AM4
MAPT ASSIGNMENT 1
2009

1 (a)(i) $F(x) = \int_x^\infty \frac{e^{-t}}{1+t}\,dt, \quad x \in \mathbb{R}$

$$F'(x) = -\frac{e^{-x}}{1+x}$$

$$F''(x) = \frac{e^{-x}}{1+x} + \frac{e^{-x}}{(1+x)^2} = -F'(x) - \frac{F'(x)}{1+x}$$

$$\therefore\ F'' + \left(1 + \frac{1}{1+x}\right)F' = 0$$

In the form $F'' + p(x)F' + q(x)F = 0$

with

$$p(x) = \frac{2+x}{1+x} \quad \text{singular at } x=-1$$

$$q(x) = 0$$

But $(1+x)\,p(x) = 2+x$ is analytic at $x=-1$
$(1+x)^2 q(x) = 0$ also analytic at $x=-1$

$\therefore\ x=-1$ is a regular singular point.

~~(b)~~ (ii) $F'(x) = -\frac{e^{-x}}{1+x}$ from (a)(i)

Let $\xi = 1+x$. Then $e^{-x} = e^{1-\xi} = e^1\left(1 - \xi + \frac{\xi^2}{2} + O(\xi^3)\right)$
as $\xi \to 0^+$.

I.e.

$$F'(x) = -\frac{e^1}{\xi}\left(1 - \xi + O(\xi^2)\right) \quad \text{as } \xi \to 0^+.$$

$$= -\frac{e^1}{\xi} + e^1 + O(\xi) \quad \text{as } \xi \to 0^+$$

$$\therefore\ F(x) = -e\ln\xi + \text{const} + e\xi + O(\xi^2)$$

To leading-order, we have $F(x) \sim -e\ln(1+x)$ as $x \to (-1)$