the integrands in (3.164) and (3.162) are the same. Therefore, (3.158) becomes

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot))(q)=\int_{-\infty}^{\infty}u_2(x;\lambda_0)^2\,dx>0.\tag{3.165}$$

As the form is positive definite, each crossing contributes $\dim\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0)$. It follows that the Maslov index counts the number of crossings (up to dimension) of the path of Lagrangian pairs $\lambda\mapsto(\mathbb{E}^u_+(\ell,\lambda),\mathbb{E}^s_+(\ell,\lambda))$, $\lambda\in[\varepsilon,\lambda_\infty]$, for $\varepsilon>0$ small enough. But this is precisely a count (with negative sign) of the number of positive eigenvalues of L_+ up to multiplicity, i.e. equation (3.156) holds.

For the path $\lambda\mapsto(\mathbb{E}^u_-(\ell,\lambda),\mathbb{E}^s_-(\ell,\lambda))$, $\lambda\in[0,\lambda_\infty]$ the argument is similar, where now the Maslov index counts, with *negative* sign, the number of crossings along Γ_2. The sign change results from the fact that λ now appears with positive sign in the first order system (3.98), so that

$$\partial_\lambda A_-(x;\lambda)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}.\tag{3.166}$$

The associated crossing form will then be negative, and by the same reasoning as before equation (3.157) follows. □

The following lemma shows that there are no crossings along Γ_3 and Γ_4.

Lemma 3.27. *We have $\mathbb{E}^u_+(x,\lambda_\infty)\cap\mathbb{E}^s_+(\ell,\lambda_\infty)=\{0\}$ for all $x\in\mathbb{R}$, provided both $\lambda_\infty>0$ and $\ell>0$ are large enough. In addition, $\mathbb{U}_+(\lambda)\cap\mathbb{E}^s_+(\ell,\lambda)=\{0\}$ for all $\lambda\geq 0$ provided $\ell>0$ is large enough. Therefore*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty),\mathbb{E}^s_+(\ell,\lambda_\infty);[-\infty,\ell])=\operatorname{Mas}(\mathbb{U}_+(\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])=0. \tag{3.167}$$

Similar statements hold for the paths $x\mapsto\big(\mathbb{E}^u_-(x,\lambda_\infty),\mathbb{E}^s_-(\ell,\lambda_\infty)\big)$ and $\lambda\mapsto\big(\mathbb{U}_+(\lambda),\mathbb{E}^s_+(\ell,\lambda)\big)$.

Proof. The strategy of the following proof mirrors the one given in [Cor19, §4] (see also [AGJ90, §3 and §5.B]).

For the first statement, we begin by noting that $\operatorname{Spec}(L_+)$ is bounded from above. To see this, note that we can write

$$L_+=D+V,\qquad D=-\partial_{xxxx}-\sigma_2\partial_{xx},\quad V=-\beta+3\phi(x)^2, \tag{3.168}$$

where $\operatorname{dom}(D)=\operatorname{dom}(V)=\operatorname{dom}(L_+))=H^4(\mathbb{R})$, so that $D=D^*$ is selfadjoint and V is bounded and symmetric on $L^2(\mathbb{R})$. It can be shown that D has no point spectrum, and moreover, $\operatorname{Spec}(D)=\operatorname{Spec}_{\text{ess}}(D)=(-\infty,1/4]$ if $\sigma_2=1$, and $\operatorname{Spec}(D)=(-\infty,0]$ if $\sigma=-1$. It then follows from [Kat80, Theorem V.4.10, p.291] that

$$\operatorname{dist}\left(\operatorname{Spec}(L_+),\operatorname{Spec}(D)\right)\leq\|V\|, \tag{3.169}$$

so that $\operatorname{Spec}(L_+)\subseteq(-\infty,\|V\|]$. Consequently, we have $\mathbb{E}^u_+(\ell,\lambda)\cap\mathbb{E}^s_+(\ell,\lambda)=\{0\}$ for all $\lambda>\|V\|$.

Next, we claim that there exists a $\lambda_\infty>\|V\|$ such that

$$\mathbb{E}^u_+(x,\lambda)\cap\mathbb{S}_+(\lambda)=\{0\} \tag{3.170}$$

for all $x\in\mathbb{R}$ and all $\lambda\geq\lambda_\infty$. Once this is shown, it follows that there exists an $\ell_\infty\gg 1$ such that

$$\mathbb{E}^u_+(x,\lambda_\infty)\cap\mathbb{E}^s_+(\ell,\lambda_\infty)=\{0\} \tag{3.171}$$

for all $x\in\mathbb{R}$ and all $\ell\geq\ell_\infty$, because $\lim_{x\to\infty}\mathbb{E}^s_+(x,\lambda)=\mathbb{S}_+(\lambda)$. It remains to prove the claim. We mimic the proof of [Cor19, Lemma 4.1]. Consider then the change of variables:

$$y=\lambda^{1/4}x,\qquad\widetilde{u}_1=u_1,\quad\widetilde{u}_2=\lambda^{1/2}u_2,\quad\widetilde{u}_3=\lambda^{1/4}u_3,\quad\widetilde{u}_4=\lambda^{-1/4}u_4, \tag{3.172}$$

under which the system (3.95) becomes

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix}=\begin{pmatrix}0&0&\frac{\sigma_2}{\sqrt{\lambda}}&1\\0&0&1&0\\1&-\frac{\sigma_2}{\sqrt{\lambda}}&0&0\\-\frac{\sigma_2}{\sqrt{\lambda}}&\frac{\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right)}{\lambda}-1&0&0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix} \tag{3.173}$$

(recall that $\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right) = 3\phi(\frac{y}{\sqrt[4]{\lambda}})^2 - \beta + 1$). Taking $y \to \pm\infty$, the asymptotic system for (3.173) is given by

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix} = \begin{pmatrix}0 & 0 & \frac{\sigma_2}{\sqrt{\lambda}} & 1\\ 0 & 0 & 1 & 0\\ 1 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 0\\ -\frac{\sigma_2}{\sqrt{\lambda}} & \frac{-\beta+1}{\lambda} - 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}. \tag{3.174}$$

Denote the stable and unstable subspaces for (3.174) by $\widetilde{\mathbb{S}}_+(\lambda)$ and $\widetilde{\mathbb{U}}_+(\lambda)$ respectively, and denote the unstable bundle of (3.173) by $\widetilde{\mathbb{E}}^u_+(y,\lambda)$. Then, we have

$$\mathbb{E}^u_+(x,\lambda) \cap \mathbb{S}_+(\lambda) = \{0\} \iff \widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x,\lambda) \cap \widetilde{\mathbb{S}}_+(\lambda) = \{0\}, \tag{3.175}$$

since $\widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x,\lambda) = M\cdot\mathbb{E}^u(x,\lambda)$ and $\widetilde{\mathbb{S}}_+(\lambda) = M\cdot\mathbb{S}_+(\lambda)$, where $M = \mathrm{diag}\{1, \lambda^{1/2}, \lambda^{1/4}, \lambda^{-1/4}\}$ is the (nonsingular) linear transformation of the dependent variables in (3.172), and "$\cdot$" represents the induced action of M on $\mathbb{R}^4$.

Both the nonautonomous system (3.173) and the autonomous system (3.174) induce flows on $\mathrm{Gr}_2(\mathbb{R}^4)$, the Grassmannian of two dimensional subspaces of $\mathbb{R}^4$. For the flow associated with (3.174), it is known [AGJ90] that $\widetilde{\mathbb{U}}_+(\lambda)$, the invariant subspace associated with eigenvalues of positive real part, is an attracting fixed point. Thus, since $\mathcal{L}(2) \subset \mathrm{Gr}_2(\mathbb{R}^4)$, there exists a trapping region $\mathcal{R} \subset \Lambda(2)$ containing $\widetilde{\mathbb{U}}_+(\lambda)$. By taking λ large enough, we can ensure that the flow induced by (3.173) is as close as we like to that induced by (3.174), because $\phi\left(\frac{y}{\sqrt[4]{\lambda}}\right)^2/\lambda$ – the nonautonomous part of (3.173) – is close to zero. It follows that $\mathcal{R} \subset \mathcal{L}(2)$ is also a trapping region for (3.173). Furthermore, we can choose $\mathcal{R}$ small enough such that $\mathbb{V} \cap \widetilde{\mathbb{S}}_+(\lambda) = \{0\}$ for all $\mathbb{V} \in \mathcal{R}$, uniformly for λ large enough. To see this, note that clearly $\widetilde{\mathbb{S}}_+(\lambda) \cap \widetilde{\mathbb{U}}_+(\lambda) = \{0\}$, while taking $\lambda \to +\infty$ in (3.174) yields

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}, \tag{3.176}$$

which has stable and unstable subspaces $\widetilde{\mathbb{S}}_{+\infty}$ and $\widetilde{\mathbb{U}}_{+\infty}$ with respective frames $(I, -W)$ and (I, W), where

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Thus, in the limit we also have $\widetilde{\mathbb{S}}_{+\infty} \cap \widetilde{\mathbb{U}}_{+\infty} = \{0\}$, so we can choose $\mathcal{R}$ as stated. Finally, we note that if $\lambda > \|V\|$ so that $\lambda \notin \mathrm{Spec}(L_+)$, then by [AGJ90, Lemma 3.7] we have $\lim_{y\to\infty}\widetilde{\mathbb{E}}^u_+(y,\lambda) = \widetilde{\mathbb{U}}_+(\lambda)$. All in all, we conclude that for any $\lambda = \lambda_\infty > \|V\|$ large enough, the trajectory $\widetilde{\mathbb{E}}^u_+(\cdot,\lambda_\infty) : [-\infty,\infty] \to \mathcal{L}(2)$, which starts and finishes at $\widetilde{\mathbb{U}}_+(\lambda_\infty)$, will remain inside $\mathcal{R}$ and thus always be disjoint from $\widetilde{\mathbb{S}}_+(\lambda_\infty)$. This proves the claim.

