PELVE: Probability Equivalent Level of VaR and ES

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VaR and ES

Value-at-Risk (VaR), $p \in (0, 1)$

$$\text{VaR}_p : L^0 \rightarrow \mathbb{R},$$

$$\text{VaR}_p(X) = F_X^{-1}(p)$$

$$= \inf \{ x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p \}.$$ (left-quantile)

Expected Shortfall (ES), $p \in (0, 1)$

$$\text{ES}_p : L^1 \rightarrow \mathbb{R},$$

$$\text{ES}_p(X) = \frac{1}{1 - p} \int_0^1 \text{VaR}_q(X) dq$$

(also: TVaR/CVaR/AVaR)
The Basel Committee on Banking Supervision

Fundamental Review of the Trading Book (FRTB), live Jan 2019

- Widely discussed since 2012
- Planned implementation
  - March 2021 (most Europe)
  - Jan 2022 (North America, some of East Asia)
VaR_{0.99} \implies \text{ES}_{0.975}

- VaR_{0.99} is replaced by \text{ES}_{0.975} as the standard risk measure for market risk in the internal model approach
- 10-day portfolio loss forecast
- In a survey in 2015, 2/3 of banks reported higher capital charge under the (back-then) proposed FTRB
- Is there a general relationship between VaR_{0.99} and \text{ES}_{0.975}?
A tiny portion of literature on VaR and ES

ES is coherent and VaR is not
  ▶ Artzner-Delbaen-Eber-Heath'99 MF; Acerbi-Tasche'02 JBF

VaR is elicitable and ES is not
  ▶ Gneiting’11 JASA

Axiomatic characterizations
  ▶ ES: W.-Zitikis’20 MS
  ▶ VaR: Chambers’09 MF; Kou-Peng’16 OR; He-Peng’18 OR; Liu-W.’20 MOR

Optimization properties
  ▶ Rockafellar-Uryasev’00/02 JR/JBF; Gaivoronski-Pflug’05 JR

Statistical inference and time series
  ▶ Scaillet’04 MF; Engle-Manganelli’04 JBES; Chen’08 JFEc

Risk aggregation
  ▶ Embrechts-Puccetti-Rüschendorf’13 JBF; Embrechts-Wang-W.’15 FS
A tiny portion of literature on VaR and ES

Investment and portfolio management

▶ Basak-Shapiro’01 RFS; Krokhmal-Palmquist-Uryasev’02 JR; Natarajan-Pachamanova-Sim’08 MS; Adrian-Shin’14 RFS

Capital allocation

▶ Dhaene-Goovaerts-Kaas’03 IME; Kalkbrener’05 MF; Dhaene-Tsanakas-Valdez-Vanduffel’12 JRI; Asimit-Peng-W.-Yu’19 MF

Insurance, reinsurance, and risk sharing

▶ Cai-Tan’07 ASTIN; Cai-Tan-Weng-Zhang’08 IME; Chi-Tan’11 ASTIN; Embrechts-Liu-W.’18 OR; Weber’18 IME

Systemic risk, CoVaR/CoES

▶ Acharya-Engle-Richardson’12 AER; Adrian-Brunnermeier’16 AER

Forecasting and backtesting

▶ Fissler-Ziegel’16 AoS; Du-Escanciano’17 MS; Kratz-Lok-McNeil’18 JBF
Progress

1. Background
2. PELVE: A tale of two risk measures
3. Theoretical properties
4. Parametric and heavy tailed distributions
5. Non-parametric estimation
6. Empirical analysis
7. Concluding remarks
Definition of PELVE

Given a random loss $X$, how do we compare $\text{VaR}_{0.99}(X)$ and $\text{ES}_{0.975}(X)$, or generally $\text{VaR}_p(X)$ and $\text{ES}_q(X)$?

