1 (b) (i) $$F(x) = \int_x^\infty \frac{e^{-t}}{1+t}\,dt$$

$$= \int_x^\infty \frac{1}{1+t} \cdot -\frac{d}{dt}\left(e^{-t}\right) dt$$

$$= \left[-\frac{e^{-t}}{1+t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{(1+t)^2}\,dt$$

$$= \frac{e^{-x}}{1+x} - \int_x^\infty \frac{e^{-t}}{(1+t)^2}\,dt$$

$\therefore$ result is true for $N = 1$.

Suppose it is true for $N = 1, \dots, K$. Then ~~integration~~

$$F(x) = \frac{e^{-x}}{1+x}\left(1 - \frac{1}{1+x} + \frac{2}{(1+x)^2} - \cdots + (-1)^{K-1}\frac{(K-1)!}{(1+x)^{K-1}}\right)$$

$$+ (-1)^K K!\, I_{K+1}(x).$$

But

$$I_{K+1}(x) = \int_x^\infty \frac{e^{-t}}{(1+t)^{K+1}}\,dt$$

$$= -\left[\frac{e^{-t}}{(1+t)^{K+1}}\right]_x^\infty - (K+1)\int_x^\infty \frac{e^{-t}}{(1+t)^{K+2}}\,dt$$

This yields

$$F(x) = \frac{e^{-x}}{1+x}\left(1 - \frac{1}{1+x} + \frac{2}{(1+x)^2} - \cdots + (-1)^K\frac{K!}{(1+x)^K}\right)$$

$$- (-1)^K (K+1)!\, I_{K+2}(x)$$

So the case $N = K+1$ is also true.

This gives the desired result.

1(b)(ii) Consider

$$R(x) := \frac{F(x)e^{x}(1+x) - \left(1 - \frac{1}{1+x} + \frac{2}{(1+x)^2}\right)}{1/(1+x)^2}$$

$$= (-1)^3\, 3!\, I_4(x)\, \frac{e^{x}(1+x)}{1/(1+x)^2}$$

$$= -e^{x}(1+x)^3\, 3!\, I_4(x).$$

Now

$$|I_4(x)| = \left| \int_0^\infty \frac{e^{-x-s}}{(1+x+s)^4}\, ds \right| \qquad \text{where } t = x+s$$

$$= \frac{e^{-x}}{(1+x)^4} \left| \int_0^\infty \frac{e^{-s}}{\left(1+\frac{s}{1+x}\right)^4}\, ds \right|$$

$$\leq \frac{e^{-x}}{(1+x)^4} \left| \int_0^\infty e^{-s}\, ds \right| \qquad \text{since } 1 + \frac{s}{1+x} > 1.$$

$$= \frac{e^{-x}}{(1+x)^4}$$

$$\therefore\ |R(x)| \leq e^{x}(1+x)^3\, 3! \cdot \frac{e^{-x}}{(1+x)^4}$$

$$= \frac{3!}{1+x} \qquad (\to 0 \text{ as } x \to +\infty.)$$

$$\therefore\ e^{x}(1+x)F(x) = 1 - \frac{1}{1+x} + \frac{2}{(1+x)^2} + O\left(\frac{1}{(1+x)^3}\right), \text{ as } x \to +\infty$$

Note that

$$O\left(\frac{1}{(1+x)^3}\right) = O\left(\frac{1}{x^3}\right) \quad \text{as } x \to +\infty.$$

1(b)(iii) Now we consider

$$R_N(x) := \frac{G(x) - S_N(x)}{1/(1+x)^N}$$

$$= (-1)^{N+1}(N+1)!\, I_{N+2}(x)\, \frac{e^x(1+x)}{1/(1+x)^N} \quad \text{by (b)(i).}$$

where

$$|I_{N+2}(x)| = \left| \frac{e^{-x}}{(1+x)^{N+2}} \int_{\infty}^{\infty} \frac{e^{-s}}{\left(1+\frac{s}{1+x}\right)^{N+2}}\, ds \right|, \quad t = s+x$$

$$\leq \frac{e^{-x}}{(1+x)^{N+2}} \int_0^\infty e^{-s}\, ds \quad \text{by similar arguments to (b)(ii)}$$

$$= \frac{e^{-x}}{(1+x)^{N+2}}$$

$$\therefore\ |R_N(x)| < \frac{(N+1)!}{1+x}$$

$$\Rightarrow\ |G(x) - S_N(x)| < \frac{(N+1)!}{(1+x)^{N+1}}$$

Since

$|R_N(x)| \to 0$ as $x \to +\infty$

we have

$$G(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k\, k!}{(1+x)^k}$$

(b) (iv) $u_k(x) = \dfrac{(k+1)!}{(1+x)^{k+1}}$

$$\therefore \quad r_k(x) = \frac{(k+2)!}{(1+x)^{k+2}} \cdot \frac{(1+x)^{k+1}}{(k+1)!}$$

$$= \frac{(k+2)}{(1+x)}$$

$$\therefore \quad r_k(x) \begin{cases} < 1 & \text{for } k+2 < 1+x \\ = 1 & \text{for } k+2 = 1+x \\ > 1 & \text{for } k+2 > 1+x. \end{cases}$$

$\therefore$ $u_k(x)$ takes on a minimum value when $k = x - 1$.

Since $\left| G(x) - S_N(x) \right| < \dfrac{(N+1)!}{(1+x)^{N+1}} = u_N(x)$

the minimum error occurs for $N = x - 1$.
$\therefore$ Best approximation arises ~~for~~ by taking $N = x-1$ terms in $S_N(x)$.

(v) $\left| G(100) - S_{100}(100) \right| \leqslant \dfrac{(101)!}{(101)^{101}} \sim 3.45 \times 10^{-43}$.

So they agree to ~~approx~~ 42 decimal places.