For the second statement of the lemma, the facts that $\mathbb{U}_+(\lambda) \cap \mathbb{S}_+(\lambda) = \{0\}$ and $\lim_{x\to\infty}\mathbb{E}^s_+(x,\lambda) = \mathbb{S}_+(\lambda)$ imply that there exists an $\ell_0 \gg 1$ such that $\mathbb{U}_+(\lambda) \cap \mathbb{E}^s_+(x,\lambda) = \{0\}$ for all $x \geq \ell_0$. Taking $\ell > \ell_0$ gives the result. □

We are now ready to prove Proposition 3.15. In what follows, we choose $\ell > 0$ and $\lambda_\infty > 0$ large enough so that the statements of Lemma 3.27 hold.

Proof of Proposition 3.15. By homotopy invariance and additivity under concatenation, we have

$$\begin{aligned}\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[-\infty,\ell]) + \operatorname{Mas}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot);[0,\lambda_\infty])\\ - \operatorname{Mas}(\mathbb{E}_+^u(\cdot,\lambda_\infty),\mathbb{E}_+^s(\ell,\lambda_\infty);[-\infty,\ell]) - \operatorname{Mas}(\mathbb{E}_+^u(-\infty,\cdot),\mathbb{E}_+^s(\ell,\cdot);[0,\lambda_\infty]) = 0.\end{aligned} \tag{3.177}$$

From Lemma 3.27 the third and fourth terms on the left hand side vanish. Again using the concatenation property, we find that

$$\begin{aligned}\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[-\infty,\ell-\varepsilon]) + \operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[\ell-\varepsilon,\ell])\\ + \operatorname{Mas}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot);[0,\varepsilon]) + \operatorname{Mas}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot);[\varepsilon,\lambda_\infty]) = 0\end{aligned} \tag{3.178}$$

where $\varepsilon > 0$ is small. The second and third terms of (3.178) represent the contributions to the Maslov index from the conjugate point $(x,\lambda) = (\ell,0)$ at the top left corner of the Maslov box in the x and λ directions respectively. From (3.155), Lemma 3.26 and Definition 3.8 we have

$$\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[\ell-\varepsilon,\ell]) = \operatorname{Mas}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot);[0,\varepsilon]) = 0. \tag{3.179}$$

Lemmas 3.18 and 3.25 imply that

$$\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[-\infty,\ell-\varepsilon]) = -\sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}_+^u(x,0)\cap\mathbb{S}_+(0)\right). \tag{3.180}$$

The previous three equations along with Lemma 3.26 now yield (3.93).

The proof for the Morse index of the L_- operator is similar. This time, crossings along Γ_1 are positive, while crossings along Γ_2 are negative. Arguing as we did for (3.155), we can show that

$$\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[\ell-\varepsilon,\ell]) = \dim\left(\mathbb{E}_+^u(\ell,0)\cap\mathbb{E}_+^s(\ell,0)\right) = 1, \tag{3.181}$$

and from Lemma 3.26 and Definition 3.8 we have

$$\operatorname{Mas}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot);[0,\varepsilon]) = \dim\left(\mathbb{E}_+^u(\ell,0)\cap\mathbb{E}_+^s(\ell,0)\right) = -1. \tag{3.182}$$

The contributions (3.181) and (3.182) to the Maslov index coming from $(x,\lambda) = (\ell,0)$ thus cancel each other out. Applying the same homotopy argument as above yields the formula for Q in the proposition. □

3.5 Proofs of the main results

We now return to the computation of the Maslov indices appearing on the left hand side of (3.90). After computing each, we provide the proofs of Theorems 3.2 and 3.5. We begin with Γ_1 (excluding its endpoint at $x = \ell$).

Lemma 3.28. $\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell-\varepsilon]) = Q - P$, *where* $\varepsilon > 0$ *is small.*

Proof. Recall that when $\lambda = 0$ the eigenvalue equations (3.11) decouple. Consequently, the equations for the u and v components in the first order system (3.22) also decouple. Hence, for each $x \in \mathbb{R}$,

$$\mathbb{E}^u(x,0) = \mathbb{E}^u_+(x,0) \oplus \mathbb{E}^u_-(x,0), \tag{3.183}$$

in the sense that for any $\mathbf{w} \in \mathbb{E}^u(x,0)$ we have

$$\mathbf{w} = \begin{pmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ v_3 \\ 0 \\ v_4 \end{pmatrix}, \tag{3.184}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^\top \in \mathbb{E}^u_+(x,0)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)^\top \in \mathbb{E}^u_-(x,0)$. By the same reasoning, for the reference plane we have

$$\mathbb{E}^s(\ell,0) = \mathbb{E}^s_+(\ell,0) \oplus \mathbb{E}^s_-(\ell,0). \tag{3.185}$$

Now using property (3) of Proposition 3.9, we have

$$\begin{aligned} \operatorname{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty, \ell-\varepsilon]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty, \ell-\varepsilon]) \\ + \operatorname{Mas}(\mathbb{E}^u_-(\cdot,0), \mathbb{E}^s_-(\ell,0); [-\infty, \ell-\varepsilon]), \end{aligned} \tag{3.186}$$

and the result follows combining equations (3.180) and (3.93) (and the accompanying statements for L_-). □

Next, we show that there are no crossings along Γ_3 and Γ_4.

Lemma 3.29. *There exists $\ell_1 \gg 1$ such that $\mathbb{E}^u(x,\lambda_\infty) \cap \mathbb{E}^s(\ell,\lambda_\infty) = \{0\}$ for all $x \in \mathbb{R}$ and all $\ell \geq \ell_1$, provided $\lambda_\infty > 0$ is large enough. Therefore, for all $\ell \geq \ell_1$,*

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty), \mathbb{E}^s(\ell,\lambda_\infty); [-\infty, \ell]) = 0.$$

In addition, $\mathbb{U}(\lambda) \cap \mathbb{E}^s(\ell,\lambda) = \{0\}$ for all $\lambda \geq 0$ provided $\ell > 0$ is large enough. Consequently,

$$\operatorname{Mas}(\mathbb{U}(\cdot), \mathbb{E}^s(\ell,\cdot); [0, \lambda_\infty]) = 0.$$

Proof. For the first assertion, note that N is a bounded perturbation of a skew-selfadjoint operator, so that its spectrum lies in a vertical strip around the imaginary axis in the complex plane. More precisely, we have that

$$iN = \widetilde{D} + \widetilde{V}, \quad \widetilde{D} = i\begin{pmatrix} 0 & \partial_{xxxx} + \sigma_2\partial_{xx} \\ -\partial_{xxxx} - \sigma_2\partial_{xx} & 0 \end{pmatrix}, \quad \widetilde{V} = i\begin{pmatrix} 0 & \beta - \phi^2 \\ -\beta + 3\phi^2 & 0, \end{pmatrix} \tag{3.187}$$

where, with $\mathrm{dom}(\widetilde{D}) = \mathrm{dom}(\widetilde{V}) = \mathrm{dom}(N)$, $\widetilde{D}^* = \widetilde{D}$ is selfadjoint and $\widetilde{V}$ is bounded. Now using [Kat80, Remark 3.2, p.208] and [Kat80, eq. (3.16), p.272], we may conclude that

$$\zeta \in \mathrm{Spec}(\widetilde{D} + \widetilde{V}) \implies |\mathrm{Im}(\zeta)| \leq \|\widetilde{V}\|. \tag{3.188}$$

By the spectral mapping theorem, $\mathrm{Spec}(iN) = i\,\mathrm{Spec}(N)$. It follows that

$$\lambda \in \mathrm{Spec}(N) \implies |\mathrm{Re}(\lambda)| \leq \|\widetilde{V}\|. \tag{3.189}$$

Thus, for all $\lambda > \|\widetilde{V}\|$ we have $\mathbb{E}^u(\ell, \lambda) \cap \mathbb{E}^s(\ell, \lambda) = \{0\}$.

The proof now follows from the same arguments used to prove the first assertion in Lemma 3.27. Namely, via the change of variables (3.172) along with

$$\widetilde{v}_1 = v_1, \quad \widetilde{v}_2 = \lambda^{1/2} v_2, \quad \widetilde{v}_3 = \lambda^{1/4} v_3, \quad \widetilde{v}_4 = \lambda^{-1/4} v_4 \tag{3.190}$$

we can rewrite (3.22) as

$$\frac{d}{dy}\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & & \frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1 & 0 \\ & 0 & & & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & & & & \\ 0 & -1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & & & & \\ -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & \frac{\alpha(x)}{\lambda} & 1 & & 0 & & \\ 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 1 & \frac{\eta(x)}{\lambda} & & & & \end{array}\right) \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix}. \tag{3.191}$$

Again, the flow of the associated asymptotic system is close to that of (3.191) for large λ. From the transversality of the four dimensional stable and unstable subspaces of the limiting system of (3.191) as $\lambda \to \infty$, i.e.

$$\frac{d}{dy}\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & & 0 & 0 & 1 & 0 \\ & 0 & & & 0 & 0 & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & & & & \\ 0 & -1 & 0 & 0 & & 0 & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \end{array}\right) \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix}, \tag{3.192}$$

one can show that there exists a $\lambda_\infty > \|\widetilde{V}\|$ such that $\mathbb{E}^u(x, \lambda)$ and $\mathbb{S}(\lambda)$ are transverse for all $x \in \mathbb{R}$ and all $\lambda \geq \lambda_\infty$. Hence $\mathbb{E}^u(x, \lambda)$ and $\mathbb{E}^s(\ell, \lambda_\infty)$ are transverse for all $x \in \mathbb{R}, \ell \geq \ell_\infty$ and $\lambda \geq \lambda_\infty$. The second assertion follows from the same arguments used to prove the second assertion in Lemma 3.27. $\square$

For the proof of Theorem 3.2, it remains to compute

$$\mathfrak{c} := \mathrm{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[\ell-\varepsilon,\ell]) + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]), \tag{3.193}$$

the contribution to the Maslov index from the conjugate point $(x,\lambda)=(\ell,0)$. For the contribution in the x direction, i.e. the arrival along Γ_1, again using property (3) of Proposition 3.9 and equations (3.179) and (3.181), we have

$$\begin{aligned}\mathrm{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[\ell-\varepsilon,\ell]) &= \mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])\\ &\quad+\mathrm{Mas}(\mathbb{E}^u_-(\cdot,0),\mathbb{E}^s_-(\ell,0);[\ell-\varepsilon,\ell]),\\ &=1.\end{aligned} \tag{3.194}$$

To determine the contribution in the λ direction given by the departure along Γ_2 (the second term on the right hand side of (3.193)), we will compute crossing forms. To that end, suppose $\lambda=\lambda_0$ is a crossing of the Lagrangian pair $\lambda\mapsto(\mathbb{E}^u(\ell,\lambda),\mathbb{E}^s(\ell,\lambda))$, $\lambda\in[0,\lambda_\infty]$. The first-order relative crossing form ((3.72) with $k=1$) is given by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = \mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) - \mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,\lambda_0))(q), \tag{3.195}$$

where $q\in\mathbb{E}^u(\ell,\lambda_0)\cap\mathbb{E}^s(\ell,\lambda_0)$ is fixed. We compute each of these terms separately.