> A number $c \in [1, 1/\epsilon]$ such that $\text{ES}_{1-c\epsilon}(X) = \text{VaR}_{1-\epsilon}(X)$

For $\epsilon = 0.01$ $\iff$ $\text{VaR}_{0.99}$ in FRTB:

- $c > 2.5 \Rightarrow \text{ES}_{0.975} > \text{VaR}_{0.99} \Rightarrow$ capital increases
- $c \approx 2.5 \Rightarrow \text{ES}_{0.975} \approx \text{VaR}_{0.99} \Rightarrow$ little or no change in capital
- $c < 2.5 \Rightarrow \text{ES}_{0.975} < \text{VaR}_{0.99} \Rightarrow$ capital decreases
Definition of PELVE

**Definition 1**

For $\epsilon \in (0, 1)$, the Probability Equivalent Level of VaR-ES (PELVE) is defined as $\Pi_\epsilon : L^1 \rightarrow \mathbb{R}$

$$
\Pi_\epsilon(X) = \inf \{ c \in [1, 1/\epsilon] : ES_{1-c\epsilon}(X) \leq \text{VaR}_{1-\epsilon}(X) \}
$$

with the convention $\inf(\emptyset) = \infty$.

- Always well defined
- Almost always $ES_{1-c\epsilon}(X) = \text{VaR}_{1-\epsilon}(X)$
- $\Pi_\epsilon(X) < \infty$ if $\epsilon$ is small
Typical values of PELVE

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Typical values of PELVE

Quick observations on $\Pi_\epsilon$:

- Common range $[2, 4]$
- $\Pi_\epsilon(X) = 1 \iff$ point-mass
- $\Pi_\epsilon(X) = 4 \iff$ Pareto(2), infinite variance
- $\Pi_\epsilon(X) \approx 2.5 \iff$ normal
  - $\text{VaR}_{0.99} \approx \text{ES}_{0.975}$ in FRTB for normal $X$
- Relatively stable across different $\epsilon$ for the same distribution
  - Constant in $\epsilon$ for degenerate, uniform, exponential and Pareto
- higher value $\iff$ heavy tails
- lower value $\iff$ light tails
- Can be used as a measure of variability
Progress

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Existence and uniqueness

Assumption 1 (Existence & uniqueness condition)

\[ \mathbb{E}[X] < \text{VaR}_{1-\epsilon}(X) \text{ and } \text{VaR}_p(X) \text{ is not a constant for } p \in [1 - \epsilon, 1). \]

Proposition 1 (Existence & uniqueness of PELVE)

Under Assumption 1, there exists a unique \( c \in [1, 1/\epsilon] \) such that

\[ \text{ES}_{1-c\epsilon}(X) = \text{VaR}_{1-\epsilon}(X). \]
Theoretical features of PELVE

Features

▶ Location-scale invariance
  • location-scale free risk assessment (e.g., Sharp ratio)

▶ Monotone in convex transformation
  • $X \sim N(0, 1) \text{ vs } e^X \sim LN(0, 1)$
  • $X \sim \text{Pareto}(4) \text{ vs } X^2 \sim \text{Pareto}(2)$

▶ Betweenness
  • quasi-convexity and quasi-concavity wrt quantile-mixture
  • combining two comonotonic losses does not give a PELVE value beyond the worse one or below the better one
  • quasi-convex programming
Theoretical features of PELVE

Theorem 1

Suppose that $X \in L^1$, $\epsilon \in (0, 1)$ and $\mathbb{E}[X] \leq \text{VaR}_{1-\epsilon}(X)$.

(i) For all $\lambda > 0$ and $a \in \mathbb{R}$, $\Pi_\epsilon(\lambda X + a) = \Pi_\epsilon(X)$.

(ii) $\Pi_\epsilon(f(X)) \leq \Pi_\epsilon(X)$ for all increasing concave functions $f : \mathbb{R} \to \mathbb{R}$ with $f(X) \in L^1$.

(iii) $\Pi_\epsilon(g(X)) \geq \Pi_\epsilon(X)$ for all strictly increasing convex functions $g : \mathbb{R} \to \mathbb{R}$ with $g(X) \in L^1$.

(iv) For all comonotonic $Y, Z \in L^1$, $\Pi_\epsilon(Y + Z)$ is between $\Pi_\epsilon(Y)$ and $\Pi_\epsilon(Z)$.

