Let k be a finite field of size q and characteristic p, and define k_r to be the unique degree r extension of k in the algebraic closure $\bar{k}$. The affine space of dimension N can be identified with $\bar{k}^N$. An *affine variety* X is a subset of $\bar{k}^N$ that consists of the zeroes of a collection of polynomials. The variety is *defined over* k if it is closed under the action of the map $F_0 : \bar{k}^N \to \bar{k}^N$ that takes the qth power of each component. The *rational points* of X over k_r, denoted by $X(k_r)$, is the set of elements of X fixed by ${F_0}^r$. A *Frobenius endomorphism* is a morphism $F : X \to X$ such that $F^r = (F_0|_X)^r$ for some integer r. The restriction of F_0 to X is called the *standard Frobenius endomorphism.*

A *linear algebraic group* is an affine variety with multiplication and inversion given by rational maps. Every linear algebraic group contains a maximal connected subgroup G°, the component of the identity. This subgroup is normal and G/G° is finite, so for many purposes it suffices to study connected groups. The most important result on connected linear algebraic groups over finite fields is:

Theorem 0.1 (Lang's Theorem). *If G is a connected linear algebraic group defined over k with Frobenius map F, then the map $G \to G$, $a \mapsto a^{-F}a$ is onto.*

Proposition 0.2. *Let G be a connected linear algebraic group defined over k. Let X be a variety with G-action defined over k. Let $\mathcal{O}$ be an F-stable orbit of X. Then*

(1) $\mathcal{O}(k)$ is nonempty;

(2) given $v \in \mathcal{O}(k)$ and $g \in G$, vg is in $\mathcal{O}(k)$ if, and only if, $g^F g^{-1}$ is in the stabiliser G_v;

(3) the $G(k)$-orbits of $\mathcal{O}(k)$ correspond to the F-conjugacy classes of G_v/G_v°.

Proof.

(1) Let u be an element of $\mathcal{O}$. Since u^F is also in $\mathcal{O}$ we have $u^F c = u$ for some c in G. By Lang's Theorem, $c = a^F a^{-1}$ for some $a \in G$. Then $(ua)^F = u^F a^F = u^F ca = ua$, so ua is in $\mathcal{O}(k)$.

(2) vg is in $\mathcal{O}(k)$ iff $vg = (vg)^F = v^F g^F = vg^F$ iff $v = vg^F g^{-1}$.

(3) If $vg = vh \in \mathcal{O}(k)$, then hg^{-1} is in G_v, and $g^F g^{-1}$ is F-conjugate to $h^F h^{-1}$ by hg^{-1}. Hence the map taking $vg \in O(k)$ to the F-class of $g^F g^{-1}$ is well defined.

Now xg and xh are in the same $G(k)$-orbit of $\mathcal{O}(k)$ iff $hg^{-1} \in G(k)$, ie, $gh^{-1} = (gh^{-1})^F$, ie, $h^{-1}h^F = g^{-1}g^F$. Hence the map taking $(xg)^{G(k)}$ to the F-class of $g^F g^{-1}$ is well defined.

The map is injective since $g^F g^{-1} = n^{-F}(h^F h^{-1})n$ for $n \in G_v$, implis $h^{-1}ng \in G(k)$ and $h^{-1}ng$ maps xh to xg. The map is onto by Lang's theorem.

Finally we show that the quotient map $G_v \to G_v/{G_v}^\circ$ induces a bijection on F-conjugacy classes. Suppose $G_v^\circ h$ is F-conjugate to $G_v^\circ h'$, ie, there exists $k \in G_v^\circ$ such that hk is F-conjugate to h' in G_v°. $hk = a^{-F}h'a$. By Lang's theorem applied to $F' : G_v^\circ \to G_v^\circ$, $k \mapsto k^{Fh}$, we get $b \in G_v^\circ$ such that $k = b^{Fh}b^{-1} = h^{-1}b^F hb^{-1}$. So $b^F hb^{-1} = hk$, and h is F-conjugate to hk, which is F-conjugate to h'.

□

Now taking the conjugacy action of G on itself, we get that every F-stable class has a retional element x, say, and if $C_G(x)$ is connected, then then the class doesn't split.