The first term concerns the path $\lambda\mapsto\mathbb{E}^u(\ell,\lambda)$ with reference plane $\mathbb{E}^s(\ell,\lambda_0)$. The first-order form (3.59) is given here by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^u(\lambda)q,q)\big|_{\lambda=\lambda_0},\quad q\in\mathbb{E}^u(\ell,\lambda_0)\cap\mathbb{E}^s(\ell,\lambda_0), \tag{3.196}$$

where $R^u(\lambda):\mathbb{E}^s(\ell,\lambda_0)\to\mathbb{E}^s(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^u(\ell,\lambda)=\mathrm{graph}(R^u(\lambda))=\{q+R^u(\lambda)q : q\in\mathbb{E}^s(\ell,\lambda_0)\}$ for all $\lambda\in[\lambda_0-\varepsilon,\lambda_0+\varepsilon]$. Fixing some $q\in\mathbb{E}^s(\ell,\lambda_0)\cap\mathbb{E}^u(\ell,\lambda_0)$, let $r(\lambda)=q+R^u(\lambda)q\in\mathbb{E}^u(\ell,\lambda)$. From the definition of $\mathbb{E}^u(\ell,\lambda)$, there exists a one-parameter family of solutions $\lambda\mapsto\mathbf{w}(\cdot;\lambda)$ to (3.22) satisfying $\mathbf{w}(x;\lambda)\to0$ as $x\to-\infty$ such that $r(\lambda)=\mathbf{w}(\ell;\lambda)$. Furthermore, $r(\lambda_0)=q=\mathbf{w}(\ell;\lambda_0)$ because $(\ker R(x_0))\cap\mathbb{E}^s(\ell,\lambda_0)=\mathbb{E}^u(\ell,\lambda_0)\cap\mathbb{E}^s(\ell,\lambda_0)$ (recall (3.53)). With this family we can write

$$\begin{aligned}\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) &= \frac{d}{d\lambda}\omega(R^u(\lambda)q,q)\big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(q+R^u(\lambda)q,q)\big|_{\lambda=\lambda_0},\\ &=\omega\Big(\frac{d}{d\lambda}\mathbf{w}(\ell,\lambda),\mathbf{w}(\ell,\lambda_0)\Big)\big|_{\lambda=\lambda_0}.\end{aligned}$$

A calculation similar to (3.160) with

$$\partial_\lambda A(x;\lambda)=\begin{pmatrix}0_4 & 0_4\\ M & 0_4\end{pmatrix},\qquad M=\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{3.197}$$

and $\mathbf{w}=(u_1,v_2,u_2,v_2,u_3,v_3,u_4,v_4)^\top$ yields

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q_0) = -2\int_{-\infty}^{\ell}u_2(x;\lambda_0)v_2(x;\lambda_0)\,dx.$$

The second term in (3.195) concerns the path $\lambda \mapsto \mathbb{E}^s(\ell, \lambda)$ with reference plane $\mathbb{E}^u(\ell, 0)$. We have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell, \cdot), \mathbb{E}^s(\ell, \lambda_0))(q) = \frac{d}{d\lambda}\omega(R^s(\lambda)q, q)\big|_{\lambda=\lambda_0}, \quad q \in \mathbb{E}^s(\ell, \lambda_0) \cap \mathbb{E}^u(\ell, \lambda_0), \tag{3.198}$$

where $R^s(\lambda) : \mathbb{E}^u(\ell, \lambda_0) \to \mathbb{E}^u(\ell, \lambda_0)^\perp$ uniquely satisfies $\mathbb{E}^s(\ell, \lambda) = \mathrm{graph}\,(R^s(\lambda))$. For the same fixed $q \in \mathbb{E}^u(\ell, \lambda_0) \cap \mathbb{E}^s(\ell, 0)$ as before, associated to the curve $t(\lambda) = q + R^s(\lambda)q \in \mathbb{E}^s(\ell, \lambda)$ is a family of solutions $\lambda \mapsto \widetilde{\mathbf{w}}(\cdot; \lambda)$ to (3.22), such that $t(\lambda) = \widetilde{\mathbf{w}}(\ell; \lambda)$ and $t(\lambda_0) = q = \widetilde{\mathbf{w}}(\ell; \lambda_0)$. Arguing as for the first term of (3.195), but noting that now $\widetilde{\mathbf{w}}(x; \lambda) \to 0$ as $x \to +\infty$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell, \cdot), \mathbb{E}^u(\ell, \lambda_0))(q) = \omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{w}}(\ell; \lambda), \widetilde{\mathbf{w}}(\ell; \lambda)\Big)\Big|_{\lambda=\lambda_0} = 2\int_\ell^\infty \widetilde{u}_2(x; \lambda_0)\widetilde{v}_2(x; \lambda_0)\, dx.$$

Using uniqueness of solutions as in the proof of Lemma 3.26, we conclude

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))(q) = -2\int_{-\infty}^\infty u_2(x; \lambda_0)v_2(x; \lambda_0)\, dx. \tag{3.199}$$

Remark 3.30. The form (3.199) is *not* sign definite, and therefore the Maslov index does not afford an exact count of the crossings of the path $\lambda \mapsto (\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))$ for $\lambda \in [0, \lambda_\infty]$. This will be the reason for the inequality (and not an equality) in (3.17) in Theorem 3.2.

Let us now evaluate the form (3.199) at $\lambda = 0$. Note that because $\dim(\mathbb{E}^u(x, 0) \cap \mathbb{E}^s(x, 0)) = 2$ (c.f. Hypothesis 3.1) where

$$\mathbb{E}^u(x, 0) \cap \mathbb{E}^s(x, 0) = \mathrm{span}\{\boldsymbol{\phi}(x), \boldsymbol{\varphi}(x)\},$$

it suffices to evaluate (3.199) on the vectors $\boldsymbol{\phi}(x)$ and $\boldsymbol{\varphi}(x)$ from (3.85). Writing $\mathbf{w}(x; 0) = \boldsymbol{\phi}(x)k_1 + \boldsymbol{\varphi}(x)k_2$ for some $k_1, k_2 \in \mathbb{R}$, so that $u_2(x; 0) = \phi'(x)k_1$ and $v_2(x; 0) = -\phi(x)k_2$, we have

$$\mathfrak{m}_0(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))(q) = 2\int_{-\infty}^\infty \phi'\phi\, dx\, k_1k_2 = \int_{-\infty}^\infty \frac{d}{dx}\phi^2\, dx\, k_1k_2 = 0, \tag{3.200}$$

since $\phi \in H^4(\mathbb{R})$. That is, the two dimensional crossing form (3.195) is identically zero at $\lambda_0 = 0$, and the conjugate point $(\ell, 0)$ is non-regular in the λ direction. We therefore need to compute higher order crossing forms.

As discussed in Section 3.3.2, in the case that the first-order form is identically zero, the second-order relative crossing form is given by

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))(q) = \mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, 0))(q) - \mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^s(\ell, \cdot), \mathbb{E}^u(\ell, 0))(q), \tag{3.201}$$

where $q \in W_2 = \mathbb{E}^u(\ell, 0) \cap \mathbb{E}^s(\ell, 0)$. Each of the crossing forms on the right hand side are computed separately with (3.63). For the first, using the same one-parameter family $\lambda \to \mathbf{w}(\cdot; \lambda)$ as we did for the corresponding first-order form (3.196) (i.e. such that $\mathbf{w}(\ell; \lambda) = r(\lambda)$; see the paragraph following (3.196)), we have

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \lambda_0))(q) = \frac{d^2}{d\lambda^2}\omega(R^u(\lambda)q, q)\big|_{\lambda=\lambda_0} = \omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell, \lambda), \mathbf{w}(\ell, \lambda_0)\Big)\big|_{\lambda=\lambda_0}.$$

Now

$$
\begin{aligned}
\omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell;\lambda),\mathbf{w}(\ell;\lambda)\Big) &= \int_{-\infty}^{\ell} \partial_x\,\omega(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))dx,\\
&= \int_{-\infty}^{\ell} \omega\left(\partial_{\lambda\lambda}\left[A(x;\lambda)\mathbf{w}(x;\lambda)\right],\mathbf{w}(x;\lambda)\right)\\
&\qquad + \omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),A(x;\lambda)\mathbf{w}(x;\lambda)\right)dx,\\
&= \int_{-\infty}^{\ell} \omega(A_{\lambda\lambda}(x;\lambda)\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\
&\qquad + 2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\
&\qquad\quad + \omega(A(x;\lambda)\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\
&\qquad\qquad + \omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),A(x;\lambda)\mathbf{w}(x;\lambda)\right)\,dx,\\
&= \int_{-\infty}^{\ell} \langle[A(x;\lambda)^\top J + JA(x;\lambda)]\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda)\rangle\\
&\qquad + 2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\,dx,\\
&= 2\int_{-\infty}^{\ell} \omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))dx,
\end{aligned}
\tag{3.202}
$$

where we used (3.34) and $A_{\lambda\lambda}(x;\lambda)=0$. Using (3.197) and evaluating at $\lambda=0$, we see that

$$
\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,0))(q) = -2\int_{-\infty}^{\ell} u_2(x;0)\partial_\lambda v_2(x;0)+v_2(x;0)\partial_\lambda u_2(x;0)\,dx. \tag{3.203}
$$

For the second form in the right hand side of (3.201), we use the same one-parameter family $\lambda\to\widetilde{\mathbf{w}}(\cdot;\lambda)$ defined in the paragraph following (3.198) (i.e. such that $\widetilde{\mathbf{w}}(\ell;\lambda)=t(\lambda)$) and the same argument used to arrive at (3.203) to obtain

$$
\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,0))(q) = 2\int_{\ell}^{\infty} \widetilde{u}_2(x;0)\partial_\lambda\widetilde{v}_2(x;0)+\widetilde{v}_2(x;0)\partial_\lambda\widetilde{u}_2(x;0)\,dx. \tag{3.204}
$$