(i/iv) $\Rightarrow \Pi_\epsilon(Y) \wedge \Pi_\epsilon(Z) \leq \Pi_\epsilon(\lambda Y + (1 - \lambda)Z) \leq \Pi_\epsilon(Y) \vee \Pi_\epsilon(Z)$, $\forall \lambda \in [0, 1]$. 
Example 1 (PELVE for time-series models)

Risk measure forecast is usually done via conditional models. Consider an AR-GARCH type of time-series for risk factors

\[ X_t = \mu_t + \sigma_t Z_t, \quad t \in \mathbb{Z}, \]

where \( \mu_t \) and \( \sigma_t \) are the conditional mean and standard deviation given a \( \sigma \)-field \( \mathcal{F}_{t-1} \), and \( Z_t \) is independent of \( \mathcal{F}_{t-1} \). Then

\[ \Pi_\epsilon(X_t|\mathcal{F}_{t-1}) = \Pi_\epsilon(Z_t). \]
Example 2 (Reducing PELVE with options)

- Losses from an asset and a European call or put (same maturity)

  \[ X_A: \text{asset}; \quad X_C: \text{call}; \quad X_{AC}: \text{asset+call}; \quad X_{AP}: \text{asset+put} \]

- Put-call parity: \[ X_C + c = X_{AP} \text{ for some } c \in \mathbb{R} \]

- All of \( X_C, X_{AC}, \) and \( X_{AP} \) are increasing concave functions of \( X_A \)

- \( X_{AP} \) is an increasing concave function of \( X_{AC} \)

Theorem 1 (ii) implies:

\[ \Pi_\epsilon(X_C) = \Pi_\epsilon(X_{AP}) \leq \Pi_\epsilon(X_{AC}) \leq \Pi_\epsilon(X_A) \]

Consistent with intuition & no model assumption
Example 3 (Comparison of PELVE using quasi-convexity/concavity)

Following the previous example:

- $X_A$ and $X_C$ are comonotonic
- Theorem 1 (iv) betweenness $\Rightarrow$

\[ \Pi_\epsilon(X_C) \wedge \Pi_\epsilon(X_A) \leq \Pi_\epsilon(X_{AC}) \leq \Pi_\epsilon(X_C) \vee \Pi_\epsilon(X_A) \]

- Remains true if the call option in $X_C$ and $X_{AC}$ is replaced by any other payoffs increasing in the asset price
- Use $\Pi_\epsilon(X_C) \leq \Pi_\epsilon(X_A) \Rightarrow$ the result in the previous example
Examples

Example 4 (Linear loss and log-loss)

For an asset with price $X_t$ at time $t = 0, 1, \ldots$ (e.g. daily prices)

- linear return: $R_t = \frac{X_t}{X_{t-1}} - 1$
- log-return: $r_t = \log\left(\frac{X_t}{X_{t-1}}\right)$
- linear loss (negative return): $-R_t = 1 - \frac{X_t}{X_{t-1}}$
- log-loss (negative log-return): $-r_t = -\log\left(\frac{X_t}{X_{t-1}}\right)$
- $y \mapsto -\log(1 - y)$ is strictly increasing and convex
- Theorem 1 (iii) $\Rightarrow \Pi_\epsilon(-r_t) \geq \Pi_\epsilon(-R_t)$
- Using log-loss $\Rightarrow$ a (slightly) higher PELVE than linear loss
Consistency

Assumption 2 (Continuous quantile)

$F_{X}^{-1}(q)$ is continuous at $q = 1 - \epsilon$.

Assumption 3 (Uniform integrability)

$\{X_n\}_{n \in \mathbb{N}}$ is uniformly integrable.

Theorem 2

Suppose that $\{X_n\}_{n \in \mathbb{N}} \subset L^1$, $X \in L^1$ and $\epsilon \in (0, 1)$ satisfy Assumptions 1-3 and $X_n \rightarrow X$ in distribution as $n \rightarrow \infty$. Then $\Pi_\epsilon(X_n) \rightarrow \Pi_\epsilon(X)$ as $n \rightarrow \infty$. 
Consistency

Corollary 1

Suppose that $X, X_1, X_2, \ldots \in L^1$ are iid, $\epsilon \in (0, 1)$, and Assumptions 1-2 hold. Then $\hat{\Pi}_\epsilon(n) \to \Pi_\epsilon(X)$ as $n \to \infty$, where $\hat{\Pi}_\epsilon(n)$ is the $\epsilon$-PELVE of the empirical distribution based on sample $X_1, \ldots, X_n$. 
Diversification effect

VaR and ES: which of them rewards diversification more?

