Computing in solluble linear algebraic groups

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August 1, 2005

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Linear algebraic groups

A subgroup of $\mathrm{GL}_n(k)$ defined by polynomial equations,
k a perfect field

eg: $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$,
group of lower triangular matrices,
group of lower unitriangular matrices

Soluble and unipotent groups

We consider k-split connected soluble groups G, ie,

$$G = G_1 \geq G_2 \geq \cdots \geq G_m \geq 1$$

with quotients G_i/G_{i+1} isomorphic to k^+ or $k^\times$

Unipotent matrix:	Only eigenvalue is 1
Unipotent group:	Every element unipotent

Triangularisation

$$\text{soluble} \iff \text{conjugate into the group of lower triangular matrices}$$
$$\text{unipotent} \iff \text{conjugate into the group of lower unitriangular matrices}$$

$G = U \rtimes T$ where
U is unipotent (conjugate into the group of unitriangular matrices),
T is a direct sum of copies of $k^\times$

Presentations for unipotent groups

Generalise the PC-presentation

Take $x_i : k^+ \to U_i$ splitting $U_i \to U_i/U_{i+1}$

Generators: $x_i(a)$ for $i = 1, \ldots, m$ and $a \in k$
Relations:

$$x_i(a)x_i(b) = x_i(a+b)$$
$$x_j(b)x_i(a) = x_i(a)x_j(b) \sum_{h \geq \max(i,j)} x_h(C_{ijh}(a,b))$$

for polynomials C_{ijh}

Root orderings for $U \leq \mathsf{SL}_4$

$$\begin{pmatrix} 1 & & & \\ a_{12} & 1 & & \\ a_{13} & a_{23} & 1 & \\ a_{14} & a_{24} & a_{34} & 1 \end{pmatrix}$$

$$x_{ij}(a) = 1 + aE_{ji}$$

Height ordering:	12, 23, 34, 13, 24, 14
Additive ordering:	12, 13, 23, 14, 24, 34
Weight ordering:	12, 13, 23, 14, 24, 34

Collection example 1

For the height ordering, collect with

$$x_{lm}(b)x_{ij}(a) = x_{ij}(a)x_{lm}(b) \cdot x_{im}(\delta_{jl}ab)x_{jl}(-\delta_{im}ab)$$

$$\begin{array}{l}
12, 23, 34, 13, 24, 14, \underline{12} \\
12, 12, 23, \color{red}{13}, \underline{34}, 13, 24, \color{red}{14}, 14 \\
12, 12, 23, 34, 13, \color{red}{14}, \underline{13}, 24, 14, 14 \\
12, 12, 23, 34, 13, 13, 14, 24, \underline{14} \\
12, 12, 23, 34, 13, 13, 14, 14, 24 \\
12, 23, 34, 13, 14, 24
\end{array}$$

Collection example 2

For the additive ordering, collect with

$$x_{lm}(b)x_{ij}(a) = x_{ij}(a) \cdot x_{lj}(-\delta_{im}ab)x_{mi}(\delta_{jl}ab) \cdot x_{lm}(b)$$

$$12, 13, 23, 14, 24, 34, \underline{12}$$
$$12, 12, 13, {\color{red}13}, 23, 14, {\color{red}14}, 24, 34$$
$$12, 13, 23, 14, 24, 34$$

Noncollection example

For the weight ordering, compute Hall polynomials from the representation:

$$x_{12}(a_{12})x_{13}(a_{13})x_{23}(a_{23})x_{14}(a_{14})x_{24}(a_{24})x_{34}(a_{34}) = \begin{pmatrix} 1 & & & \\ a_{12} & 1 & & \\ a_{13} & a_{23} & 1 & \\ a_{14} & a_{24} & a_{34} & 1 \end{pmatrix}$$

Extensions

- Soluble groups: action of the torus on the unipotent
- Disconnected groups: action of finite group on connected
- Group schemes: Frobenius kernel μ_p
- Nonsplit groups: Twisted tori
- Converting between presentation and representation

Selected references

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