By uniqueness of solutions we have $\mathbf{w}(\cdot;0)=\widetilde{\mathbf{w}}(\cdot;0)$. On the other hand, it is not immediately obvious whether the same is true for the functions $\widehat{u}_2(x)=\partial_\lambda u_2(x;0)$ and $\widehat{v}_2(x)=\partial_\lambda v_2(x;0)$. However, observe that with (3.196) and (3.198), we can write the relative crossing form (3.195) as

$$
\mathfrak{m}_{\lambda_0}\left(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot)\right)(q) = \omega(q,\big(\dot{R}^u(0)-\dot{R}^s(0)\big)q), \qquad q\in\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0),
$$

where dot denotes $d/d\lambda$. This form is identically zero if and only if $J(\dot{R}^u(0)-\dot{R}^s(0))$ is the zero operator on $\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$. From the invertibility of J, it follows that $\dot{R}^u(0)q=\dot{R}^s(0)q$ for all $q\in\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$. Recalling that $\mathbf{w}(\ell;\lambda)=r(\lambda)=q+R^u(\lambda)q$ and $\widetilde{\mathbf{w}}(\ell;\lambda)=t(\lambda)=q+R^s(\lambda)q$, taking λ derivatives and evaluating at $\lambda=0$ yields

$$
\partial_\lambda\mathbf{w}(\ell;0)=\dot{r}(0)=\dot{R}^u(0)q=\dot{R}^s(0)q=\dot{t}(0)=\partial_\lambda\widetilde{\mathbf{w}}(\ell;0). \tag{3.205}
$$

Now, both $\partial_\lambda\mathbf{w}(\cdot;0)$ and $\partial_\lambda\widetilde{\mathbf{w}}(\cdot;0)$ solve the inhomogeneous differential equation

$$
\frac{d}{dx}\left(\partial_\lambda\mathbf{w}\right)=A\left(\partial_\lambda\mathbf{w}\right)+A_\lambda\left(\boldsymbol{\phi}\,k_1+\boldsymbol{\varphi}\,k_2\right), \tag{3.206}
$$

obtained by differentiating (3.23) with respect to λ and evaluating at $\lambda = 0$, and using that $\mathbf{w}(\cdot;0) = \boldsymbol{\phi}\, k_1 + \boldsymbol{\varphi}\, k_2$. (Note that $k_1, k_2 \in \mathbb{R}$ are determined by the fixed vector q, where $q = \mathbf{w}(\ell;0) = \boldsymbol{\phi}(\ell)k_1 + \boldsymbol{\varphi}(\ell)k_2$.) It follows from (3.205) and uniqueness of solutions of (3.206) that indeed $\partial_\lambda \mathbf{w}(x;0) = \partial_\lambda \widetilde{\mathbf{w}}(x;0)$ for *all* $x \in \mathbb{R}$. Collecting (3.203) and (3.204) together, (3.201) becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = -2\int_{-\infty}^{\infty} u_2(x;0)\partial_\lambda v_2(x;0) + v_2(x;0)\partial_\lambda u_2(x;0)\,dx. \tag{3.207}$$

We need to understand the function $\partial_\lambda \mathbf{w}(\cdot;0)$. Notice that it solves the inhomogeneous equation (3.206) if and only if its third and fourth entries $\partial_\lambda u_2(\cdot;0)$ and $\partial_\lambda v_2(\cdot;0)$ solve

$$N\begin{pmatrix}\partial_\lambda u_2(\cdot;0)\\ -\partial_\lambda v_2(\cdot;0)\end{pmatrix} = \begin{pmatrix}\phi_x\, k_1\\ -\phi\, k_2\end{pmatrix}. \tag{3.208}$$

This follows from differentiating the eigenvalue equation (3.12) with respect to λ, evaluating at $\lambda = 0$ and making the substitutions

$$\partial_\lambda u(\cdot;0) = \partial_\lambda u_2(\cdot;0), \quad \partial_\lambda v(\cdot;0) = -\partial_\lambda v_2(\cdot;0), \quad u(\cdot;0) = \phi_x\, k_1, \quad v(\cdot;0) = -\phi\, k_2.$$

Now, both equations

$$\begin{aligned} -L_-\partial_\lambda v_2(\cdot;0) &= -\phi_x\, k_1,\\ L_+\partial_\lambda u_2(\cdot;0) &= -\phi\, k_2, \end{aligned} \tag{3.209}$$

are solvable by virtue of the Fredholm alternative, since $\langle \phi', \phi\rangle_{L^2(\mathbb{R})} = 0$ and hence $\phi_x \in \ker(L_-)^\perp$ and $\phi \in \ker(L_+)^\perp$. Denoting by $\widehat{v}$ and $\widehat{u}$ any solutions to

$$-L_- v = \phi_x \qquad \text{and} \qquad L_+ u = \phi \tag{3.210}$$

in $H^4(\mathbb{R})$ respectively (note the sign change in both equations from (3.209)), (3.207) becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = 2\left(\int_{-\infty}^{\infty}\phi_x\,\widehat{v}\,dx\right)k_1^2 - 2\left(\int_{-\infty}^{\infty}\phi\,\widehat{u}\,dx\right)k_2^2, \tag{3.211}$$

recalling that $u_2 = \phi_x k_1$ and $v_2 = -\phi\, k_2$. Having computed the form, we count the number of negative squares. Using (3.75), and defining $\mathcal{I}_1$ and $\mathcal{I}_2$ to be the integrals appearing in the first and second terms of (3.211) respectively (as in (3.16)), we find that

$$\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]) = -n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 0 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0,\\ -1 & \mathcal{I}_1\mathcal{I}_2 > 0,\\ -2 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0.\end{cases} \tag{3.212}$$

Recalling the definition of $\mathfrak{c}$ in (3.193) and using (3.194) yields the following.

Lemma 3.31. *The value of $\mathfrak{c}$ is given by*

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0, \\ 0 & \mathcal{I}_1\mathcal{I}_2 > 0, \\ -1 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0. \end{cases} \tag{3.213}$$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. By homotopy invariance and additivity under concatenation, we have

$$\begin{aligned} &\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell]) + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty]) \\ &\qquad - \operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty),\mathbb{E}^s(\ell,\lambda_\infty);[-\infty,\ell]) - \operatorname{Mas}(\mathbb{E}^u(-\infty,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty]) = 0. \end{aligned}$$

By Lemma 3.29 the last two terms on the left hand side vanish. Recalling the definition of $\mathfrak{c}$ from (3.91) and using the concatenation property once more,

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell-\varepsilon]) + \mathfrak{c} + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty]) = 0. \tag{3.214}$$

Since the Maslov index counts *signed* crossings, the number of crossings along Γ_2 for $\lambda > 0$ is bounded from below by the absolute value of the Maslov index of this piece, i.e.

$$n_+(N) \geq |\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty])|. \tag{3.215}$$

Combining (3.214) and (3.215) with Lemma 3.28, the inequality (3.17) follows. The statement of the theorem then follows from the computation of $\mathfrak{c}$ in Lemma 3.31. □

Remark 3.32. It may be more tractable to compute P and Q via Proposition 3.15. Thus, an alternate form of (3.17), which may be more useful in practice, is given by

$$n_+(N) \geq \left| \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right) - \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_-(x,0)\cap\mathbb{S}_-(0)\right) - \mathfrak{c} \right|. \tag{3.216}$$

We conclude with the proof of Theorem 3.5, for which we will need the following lemma. The first assertion gives a sufficient condition for monotonicity of the Maslov index along Γ_2, and is adapted from [CCLM23, Lemma 5.1]. The second assertion is given in [CCLM23, Lemma 5.2].

Lemma 3.33. *If L_- is a nonpositive operator, then each crossing $\lambda = \lambda_0 > 0$ of the path $\lambda \mapsto (\mathbb{E}^u(\ell,\lambda),\mathbb{E}^s(\ell,\lambda))$ is positive. Moreover, in this case $\operatorname{Spec}(N) \subset \mathbb{R}\cup i\mathbb{R}$.*

Proof. If $\lambda = \lambda_0$ is a crossing then the eigenvalue equations

$$-L_-v = \lambda_0 u, \qquad L_+u = \lambda_0 v \tag{3.217}$$

are satisfied for some $\tilde{u},\tilde{v} \in H^4(\mathbb{R})$. Notice that $\lambda_0 > 0$ necessitates that *both* $\tilde{u}$ and $\tilde{v}$ are nontrivial.

Solving the first equation in (3.217) yields $\tilde{v} = \alpha\phi + \tilde{v}_\perp$ for some $\alpha \in \mathbb{R}$, where $\ker(L_-) = \operatorname{span}\{\phi\}$ and $\tilde{v}_\perp \in \ker(L_-)^\perp$. Therefore

$$\langle L_-\tilde{v}, \tilde{v}\rangle_{L^2(\mathbb{R})} = \langle L_-(\alpha\phi + \tilde{v}_\perp), \alpha\phi + \tilde{v}_\perp\rangle_{L^2(\mathbb{R})} = \langle L_-\tilde{v}_\perp, \tilde{v}_\perp\rangle_{L^2(\mathbb{R})} < 0 \tag{3.218}$$

because L_- is nonpositive and $\hat{v}_\perp \in \ker(L_-)^\perp$. Now analysing the crossing form (3.199) for the path $\lambda \mapsto (\mathbb{E}^u(\ell, \lambda), \mathbb{E}^s(\ell, \lambda))$, where $v_2 = -\tilde{v}$ and $u_2 = \tilde{u}$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))(q) = -\frac{2}{\lambda_0}\int_{-\infty}^{\infty} (\lambda_0\, u_2)\, v_2\, dx = -\frac{2}{\lambda_0}\langle L_-\tilde{v}, \tilde{v}\rangle_{L^2(\mathbb{R})} > 0,$$

which was to be proven. The second statement may be proven using similar arguments as in the proof of [CCLM23, Lemma 5.1]. Namely, we can rewrite (3.12) as the selfadjoint eigenvalue problem

$$(-L_-|_{X_c})^{1/2}\, \Pi L_+ \Pi\, (-L_-|_{X_c})^{1/2}\, w = \lambda^2 w, \tag{3.219}$$

where $X_c = \ker(L_-)^\perp$, Π is the orthogonal projection in $L^2(\mathbb{R})$ onto X_c, $(-L_-|_{X_c})^{1/2}$ is well-defined because $-L_-$ is nonnegative, and $w = (-L_-|_{X_c})^{1/2}\, \Pi v$. It follows that $\lambda^2 \in \mathbb{R}$. For more details on the equivalence of (3.12) with (3.219), see [CCLM23, Lemma 3.21]. We omit the details here. □