- Elliptical family: the same
- Generally: unclear

Consider a general setting

- $X_1, \ldots, X_n$: individual losses with finite variances
- $S_n = \sum_{i=1}^{n} w_i X_i$ for some weights $w_1, \ldots, w_n \geq 0$
- Allow $n \to \infty$
Diversification effect

- Assume CLT: \( \frac{S_n - a_n}{b_n} \rightarrow N \sim N(0, 1) \) for some \( a_n \in \mathbb{R} \) and \( b_n > 0 \)
- Theorems 1-2:
  \[
  \Pi_\epsilon(S_n) = \Pi_\epsilon \left( \frac{S_n - a_n}{b_n} \right) \rightarrow \Pi_\epsilon(N)
  \]
- most asset PELVE \( \geq \Pi_\epsilon(N) \), e.g., \( \Pi_{0.01}(N) = 2.58 \)
- Aggregate PELVE is likely smaller than individual PELVE
- True even without CLT
### Example 5 (Diversification ratio)

- **Diversification ratio:** for a risk measure $\rho$, 

\[
\Delta(\rho) = \frac{\rho(S_n)}{\sum_{i=1}^{n} w_i \rho(X_i)} = \frac{\text{with diversification}}{\text{without diversification}}
\]

- $c \geq 1$: $\sum_{i=1}^{n} w_i \text{VaR}_{1-\epsilon}(X_i) = \sum_{i=1}^{n} w_i \text{ES}_{1-c\epsilon}(X_i)$

- $c' = \Pi_{\epsilon}(S_n)$: $\text{VaR}_{1-\epsilon}(S_n) = \text{ES}_{1-c'\epsilon}(S_n)$

- Assume $c > \Pi_{\epsilon}(N)$ (commonly observed)

- **CLT:** $c' < c$ for $n$ large $\Rightarrow \text{ES}_{1-c\epsilon}(S_n) < \text{VaR}_{1-\epsilon}(S_n)$

- $\Delta(\text{ES}_{1-c\epsilon}) < \Delta(\text{VaR}_{1-\epsilon})$ and $\Delta(\text{ES}_{1-c'\epsilon}) < \Delta(\text{VaR}_{1-\epsilon})$

**Remark.** The result will be flipped if we assume $c < \Pi_{\epsilon}(N)$, e.g., uniform individual losses (nothing to do with the coherence of ES).
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Example 6 (Uniform/Exponential/Pareto distributions)

- $X \sim \text{uniform}: \Pi_\epsilon(X) = 2$ for $\epsilon \leq 1/2$.
- $X \sim \text{exponential}: \Pi_\epsilon(X) = e$ for $\epsilon \leq 1/e$.
- $X \sim \text{Pareto}(\alpha), \alpha > 1$ (i.e., $\mathbb{P}(X > x) = x^{-\alpha}, \ x \geq 1$):

\[
\Pi_\epsilon(X) = \left(\frac{\alpha}{\alpha - 1}\right)^\alpha \quad \text{for} \quad \epsilon \leq \left(\frac{\alpha}{\alpha - 1}\right)^{-\alpha}.
\]

- For Pareto($\alpha$), $\alpha \mapsto \left(\frac{\alpha}{\alpha - 1}\right)^\alpha$ is decreasing in $\alpha$ and

\[
\Pi_\epsilon(X) \geq \lim_{\alpha \to \infty} \left(\frac{\alpha}{\alpha - 1}\right)^\alpha = e \approx 2.718.
\]

- $e$ is a threshold for heavy and light tails.
Example 7 (Normal/t/log-normal distributions)

\( \Pi_\epsilon \) has no explicit formula. VaR and ES have explicit formulas.