Proof of Theorem 3.5. If $Q = 0$ then it follows from Lemma 3.33 that

$$\operatorname{Mas}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot); [\varepsilon, \lambda_\infty]) = n_+(N) \tag{3.220}$$

for ε small enough. Using this and Lemma 3.28 in (3.214), we obtain

$$n_+(N) = P - Q - \mathfrak{c} = 1 - \mathfrak{c}. \tag{3.221}$$

For the evaluation of $\mathfrak{c}$, using (3.210) we can write

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} \phi_x\, \widehat{v}\, dx = -\int_{-\infty}^{\infty} (L_-\widehat{v})\, \widehat{v}\, dx, \tag{3.222}$$

so that if $Q = 0$ then $\mathcal{I}_1 \geq 0$. An argument similar to (3.218) shows that in fact $\mathcal{I}_1 > 0$. Lemma 3.31 now yields the value of $\mathfrak{c}$. In particular, if $\mathcal{I}_2 > 0$ then $\mathfrak{c} = 0$ and $n_+(N) = 1$, and the standing wave $\widehat{\psi}$ is unstable. If, on the other hand, $\mathcal{I}_2 < 0$, then $\mathfrak{c} = 1$ and $n_+(N) = 0$. By the second assertion of Lemma 3.33, this means $\operatorname{Spec}(N) \subset i\mathbb{R}$, so that $\widehat{\psi}$ is spectrally stable. □

Remark 3.34. If either $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$, the second order form (3.211) is degenerate. In this case one would need to determine the signature of crossing forms $\mathfrak{m}^{(k)}_{\lambda_0}(q)$ with $k \geq 3$ in order to compute $\mathfrak{c}$. If both $\mathcal{I}_1 = \mathcal{I}_2 = 0$ then (3.211) is identically zero. In this case the third-order form will in fact also be identically zero. One would then need to determine the number of negative squares of the fourth-order form, provided it is nondegenerate.

Chapter 4

Additional notes and future directions

4.1 Notes on the second-order problem

4.1.1 Alternate boundary conditions

In Chapter 2, the lower bound in Theorem 2.2, i.e.

$$n_+(N) \geq |P - Q - \mathfrak{c}|, \tag{4.1}$$

was derived for the eigenvalue problem (2.5) with Dirichlet boundary conditions. This inequality will also hold for the case of Neumann boundary conditions (which falls under the general framework of [KKS04, KP13]), and it may be possible to show this using the Maslov index. In the Dirichlet case, the appearance of "$P - Q$" in the right hand side of (4.1) is a consequence of the fact that the Lagrangian path $s \mapsto \Lambda(0, s) \in \mathcal{L}(4)$ (see (2.35)) is the direct sum of two Lagrangian paths in $\mathcal{L}(2)$, which are monotone but oppositely oriented with respect to the train of the vertical subspace $\{0\} \times \mathbb{R}^2$ of $\mathbb{R}^4$[1] (which encodes Dirichlet conditions). On the other hand, monotonicity of these two paths in $\mathcal{L}(2)$ with respect to the horizontal subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^4$ (which encodes Neumann boundary conditions) is certainly not guaranteed. Nonetheless, it might still be possible to recover the right hand side of (4.1) in the Neumann case through the use of *Hörmander's index*, an enlightening study of which is given in [How21]. In that paper, it is shown how to exchange the reference plane of a Lagrangian path with one with respect to which the path is monotonic (the Maslov index of the monotonic path being straightforward to compute). In the current context, this allows one to exchange the horizontal subspace with the vertical subspace as the reference plane. The only possible issue lies in the requirement of a frame for the endpoint $\Lambda(0, 1)$ of the path $s \mapsto \Lambda(0, s)$, which is required for the computation of Hörmander's index. If this can be resolved, then the approach should in fact be able to handle *any* separated boundary conditions for which the linear operators L_+ and L_- are selfadjoint, since the inequality of [KKS04, Eq. (3.9)] (see (2.172)) i.e.

$$n_+(N) \geq |P - Q - n_-(D_+) + n_-(D_-)| \tag{4.2}$$

[1] a similar argument is used for the Lagrangian path $x \mapsto \mathbb{E}_\pm(x, 0)$ in Section 3.5

holds in these cases. (That the right hand sides of (4.1) and (4.2) are equivalent outside of the Dirichlet case will hold using the same arguments as in the proof of Proposition 2.61.)

The Maslov index may also be able to handle the nonseparated quasi-periodic boundary conditions

$$\begin{pmatrix} u(\ell) \\ v(\ell) \end{pmatrix} = e^{i\theta} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \qquad \begin{pmatrix} u'(\ell) \\ v'(\ell) \end{pmatrix} = e^{i\theta} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}, \qquad \theta \in [0, 2\pi). \tag{4.3}$$

The Morse index for matrix-valued Schrödinger operators with a periodic real-valued symmetric potential and boundary conditions of this type was given as the Maslov index of a related Lagrangian path in the papers [JLM13, JLS17]. In [JLM13], the boundary conditions are encoded as the solutions to a system of differential equations; this system is then appended to the differential equations describing the eigenvalue equations. By rescaling the domain as in [DJ11] (and Chapter 2) with a spatial rescaling parameter s, an eigenvalue is then encoded as the nontrivial intersection of a path of Lagrangian planes, described by solutions to the augmented system, with a certain fixed reference plane. Here, $\theta \in [0, 2\pi)$ is fixed, and two different formulas for the Morse index are given, depending on whether $\theta = 0$ or $\theta \in (0, 2\pi)$. In [JLS17], the authors use a different approach. There, θ is used as the "homotopy" parameter instead of s. In this case, two paths of Lagrangian planes are constructed, one describing the λ-dependent general solutions to the eigenvalue equations, and one describing the θ-dependent boundary conditions. By using a homotopy argument in the $\theta\lambda$-plane, the authors derive a formula for the difference in the eigenvalue counting functions of two Schrödinger operators with the boundary conditions (4.3), which differ only in values of θ. This formula is given in terms of the Maslov index of the θ-dependent Lagrangian path.

Using either of the approaches in [JLM13, JLS17], it should be possible to derive a lower bound of the form of (4.1) for the problem (2.5) with the boundary conditions (4.3). In the context of the linearisation of the NLS equation about a standing wave, for the conditions (4.3) one has $0 \in \mathrm{Spec}(L_-) \cap \mathrm{Spec}(L_+)$. By constructing the eigenvalue curves in the λs- or $\lambda\theta$-plane and analysing them in a neighbourhood of $\lambda = 0$, a computation of $\mathfrak{c}$ akin to that in Section 2.3.3 should be possible. It is worth emphasising, however, that the Maslov index is only able to detect *real* eigenvalues. Complex eigenvalues are therefore invisible to the analysis; it remains an open question how to resolve this issue.

4.1.2 The Maslov index and algebraic multiplicity

Having discussed higher-order crossing forms in Section 3.3.2, in this section we investigate a relationship between the degree of vanishing of the eigenvalue curve $s(\lambda) - s_0$ at some $\lambda = \lambda_0$, the algebraic multiplicity of $s_0^2\lambda_0$ as an eigenvalue of N_{s_0}, and these higher-order crossing forms. We focus only on the case that λ_0 is geometrically simple.

Recall the following notation from Chapter 2. The two-parameter family of Lagrangian planes $\mathbb{R} \times (0, 1] \ni (\lambda, s) \mapsto \Lambda(\lambda, s) \in \mathcal{L}(4)$ is given by (2.35), and the fixed reference plane $\mathcal{D} = 0 \times \mathbb{R}^4$ is the vertical subspace of $\mathbb{R}^8$. Let $(\lambda, s) = (\lambda_0, s_0)$ be a crossing, i.e. $\Lambda(\lambda_0, s_0) \cap \mathcal{D} \neq \{0\}$. Fixing s_0, I write $\Lambda(\lambda, s_0) = \operatorname{graph} R(\lambda) = \{q + R(\lambda)q : q \in \mathcal{D}\}$, where $R(\lambda) : \mathcal{D} \to \mathcal{D}^\perp$. Since $\lambda \mapsto \Lambda(1, \lambda)$ is analytic (Λ is the trace map for solutions of an ordinary differential equation which is analytic in λ), the mapping $\lambda \mapsto R_\lambda$ is also analytic. The kth order crossing form (see (3.54)) is therefore well-defined,

$$\mathfrak{m}_{\lambda_0}^{(k)}(q_0) = \frac{d^k}{d\lambda^k}\,\omega(q_0, R(\lambda)q(\lambda))\Big|_{\lambda=\lambda_0} \qquad q_0 \in W_k, \tag{4.4}$$

where q is any root function of the curve of symmetric bilinear forms $\lambda \mapsto \omega(\cdot, R(\lambda)\cdot)|_{\mathcal{D}\times\mathcal{D}}$ at $\lambda = \lambda_0$, with $\operatorname{ord}(q) \geq k$. (Note we have written R in the second entry of (4.4) to be consistent with Chapter 2; this form will therefore be the negative of (3.54). In addition, we have written $R(\lambda) : \mathcal{D} \to \mathcal{D}^\perp$ instead of $R(\lambda) : \Lambda(\lambda_0, s_0) \to \mathcal{D}^\perp$; this is inconsequential to the calculation of the forms (4.4), which are evaluated on $\mathcal{D} \cap \Lambda(\lambda_0, s_0)$.)