- Normal: \( \approx 2.5 \)
- \( t(\nu) \): > 2.72 for \( \epsilon \approx 0 \) and \( \nu \) not too large
- log-normal: various possibilities
PELVE of parametric models ($\epsilon = 0.01$ if unspecified)

(a) Pareto($\alpha$), $\alpha \in [1.5, 50]$

(b) $t(\nu)$, $\epsilon \in (0, 0.2]$

(c) LN($\sigma^2$), $\epsilon \in (0, 0.2]$

(d) $N(\mu, \sigma^2)$, $\epsilon \in (0, 0.2]$

(e) $t(\nu)$, $\nu \in [1.5, 50]$

(f) LN($\sigma^2$), $\sigma^2 \in (0, 2]$
PELVE of regularly varying distributions

A survival function $\overline{F}$ is regularly varying (RV) with a tail index $\alpha > 0$, denoted by $f \in \text{RV}_{-\alpha}$, if

$$
\lim_{x \to \infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\alpha}, \quad \text{for all } t > 0.
$$

▶ e.g., Pareto($\alpha$), t($\alpha$).

**Theorem 3**

*Suppose that the function $\overline{F}(x) = \mathbb{P}(X > x)$ is RV$_{-\alpha}$, $\alpha > 1$. Then*

$$
\lim_{\epsilon \downarrow 0} \prod_{\epsilon} \mathbb{P}(X) = \left(\frac{\alpha}{\alpha - 1}\right)^\alpha.
$$
Comparing PELVE and the tail index:

- **Both** location-scale invariant

- **Assumptions**
  - The tail index requires regular variation *(difficult to check!)*
  - PELVE only requires **finite mean**, well defined for bounded rvs

- **Estimation**
  - The tail index needs an **ad-hoc threshold** (e.g., Hill estimator)
  - PELVE needs only $\epsilon$ which has a physical meaning

- **Interpretation**
  - The tail index **remains the same** for the average of iid risks
  - PELVE can **reflect diversification** (e.g., CLT)
  - $\Pi_{0.01}$ has a meaning for **banking regulation** ($\text{VaR} \Rightarrow \text{ES}$)
PELVE vs tail index

Proposition 2

Suppose that the function \( F(x) = \mathbb{P}(X > x) \) is \( \text{RV}_{-\alpha} \), \( \alpha > 1 \).

Then, for \( c > 1 \),

\[
\lim_{\epsilon \downarrow 0} \frac{\text{ES}_{1-c\epsilon}(X)}{\text{VaR}_{1-c\epsilon}(X)} = \frac{\alpha}{\alpha - 1} c^{-1/\alpha} = \lim_{\epsilon \downarrow 0} \left( \frac{\Pi_{\epsilon}(X)}{c} \right)^{1/\alpha}.
\]

- In FRTB, \( c = 2.5 \) leads to

\[
R(\alpha) := \frac{\alpha}{\alpha - 1} 2.5^{-1/\alpha} = \lim_{\epsilon \downarrow 0} \left( \frac{\Pi_{\epsilon}(X)}{2.5} \right)^{1/\alpha} > 1
\]

- \( R(\alpha) \approx \text{ES}_{0.975}/\text{VaR}_{0.99} \)

- \( R(8) = 1.02; \ R(4) = 1.06; \ R(2) = 1.26 \)
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Empirical PELVE estimators

Empirical PELVE estimator $\hat{\Pi}_\epsilon(n)$: solve

$$\hat{\text{ES}}_{1-c\epsilon} = \hat{\text{VaR}}_{1-\epsilon} \quad \text{for} \; c \in [1, 1/\epsilon],$$

where $\hat{\text{ES}}$ and $\hat{\text{VaR}}$ are the empirical ES and VaR, respectively

- Let $X[1] \leq \cdots \leq X[n]$ be the order statistics of $X_1, \ldots, X_n$.

  $\hat{\text{VaR}}_p = X[i] \quad \text{for} \; p \in \left( \frac{i - 1}{n}, \frac{i}{n} \right], \; i = 1, \ldots, n,$

  $$\hat{\text{ES}}_p = \frac{1}{1 - p} \int_p^1 \hat{\text{VaR}}_q \, dq, \quad p \in (0, 1).$$

- Safely assume that $c$ above is unique
Empirical PELVE estimators

Smoothed empirical PELVE estimator $\tilde{\Pi}_\epsilon(n)$:

- $\hat{\ES}_p$ is continuous in $p$
- $\hat{\VaR}_p$ has jumps
- Quick fix: use $\tilde{\VaR}_p$, the standard linearly interpolated quantile (McNeil-Frey-Embrechts’15, Section 9.2.6)
- Calculate $\tilde{\ES}_p$ based on $\tilde{\VaR}_p$

Assumption 4 (Regularity)

$X \sim F$ has a density function $f$ which is positive and continuous at $F^{-1}(1 - \epsilon)$, and $\mathbb{E}[|X|^{2+\delta}] < \infty$ for some $\delta > 0$. 