Suppose that the first-order crossing form

$$\mathfrak{m}_{\lambda_0}(q_0) = \frac{d}{d\lambda}\,\omega(q_0, R(\lambda)q(\lambda))\Big|_{\lambda=\lambda_0} = \omega(q_0, \dot{R}(\lambda_0)q_0), \qquad q_0 \in W_1 = \ker R(\lambda_0) = \Lambda(\lambda_0, s_0) \cap \mathcal{D},$$

is identically zero. (Dot denotes $d/d\lambda$.) Then $\dot{R}(\lambda_0)|_{W_1} = 0$, $W_2 = \ker \dot{R}(t_0) = \ker R(t_0) = \Lambda(\lambda_0, s_0) \cap \mathcal{D}$ (see (3.38)), and

$$\begin{aligned}
\mathfrak{m}_{\lambda_0}^{(2)}(q_0) &= \frac{d^2}{d\lambda^2}\,\omega(q_0, R(\lambda)q(\lambda))\Big|_{\lambda=\lambda_0}, && (4.5)\\
&= \omega(q_0, \ddot{R}(\lambda_0)q_0) + \omega(q_0, \dot{R}(\lambda_0)\dot{q}(\lambda_0)) + \omega(q_0, R(\lambda_0)\ddot{q}(\lambda_0)), && (4.6)\\
&= \omega(q_0, \ddot{R}(\lambda_0)q_0), && (4.7)
\end{aligned}$$

where $q_0 \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$ and the final two terms in (4.6) vanish because $q_0 \in \ker R(\lambda_0) \cap \ker \dot{R}(\lambda_0)$. By the same reasoning, if both $\mathfrak{m}_{\lambda_0}(q_0)$ and $\mathfrak{m}_{\lambda_0}^{(2)}(q_0)$ are identically zero, then

$$\mathfrak{m}_{\lambda_0}^{(3)}(q_0) = \omega(q_0, \partial_\lambda^3 R(\lambda_0)q_0), \qquad q_0 \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}.$$

Arguing inductively leads to the following lemma.

Lemma 4.1. *Suppose the kth order crossing form $\mathfrak{m}_{\lambda_0}^{(k)}(q_0)$ given by (4.4) is identically zero for $k = 1, \dots, n-1$. Then the nth order crossing form is given by*

$$\mathfrak{m}_{\lambda_0}^{(n)}(q_0) = \omega\left(q_0, \partial_\lambda^n R(\lambda_0)\, q_0\right), \qquad q_0 \in W_n = \ker R(\lambda_0) = \Lambda(\lambda_0, s_0) \cap \mathcal{D}. \tag{4.8}$$

In the case that $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$, we have the following expression for (4.8) in terms of the Jordan chains of N at $s = s_0$.

Lemma 4.2. *Let (λ_0, s_0) be a crossing such that $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$, and fix any nonzero $q_0 \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$. If the kth order crossing forms (4.4) for $k = 1, \dots, n-1$ $(n \geq 2)$ are all zero, so that (4.8) holds, then the algebraic multiplicity of $s_0^2\lambda_0 \in \operatorname{Spec}(N_{s_0})$ is at least n, and the nth order crossing form (for $n \geq 2$) is given by*

$$\mathfrak{m}_{\lambda_0}^{(n)}(q_0) = -n!\,s_0^{2n-1}\langle \mathbf{v}_{s_0}^{(n-1)}, S\mathbf{u}_{s_0}\rangle, \qquad q_0 = \operatorname{Tr}_{s_0}\mathbf{u}_{s_0}, \tag{4.9}$$

where $\mathbf{u}_{s_0} \in \ker(N_{s_0} - s_0^2\lambda_0)$, and $\mathbf{v}_{s_0}^{(k)} \in \operatorname{dom}(N_{s_0})$ is the kth generalised eigenvector in the Jordan chain associated with $s_0^2\lambda_0 \in \operatorname{Spec} N_{s_0}$).

Proof. Consider an analytic family of vectors $\lambda \mapsto \mathbf{w}_\lambda$ satisfying (2.52), i.e.

$$N_{s_0}\mathbf{w}_\lambda = s_0^2\lambda\mathbf{w}_\lambda, \qquad x \in [0,\ell], \;\; \lambda \in (\lambda_0-\varepsilon, \lambda_0+\varepsilon) \tag{4.10a}$$
$$\operatorname{Tr}_{s_0}\mathbf{w}_\lambda = \operatorname{Tr}_{s_0}\mathbf{u}_{s_0} + R(\lambda)\operatorname{Tr}_{s_0}\mathbf{u}_{s_0}, \qquad \mathbf{w}_{\lambda_0} = \mathbf{u}_{s_0}, \tag{4.10b}$$

where $R(\lambda): \mathcal{D} \longrightarrow \mathcal{D}^\perp$ is such that $\Lambda(\lambda, s_0) = \operatorname{graph}(R(\lambda))$. Setting $q_0 = \operatorname{Tr}_{s_0}\mathbf{u}_{s_0} \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$, from (4.8) and (4.10b) we have

$$\mathfrak{m}^{(n)}_{\lambda_0}(q_0) = \omega\left(q_0, \partial^n_\lambda R(\lambda_0)\, q_0\right) = \omega\left(\operatorname{Tr}_{s_0}\mathbf{u}_{s_0}, \operatorname{Tr}_{s_0}\partial^n_\lambda\mathbf{w}_\lambda\right)\big|_{\lambda=\lambda_0}.$$

Differentiating (4.10a) n times with respect to λ, applying $\langle\cdot, S\mathbf{w}_\lambda\rangle$ and rearranging yields

$$\left\langle (N_{s_0} - s_0^2\lambda)\partial^n_\lambda\mathbf{w}_\lambda, S\mathbf{w}_\lambda\right\rangle = ns_0^2\left\langle\partial^{n-1}_\lambda\mathbf{w}_\lambda, S\mathbf{w}_\lambda\right\rangle.$$

Now using the modified Green's identity (2.33) with $\mathbf{u} = \mathbf{w}_\lambda$ and $\mathbf{v} = \partial^n_\lambda\mathbf{w}_\lambda$, we have

$$s_0\,\omega\left(\operatorname{Tr}_{s_0}\mathbf{w}_\lambda, \operatorname{Tr}_{s_0}\partial^n_\lambda\mathbf{w}_\lambda\right) = \langle (N_{s_0} - s_0^2\lambda)\mathbf{w}_\lambda, S\partial^n_\lambda\mathbf{w}_\lambda\rangle - \langle S\mathbf{w}_\lambda, (N_{s_0} - s_0^2\lambda)\partial^n_\lambda\mathbf{w}_\lambda\rangle.$$

Combining (4.10a) with the previous two equations, we get

$$s_0\,\omega(\operatorname{Tr}_{s_0}\mathbf{w}_\lambda, \operatorname{Tr}_{s_0}\partial^n_\lambda\mathbf{w}_\lambda) = -ns_0^2\left\langle\partial^{n-1}_\lambda\mathbf{w}_\lambda, S\mathbf{w}_\lambda\right\rangle.$$

Evaluating this last equation at $\lambda = \lambda_0$ and dividing through by s_0, we see that

$$\mathfrak{m}^{(n)}_{\lambda_0}(q_0) = \omega(\operatorname{Tr}_{s_0}\mathbf{u}_{s_0}, \operatorname{Tr}_{s_0}\partial^n_\lambda\mathbf{w}_\lambda)\big|_{\lambda=\lambda_0} = -ns_0\left\langle\partial^{n-1}_\lambda\mathbf{w}_{\lambda_0}, S\mathbf{u}_{s_0}\right\rangle. \tag{4.11}$$

Seeking an expression for $\partial^{n-1}_\lambda\mathbf{w}_{\lambda_0}$, we differentiate (4.10a) $n-1$ times with respect to λ, evaluate at $\lambda = \lambda_0$ and rearrange:

$$\left(N_{s_0} - s_0^2\lambda_0\right)\partial^{n-1}_\lambda\mathbf{w}_{\lambda_0} = s_0^2(n-1)\partial^{n-2}_\lambda\mathbf{w}_{\lambda_0}. \tag{4.12}$$

A similar procedure yields inhomogeneous equations satisfied by the functions $\partial^{n-2}_\lambda\mathbf{w}_{\lambda_0}$, $\partial^{n-3}_\lambda\mathbf{w}_{\lambda_0}$, $\dots, \dot{\mathbf{w}}_{\lambda_0}$. Setting

$$\mathbf{v}^{(k)}_{s_0} := \frac{1}{k!s_0^{2k}}\,\partial^k_\lambda\mathbf{w}_\lambda, \qquad k = 1, \dots, n-1, \tag{4.13}$$

and using (4.13) for $k = n-1$ in (4.11), we arrive at the expression (4.9), where the functions $\mathbf{v}^{(k)}_{s_0}$ satisfy

$$(N_{s_0} - s_0^2\lambda_0)\mathbf{v}^{(k)}_{s_0} = \mathbf{v}^{(k-1)}_{s_0}, \qquad k = 2, \dots, n-1, \tag{4.14a}$$
$$(N_{s_0} - s_0^2\lambda_0)\mathbf{v}^{(1)}_{s_0} = \mathbf{u}_{s_0}. \tag{4.14b}$$

We already saw in Lemma 2.23 that $\mathfrak{m}_{\lambda_0}(q_0) = \langle\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle$. Now, if $\mathfrak{m}_{\lambda_0}(q_0) = \langle\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle = 0$, then the Fredholm alternative guarantees that a unique solution $\mathbf{v}^{(1)}_{s_0} \in \operatorname{dom}(N_{s_0})$ to (4.14b) exists, since $\ker(N^*_{s_0} - s_0^2\lambda_0) = \operatorname{span}\{S\mathbf{u}_{s_0}\}$. Similarly, if $\mathfrak{m}^{(2)}_{\lambda_0}(q) = -2s_0^3\langle\mathbf{v}^{(1)}_{s_0}, S\mathbf{u}_{s_0}\rangle = 0$, then (4.14a) for $k = 2$ has a solution $\mathbf{v}^{(2)}_{s_0} \in \operatorname{dom}(N_{s_0})$. Arguing in this way, we find that $s_0^2\lambda_0 \in \operatorname{Spec}(N_{s_0})$ has algebraic multiplicity at least n, with the vectors $\{\mathbf{u}_{s_0}, \mathbf{v}^{(1)}_{s_0}, \dots, \mathbf{v}^{(n-1)}_{s_0}\}$ comprising the first n elements of an associated Jordan chain. □

Next, we write down a Hadamard-type formula for the nth derivative of the eigenvalue curve $s(\lambda)$, at a point (λ_0, s_0) where $s(\lambda) - s_0$ has a critical point of order (at least) n. The following lemma is an extension of the second part of Corollary 2.39, and its proof similarly follows from an application of the implicit function theorem.