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Asymptotic normality

**Theorem 4 (Asymptotic normality)**

Suppose that \( X, X_1, X_2, \cdots \in L^1 \) are iid, \( \epsilon \in (0, 1) \), and Assumptions 1 and 4 hold. Let \( c = \prod_\epsilon(X) \) and \( \hat{c}_n = \hat{\Pi}(n) \) or \( \tilde{c}_n = \tilde{\Pi}(n) \). Then

\[
\sqrt{n}(\hat{c}_n - c) \xrightarrow{p} \frac{1}{b} \left( \int_q^1 \frac{W(1-t)}{\epsilon f(F^{-1}(t))} \, dt - aW(\epsilon) \right) \sim N(0, \sigma^2),
\]

where \( W \) is a standard Brownian bridge on \([0, 1]\), \( p = 1 - \epsilon \), \( q = 1 - c\epsilon \), \( a = c/f(F^{-1}(p)) \), \( b = F^{-1}(p) - F^{-1}(q) \), and \( \sigma^2 \) can be computed as

\[
\sigma^2 = \frac{1}{b^2} \left( a^2(\epsilon - \epsilon^2) + \frac{2}{\epsilon^2} \int_{F^{-1}(q)}^\infty E_{F(x)}F(x) \, dx - \frac{2a}{\epsilon} E_p + 2a(E_q - b) \right),
\]

where \( E_t = \mathbb{E}[(X - F^{-1}(t))_+] \) for \( t \in (0, 1) \).
Asymptotic normality

Remarks.

- $\sigma^2 \approx O(\epsilon^{-1})$ as $\epsilon \downarrow 0$
  - $\hat{ES}_{1-\epsilon}$ and $\hat{VaR}_{1-\epsilon}$ effectively use $O(n\epsilon)$ data points
- Very small value of $\epsilon \Rightarrow$ large estimation error of $\Pi_\epsilon$
- $\Pi_\epsilon$ typically stable in $\epsilon \Rightarrow$ no need $\epsilon$ too small
- $\tilde{\Pi}_\epsilon(n)$ seems to have a negative bias whereas $\hat{\Pi}_\epsilon(n)$ does not
- Similar results on $\alpha$-mixing data
  - proof is based on Asimit-Peng-W.-Yu’19 MF
Simulation of PELVE estimators for Pareto(4), $\epsilon = 0.05$

(a) $\hat{\Pi}_\epsilon(n), n = 1000$

(b) $\tilde{\Pi}_\epsilon(n), n = 1000$

(c) $\hat{\Pi}_\epsilon(n), n = 5000$

(d) $\tilde{\Pi}_\epsilon(n), n = 5000$
Data description

- Price data of S&P 500 constituents
- Daily log-losses (log-transform is negligible)
- 01/04/1999 to 05/30/2020 (≈ 21.4 years)
- Empirical estimators based on a moving window of 500 days
- We report sector averages of S&P 500
5% PELVE of log-loss vs linear loss, S&P 500, Jan 01 - May 20
Empirical PELVE, VaR and ES for S&P Sectors

(a) 5% PELVE of S&P 500 Financials

(b) 5% PELVE of S&P 500 IT

(c) VaR and ES of S&P 500 Financials

(d) VaR and ES of S&P 500 IT

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Empirical PELVE, VaR and ES for S&P Sectors

(a) 5% PELVE of S&P 500 Energy
(b) 5% PELVE of S&P 500 Real Estate
(c) 5% PELVE of S&P 500 Utilities

(d) VaR & ES of S&P 500 Energy
(e) VaR & ES of S&P 500 Real Estate
(f) VaR & ES of S&P 500 Utilities
Empirical PELVE for other S&P Sectors

(a) S&P 500 Consumer Discretionary
(b) S&P 500 Consumer Staples
(c) S&P 500 Communication Services
(d) S&P 500 Materials
(e) S&P 500 Healthy Care
(f) S&P 500 Industrials
Summary of findings

Findings:

▷ Most PELVE are between 2.8 and 3.4 prior to COVID-19 and usually above $e \approx 2.72$ (average $= 2.98$) $\Rightarrow$ Heavy tails
  - Tail index between 3 and 5 (Jansen-De Vries’91 RES, Cont’01 QF) $\Rightarrow$ PELVE between $3.05$ and $3.38$ for small $\epsilon$

▷ Overall PELVE values are quite stable during the past 20 years except for peaks around the two crises

▷ VaR/ES:
  - IT: dot-com bubble $\gg$ subprime crisis/covid crisis
  - Financials/Real Estate: subprime crisis $\gg$ covid crisis (so far)

▷ PELVE strongly disagrees with VaR/ES
Well-diversified portfolios

1/\(N\) portfolio (DeMiguel-Garlappi-Uppal’09 RFS):

- \(N = 500\) constituents of S&P 500
- Monthly rebalanced
- The constituents of S&P 500 change over time \(\Rightarrow\) two types
  - (a) with replacement, \(N = 500\)
  - (b) without replacement, \(N \downarrow 186\) (May 2020)
- S&P 500 index is a diversified portfolio (with survival bias)
- Jan 1999 to May 2020
5% PELVE of $1/N$ portfolios

(g) $1/N$ portfolio with replacement

(h) $1/N$ portfolio without replacement
Summary of findings

<table>
<thead>
<tr>
<th></th>
<th>average PELVE</th>
<th>average log-return</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>2.75</td>
<td>3.94%</td>
</tr>
<tr>
<td>1/N with repl.</td>
<td>2.72</td>
<td>8.15%</td>
</tr>
<tr>
<td>1/N without repl.</td>
<td>2.74</td>
<td>9.02%</td>
</tr>
</tbody>
</table>

- All curves fluctuate around $e \approx 2.72$
- Well-diversified portfolios have a smaller PELVE, close to $e$.
- A rough explanation: a well-diversified portfolio is more “Gaussian-like” $\Rightarrow$ its PELVE is closer to Gaussian ($\approx 2.5$)
- High return and low PELVE of the $1/N$ portfolios
Implications for risk management

Switching from 99% VaR to 97.5% ES
- diversified portfolio ⇒ somewhat fine
- non-diversified portfolio ⇒ significant capital increase

For a well-diversified portfolio $X$ and a non-diversified one $Y$:
- $\text{VaR}_{0.99}(X) = \text{VaR}_{0.99}(Y) \Rightarrow \text{ES}_{0.975}(X) < \text{ES}_{0.975}(Y)$.

ES rewards portfolio diversification more than the VaR
- Hidden feature: not mentioned by FRTB or previous research
- Subadditivity (coherence) does not imply this
  - elliptical risk factor models
  - time-series models with Gaussian white noise
Progress

1. Background
2. PELVE: A tale of two risk measures
3. Theoretical properties
4. Parametric and heavy tailed distributions
5. Non-parametric estimation
6. Empirical analysis
7. Concluding remarks
Concluding remarks

PELVE in banking regulation

- Motivated by 2019 Basel FRTB
- \( \text{VaR}_{0.99} \Rightarrow \text{ES}_{0.975} \) increases capital for heavy-tailed losses
- Loss distributions in the US equity market are heavy-tailed
- Well-diversified portfolios have lower PELVE

Theory

- Location-scale invariant and quasi-convex/concave
- Monotone in convex transform and in tail index (asymptotically)
- Well defined for all commonly used distributions
- Empirical estimation is standard and simple
Thank you for your kind attention

The manuscript is available at SSRN: 3489566
Comments are welcome
PELVE vs volatility measures

- The average PELVE of S&P 500 constituents
- Left Tail Volatility index (LTV) during the period Jan 2001 to Dec 2017 (Andersen-Todorov-Ubukata’20 JoE)
- CBOE Volatility index (VIX) during the period Jan 2001 to Dec 2018
Appendix

PELVE of average S&P 500 constituents vs LTV and VIX

(i) 5% PELVE vs LTV (Jan 01 - Dec 17)

(j) 5% PELVE vs VIX (Jan 01 - Dec 18)