Lemma 4.3. *Suppose (λ_0, s_0) is a crossing with $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$ and $\mathfrak{m}_{s_0} \neq 0$. Then, there is an analytic curve $s(\lambda)$ passing through (λ_0, s_0). In addition, if $\dot{s}(\lambda_0) = \ddot{s}(\lambda_0) = ... = \partial_\lambda^{n-1}s(\lambda_0) = 0$ $(n \geq 2)$, then*

$$\partial_\lambda^n s(\lambda_0) = -\frac{\mathfrak{m}_{\lambda_0}^{(n)}(q)}{\mathfrak{m}_{s_0}(q)}. \tag{4.15}$$

Proof. By Proposition 2.37, if $\dim\ker(N_{s_0} - s_0^2\lambda_0) = 1$ then, in a neighbourhood of (λ_0, s_0), there is a scalar-valued function $M(\lambda, s)$, defined in (2.94) and given here by

$$M(\lambda, s) = \left\langle (N_s - s^2\lambda)\big(I + A(\lambda, s)\big)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0} \right\rangle, \tag{4.16}$$

such that $M(\lambda, s) = 0$ if and only if $s^2\lambda \in \operatorname{Spec}(N_s)$. It follows from the definition of $A(\lambda, s)$ (see (2.89)) that $\lambda \mapsto M(\lambda, s)$ is analytic, and from (2.100) we have $\partial_s M(\lambda_0, s_0) = s_0\mathfrak{m}_{s_0}(q)$, which is nonzero by assumption. It follows from the implicit function theorem (see [FG02, pp. 34]) that there is an analytic curve $s(\lambda)$ passing through (λ_0, s_0). Differentiating the equation $M(\lambda, s(\lambda)) = 0$ and evaluating at $\lambda = \lambda_0$, we find

$$\dot{s}(\lambda_0) = -\frac{\partial_\lambda M(\lambda_0, s_0)}{\partial_s M(\lambda_0, s_0)} = -\frac{\mathfrak{m}_{\lambda_0}(q_0)}{\mathfrak{m}_{s_0}(q_0)}, \tag{4.17}$$

for any $q_0 \in \Lambda(\lambda_0, s_0) \cap \mathcal{D}$, where the second equality follows from (2.100). Therefore, $\dot{s}(\lambda_0) = 0$ if and only if $\mathfrak{m}_{\lambda_0} = 0$.

Now differentiating $M(\lambda, s(\lambda)) = 0$ n times with respect to λ, using that $\dot{s}(\lambda_0) = \ddot{s}(\lambda_0) = ... = \partial_\lambda^{n-1}s(\lambda_0) = 0$ and rearranging, we obtain

$$\partial_\lambda^n s(\lambda_0) = -\frac{\partial_\lambda^n M(\lambda_0, s_0)}{\partial_s M(\lambda_0, s_0)}. \tag{4.18}$$

Since $\partial_s M(\lambda_0, s_0) = s_0\mathfrak{m}_{s_0}(q)$, in order to arrive at (4.15) it remains to show that $\partial_\lambda^n M(\lambda_0, s_0) = s_0\mathfrak{m}_{\lambda_0}^{(n)}(q_0)$. Note this also shows that $\partial_\lambda^n M(\lambda_0, s_0) = 0 \iff \partial_\lambda^n s(\lambda_0) = 0$.

Differentiating (4.16) with respect to λ and using that $S\mathbf{u}_{s_0} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$ (as in the proof of Proposition 2.37), we find that

$$\partial_\lambda^k M(\lambda_0, s_0) = -ks_0^2\langle \partial_\lambda^{k-1}A(\lambda_0, s_0)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle, \qquad k = 2, \dots, n-1, \tag{4.19a}$$

$$\partial_\lambda M(\lambda_0, s_0) = -s_0^2\langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle. \tag{4.19b}$$

It follows from the definition of $A(\lambda, s)$ in (2.89) that

$$T(\lambda, s)A(\lambda, s)\mathbf{u}_{s_0} = -(I - P)(N_s - s^2\lambda)\mathbf{u}_{s_0} \tag{4.20}$$

(where T is defined in (2.88), and P is the orthogonal projection onto $\ker(N_{s_0}^* - s_0^2\lambda_0)$). Differentiating (4.20) k times with respect to λ, and using that $\partial_\lambda^k T(\lambda_0, s_0) \equiv 0$ for $k \geq 2$ (which follows from (2.88)) and $A(\lambda_0, s_0)\mathbf{u}_{s_0} = 0$, we find that

$$T(\lambda_0, s_0)\partial_\lambda^k A(\lambda_0, s_0)\mathbf{u}_{s_0} = ks_0^2(I-P)\partial_\lambda^{k-1}A(\lambda_0, s_0)\mathbf{u}_{s_0} \qquad k = 2, \ldots, n-1, \tag{4.21a}$$

$$T(\lambda_0, s_0)\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0} = s_0^2(I-P)\mathbf{u}_{s_0} = s_0^2\mathbf{u}_{s_0}. \tag{4.21b}$$

The second equality in (4.21b) follows since $\dot{s}(\lambda_0) = 0$ implies $\mathfrak{m}_{\lambda_0} = \langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle = 0$ and thus $\mathbf{u}_{s_0} \in \ker(N_{s_0}^* - s_0^2\lambda_0))^\perp = \ker(P)$ (as was already observed in the argument following (2.98)). Defining $\mathbf{y}_{s_0}^{(1)} := \frac{1}{s_0^2}\partial_\lambda A(\lambda_0, s_0)\mathbf{u}_{s_0}$, and noting that

$$T(\lambda_0, s_0)\mathbf{y}_{s_0}^{(1)} = (I-P)(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(1)} = (N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(1)}, \tag{4.22}$$

we see from (4.21b) that $\mathbf{y}_{s_0}^{(1)}$ satisfies $(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(1)} = \mathbf{u}_{s_0}$. Since $\langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle = 0$, the Fredholm alternative guarantees a solution $\mathbf{y}_{s_0}^{(1)} \in \mathrm{dom}(N_{s_0})$ exists, and in this case we have

$$\partial_\lambda^2 M(\lambda_0, s_0) = -2s_0^4\langle \mathbf{y}_{s_0}^{(1)}, S\mathbf{u}_{s_0}\rangle. \tag{4.23}$$

Now set $\mathbf{y}_{s_0}^{(2)} := \frac{1}{2s_0^4}\partial_\lambda^2 A(\lambda_0, s_0)\mathbf{u}_{s_0}$, and consider equation (4.21a) with $k = 2$. For the left hand side, equation (4.22) holds with $\mathbf{y}_{s_0}^{(1)}$ replaced by $\mathbf{y}_{s_0}^{(2)}$, while for the right hand side, we note that in the case that $\partial_\lambda^2 M(\lambda_0, s_0) = -2s_0^4\langle \mathbf{y}_{s_0}^{(1)}, S\mathbf{u}_{s_0}\rangle = 0$, we have $\mathbf{y}_{s_0}^{(1)} \in \ker(N_{s_0}^* - s_0^2\lambda_0)$ and therefore $(I-P)\mathbf{y}_{s_0}^{(1)} = \mathbf{y}_{s_0}^{(1)}$. Thus, (4.21a) with $k = 2$ simplifies to

$$(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(2)} = \mathbf{y}_{s_0}^{(1)}, \tag{4.24}$$

where the Fredholm alternative again guarantees a solution $\mathbf{y}_{s_0}^{(2)} \in \mathrm{dom}(N_{s_0})$ because $\partial_\lambda^2 M(\lambda_0, s_0) = -2s_0^4\langle \mathbf{y}_{s_0}^{(1)}, S\mathbf{u}_{s_0}\rangle = 0$. Moreover, we have

$$\partial_\lambda^3 M(\lambda_0, s_0) = -3s_0^2\left\langle \partial_\lambda^2 A(\lambda_0, s_0)\mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\right\rangle = -6s_0^6\langle \mathbf{y}_{s_0}^{(2)}, S\mathbf{u}_{s_0}\rangle. \tag{4.25}$$

Continuing this pattern, we conclude that, if $\partial_\lambda^k M(\lambda_0, s_0) = 0$ for all $k = 1, \ldots, n-1$, then

$$\partial_\lambda^n M(\lambda_0, s_0) = -n!s_0^{2n}\langle \mathbf{y}_{s_0}^{(n-1)}, S\mathbf{u}_{s_0}\rangle, \tag{4.26}$$

where $(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(k)} = \mathbf{y}_{s_0}^{(k-1)}$ for $k = 2, \ldots, n-1$, and $(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(1)} = \mathbf{u}_{s_0}$. That is, $\mathbf{y}_{s_0}^{(k)} = \mathbf{v}_{s_0}^{(k)}$, and from (4.9) we obtain

$$\partial_\lambda^n M(\lambda_0, s_0) = -n!s_0^{2n}\langle \mathbf{v}_{s_0}^{(n-1)}, S\mathbf{u}_{s_0}\rangle = s_0\mathfrak{m}_{\lambda_0}^{(n)}(q). \tag{4.27}$$

□

Remark 4.4. As a corollary to Lemma 4.3, notice that this proves that $\dot{s}(\lambda_0) = \ddot{s}(\lambda_0) = \ldots = \partial_\lambda^{k-1}s(\lambda_0) = 0$ if and only if $\mathfrak{m}_{\lambda_0} = \mathfrak{m}_{\lambda_0}^{(2)} = \cdots = \mathfrak{m}_{\lambda_0}^{(k-1)} = 0$, and in this case (4.15) holds with $n = k$. Indeed, we already saw from (4.17) that $\dot{s}(\lambda_0) = 0 \iff \mathfrak{m}_{\lambda_0} = 0$. But if $\dot{s}(\lambda_0) = 0$, then from Lemma 4.3 with $n = 2$ we have $\ddot{s}(\lambda_0) = 0 \iff \mathfrak{m}_{\lambda_0}^{(2)} = 0$. Continuing this pattern, the result follows.

We are now lead to the following result.

Theorem 4.5. *Assume the condition of Lemma 4.3. Then, for $n \geq 2$, the following are equivalent:*

(i) $\dot{s}(\lambda_0) = \ddot{s}(\lambda_0) = ... = \partial_\lambda^{n-1} s(\lambda_0) = 0$ *and* $\partial_\lambda^n s(\lambda_0) \neq 0$,

(ii) $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ *has algebraic multiplicity* n,

(iii) *The kth order crossing forms (4.4) are zero for* $k = 1, ..., n-1$, *while the nth order crossing form is nonzero.*

Proof. It follows from Lemma 4.3 and Remark 4.4 that (i) $\iff$ (iii).

To see that (iii) $\implies$ (ii), we have from Lemma 4.2 that the algebraic multiplicity of $s_0^2\lambda_0$ is at least n. Now since $\mathfrak{m}^{(n)}(q) = -n! s_0^{2n-1}\langle \mathbf{v}_{s_0}^{(n-1)}, S\mathbf{u}_{s_0}\rangle \neq 0$, the Fredholm alternative implies that the equation $(N_{s_0} - s_0^2\lambda_0)\mathbf{y}_{s_0}^{(n)} = \mathbf{y}_{s_0}^{(n-1)}$ has no solution. Therefore, the Jordan chain $\{\mathbf{u}_{s_0}, \mathbf{v}_{s_0}^{(1)}, \dots, \mathbf{v}_{s_0}^{(n-1)}\}$ is maximal, and $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ has algebraic multiplicity n.

To see that (ii) $\implies$ (iii), if $s_0^2\lambda_0 \in \mathrm{Spec}(N_{s_0})$ has algebraic multiplicity n, then from the solvability of (4.14b) we have $\langle \mathbf{u}_{s_0}, S\mathbf{u}_{s_0}\rangle = 0$. Hence, $\mathfrak{m}_{\lambda_0} = 0$. Now using the expression for $\mathfrak{m}_{\lambda_0}^{(2)}$ given by Lemma 4.2, it follows from the solvability of (4.14a) with $k = 2$ that $\mathfrak{m}_{\lambda_0}^{(2)} = 0$. Arguing in this way we find that $\mathfrak{m}_{\lambda_0}^{(k)} = 0$ for $k = 1, \dots, n-1$. Since the equation $(N_{s_0} - s_0^2\lambda_0)\mathbf{v}_{s_0}^{(n)} = \mathbf{v}_{s_0}^{(n-1)}$ is not solvable, it follows from Lemma 4.2 that $\mathfrak{m}_{\lambda_0}^{(n)} \neq 0$. □

Remark 4.6. The previous theorem shows that the algebraic multiplicity of an eigenvalue (which is geometrically simple) is encoded in the degree of vanishing of the spectral curve $s = s(\lambda)$ at the point (λ_0, s_0). The same is *not* true for the s-direction. That is, if $\mathfrak{m}_{\lambda_0} \neq 0$ and there exists an analytic curve $\lambda(s)$ through (λ_0, s_0), then the algebraic multiplicity of $s_0^2\lambda_0$ is *not* equal to the order of vanishing of $\lambda(s)$. It will be the subject of future work to extend the result of Theorem 4.5 to eigenvalues with higher geometric multiplicity.

Using the previous result, we have the following analogue of Proposition 2.50 for determining the contribution to the Maslov index from the crossing at $(0, 1)$ in the λ direction (i.e. the quantity $\mathfrak{b}$), in the case when the second-order crossing form (and possibly higher-order crossing forms) are degenerate. This agrees with the formula of [GPP04b] given in Definition 3.8 in this case.

Proposition 4.7. *Suppose* $\dim\ker N = 1$, *and assume the kth order crossing forms* $\mathfrak{m}_{\lambda_0}^{(k)}$ *for* $k = 1, \dots, n-1$ *are all identically zero at the crossing* $(\lambda_0, s_0) = (0, 1)$. *If the nth order crossing form* $\mathfrak{m}_{\lambda_0}^{(n)}$ *is nondegenerate, then*

$$\mathrm{Mas}(\Lambda(\lambda, 1), \mathcal{D}; \lambda \in [0, \varepsilon]) = -n_-(\mathfrak{m}_{\lambda_0}^{(n)}). \tag{4.28}$$

Proof. The proof is similar to that of Proposition 2.50. Note firstly that by Theorem 4.5, under the conditions of the proposition we have $\dot{s}(\lambda_0) = \ddot{s}(\lambda_0) = ... = \partial_\lambda^{n-1} s(\lambda_0) = 0$ and $\partial_\lambda^n s(\lambda_0) \neq 0$. Now for the right hand side of (4.28), if $\dim\ker(N) = 1$, recall using (2.44) that $\mathfrak{m}_{s_0} > 0$ if $0 \in \mathrm{Spec}(L_-^{s_0}) \setminus \mathrm{Spec}(L_+^{s_0})$, and $\mathfrak{m}_{s_0} < 0$ if $0 \in \mathrm{Spec}(L_+^{s_0}) \setminus \mathrm{Spec}(L_-^{s_0})$. Then, using (4.15), we find that

(i) If $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ then $n_-(\mathfrak{m}_{\lambda_0}^{(n)}) = \begin{cases} 0 & \partial_\lambda^n s(0) > 0, \\ 1 & \partial_\lambda^n s(0) < 0. \end{cases}$

(ii) If $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ then $n_-(\mathfrak{m}_{\lambda_0}^{(n)}) = \begin{cases} 1 & \partial_\lambda^n s(0) > 0, \\ 0 & \partial_\lambda^n s(0) < 0. \end{cases}$

For the left hand side of (4.28), just as in the proof of Proposition 2.50, we have $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$, where $\mathfrak{a} := \operatorname{Mas}(\Lambda(s,0), \mathcal{D}; s \in [1-\varepsilon, 1]) = \dim\ker(L_-)$ and $\mathfrak{b} := \operatorname{Mas}(\Lambda(\lambda,1), \mathcal{D}; \lambda \in [0,\varepsilon])$. Thus $\mathfrak{b} = \mathfrak{c} - \dim\ker(L_-)$. To determine $\mathfrak{c}$, note that we have $s^\sharp(0) = \operatorname{sign} \partial_\lambda^n s(0)$ (see (2.133) and the accompanying Remark 2.48). Now using the values of $\mathfrak{c}$ computed in Theorem 2.49, we confirm that $\mathfrak{b} = -n_-(\mathfrak{m}_{\lambda_0}^{(2)})$ in cases (i) and (ii) described above, as claimed. □

4.2 Notes on the fourth-order problem

The two main issues in Chapter 3 that need to be resolved are the requirements of Hypotheses 3.16 and 3.17. The former states that $\phi(x_0) \neq 0$ for any $x_0 \in \mathbb{R}$ where $\mathbb{E}_\pm^u(x_0, 0) \cap \mathbb{S}_\pm(0) \neq \{0\}$, while the latter says that any one-dimensional intersection of the unstable bundle $\mathbb{E}_\pm(\cdot, 0)$ with the stable subspace $\mathbb{S}_\pm(0)$ of the asymptotic system cannot coincide purely with the span of the first column of the frame $\mathbf{S}_+$ in (3.104).

If Hypothesis 3.16 fails, then all of the crossing forms computed in the proof of Lemma 3.18 are identically zero. Specifically, for one-dimensional crossing under Hypothesis 3.17 we have $\mathfrak{m}_{x_0} = 0$, while for two-dimensional crossings we have $\mathfrak{m}_{x_0}^{(k)} = 0$ for $k = 1, 2, 3$. Therefore, higher-order crossing forms are needed. Such forms will involve higher-order derivatives of $\phi(x)$ at $x = x_0$. Given ϕ solves a fourth-order differential equation, there must exist a $k \in \{0, 1, 2, 3\}$ such that $\phi^{(k)}(x_0) \neq 0$. Taking sufficiently many higher-order crossing forms will therefore yield enough nondegenerate forms. To prove negativity of the crossings (for the L_+ problem, and positivity for the L_- problem), thereby recovering the monotonicity property of Lemma 3.18, one would need to prove sign-definiteness of the forms of odd order.

If Hypothesis 3.17 fails, then for such an intersection the calculation of the signatures of the second and third-order crossing forms becomes intractable without knowing the matrices $X(x_0)$ and $Y(x_0)$ explicitly. A possible fix, as discussed in Section 3.1, may be to use Hörmander's index [How21] to swap the reference plane $\mathbb{S}_+(0)$ for a reference plane with respect to which the path $x \mapsto \mathbb{E}_\pm(x, 0)$ is monotonic. One such plane is that with the frame

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting this subspace by $\mathbb{V}$, in this case it can be shown that the crossing form is given by

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot, 0), \mathbb{V})(q) = \langle JA_+(x_0)q, q \rangle_{\mathbb{R}^4} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\rangle_{\mathbb{R}^2} = -k_1^2,$$

(just as for the fourth-order problems studied in [How23, §6] and [How21, §5.2], although the words "crossing form" are not used) where $q = \mathbf{V}k \in \mathbb{E}_+^u(x_0, 0) \cap \mathbb{V}$ and $k = (k_1, k_2) \in \mathbb{R}^2$. While the form is therefore degenerate, it is possible to exploit a certain nonpositivity in a neighbourhood of the crossing $x = x_0$, and using a result such as [HS22, Lemma 3.2] yields the required monotonicity (see, for example, [How21, Lemma 2.4]). The issue with this approach for the current problem is the requirement of an explicit form for the frame of the unstable bundle $\mathbb{E}_\pm(x, 0)$ at $x = +\infty$. Since $\lambda = 0$ is a simple eigenvalue, $\lim_{x\to\infty} \mathbb{E}_\pm(x, 0)$ will have a one-dimensional intersection with $\mathbb{S}_\pm(0)$. The problem lies in determining *which* subspace of $\mathbb{S}_\pm(0)$ this intersection occurs, as well as which subspace of $\mathbb{U}_\pm(0)$ (the unstable subspace of the asymptotic system) the remaining part of the unstable bundle tends to.

Finally, the cases when one or both of $\mathcal{I}_1$ and $\mathcal{I}_2$ are zero can also be dealt with via higher-order crossing forms (see Remark 3.34). Resolving the above listed issues is the subject of ongoing work.

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